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AN ODDS-RATIO INDIFFERENCE-ZONE SELECTION PROCEDURE FOR BERNOULLI POPULATIONS

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ABSTRACT

We present a new sequential, eliminating procedure for selecting the best system in a single-factor Bernoulli-response experiment with an odds-ratio indifference zone, where best refers to the system with the largest probability of success on any given trial. Analytical results show that the proposed procedure is more efficient than existing procedures in that it requires fewer observations to identify the best system when only two systems are being considered. Empirical results show that the procedure is more efficient for comparing up to five systems when the specified odds-ratio indifference zone is greater than two.

1 INTRODUCTION

The Bernoulli selection problem is a special case of the more general ranking-and-selection problem of designing an experiment to select the best out of a group of K systems. The term "best" often refers to the system having the largest expected value. In the Bernoulli selection problem this is the system with the largest probability of success on any given trial, where trials are binary observations sampled from each of the K Bernoulli systems being considered. Traditional applications for Bernoulli selection are reliability and quality control, where outcomes are considered successes if they survive the mission or meet the quality standard, and they are failures otherwise. The reliability context is the most common in discrete-event, stochastic simulation.

A procedure used for solving the Bernoulli selection problem must specify how many samples to take from each system to guarantee a probability of correct selection (PCS) $\geq P^*$. In addition, if it is possible to obtain observations from systems individually (sequential sampling) and in different orders (switching), rather than collecting observations in a single stage, then the procedure must also specify how often and in what order to switch sampling between systems.

We present a procedure for selecting the best of K systems in a single-factor Bernoulli-response experiment using an odds-ratio indifference zone $\theta > 1$, which is the minimal odds ratio worth detecting. This procedure is a sequential, eliminating procedure that is superior to existing Bernoulli selection procedures developed by Bechhofer, Kiefer, and Sobel (1968) and Paulson (1994) in that it reduces the expected number of observations required to correctly select the best system with a pre-specified minimum probability P^* when the number of systems K is small ($K \leq 3$). In cases where the minimal odds ratio worth detecting is large ($\theta \geq 2$) our procedure is superior to the existing procedure when comparing up to five systems ($K \leq 5$).

1.1 Problem Statement

The following three inputs are assumed to be given: (1) $K \geq 2$ Bernoulli systems $\pi_1, \pi_2, \ldots, \pi_K$ with unknown probabilities of success $p_1, p_2, \ldots, p_K$, where independent random samples can be drawn from each of the systems; (2) PCS P^* such that $1/K < P^* < 1$; and (3) odds-ratio indifference zone θ such that $\theta > 1$. For notational convenience, we will assume $p_1 \leq p_2 \leq \cdots \leq p_K$ from here on, so that (unknown to us) system K is the best. A correct selection is said to be made if the system associated with p_K is selected whenever

$$\frac{p_K(1-p_{K-1})}{p_{K-1}(1-p_K)} \geq \theta. \tag{1}$$

The left-hand side of (1) is the odds ratio, since it is the odds of success vs. failure for system K, divided by the odds of success vs. failure for system $K-1$.

The metric used to evaluate procedures is the expected number of observations required to select the best system with probability $\geq P^*$. Procedures that, on average, require fewer observations to make a correct selection are preferred. An alternative metric used by Hong and Nelson (2003) to evaluate selection procedures is total computational cost,

which includes the costs of sampling and of switching between systems. This metric has not been applied to the Bernoulli selection problem and will not be considered here.

1.2 Preliminaries

This paper focuses on Bernoulli selection procedures that are eliminating and utilize an odds-ratio indifference zone. Eliminating procedures cease sampling from systems that are inferior once the data collected indicate that the system in question is no longer in contention for the best, as opposed to non-eliminating procedures where each system remains in the pool of candidates for selection until the experiment is run to completion. Elimination can significantly decrease the total number of observations required to make a correct selection, and computer simulation is the ideal environment for sequential, eliminating procedures because sampling can be automated.

Three types of indifference-zone formulations are used in Bernoulli selection experiments. In each case the goal is to have a PCS $\geq P^*$ when condition (2), (3) or (4) holds: Absolute:

$$p_K - p_{K-1} \geq \delta > 0, \tag{2}$$

Odds Ratio:

$$\frac{p_K(1-p_{K-1})}{p_{K-1}(1-p_K)} \geq \theta > 1, \tag{3}$$

Relative Risk:

$$\frac{p_K}{p_{K-1}} \geq \psi > 1. \tag{4}$$

We briefly discuss the absolute and odds-ratio indifference zones. For a more comprehensive discussion refer to Bechhofer, Santner, and Goldsman (1995). The absolute indifference zone is fairly intuitive. When specifying an absolute indifference zone, the magnitude of the individual probabilities p_K and p_{K-1} are irrelevant; the only consideration in choosing δ is the difference between p_K and p_{K-1}. However, the odds-ratio indifference zone is dependent on the the magnitude of the individual probabilities p_K and p_{K-1}. For example, in the case where $p_1 = 0.10$, $p_2 = 0.30$, and $p_3 = 0.50$, the absolute difference between p_2 and p_1 is 0.20, which is the same as the absolute difference between p_3 and p_2. However, the odds ratio between p_2 and p_1 is 3.86, which is not the same as the odds ratio between p_3 and p_2, which is 2.33. The odds ratio magnifies small differences when the probabilities are near 0 or 1.

An important distinction between the absolute and odds-ratio indifference zone is that when an absolute indifference zone is specified, the maximum number of observations required to make a correct selection with probability $\geq P^*$ can be pre-determined. This type of procedure is referred to as a closed procedure. Alternatively, when an odds-ratio indifference zone is specified, an open procedure must be used. This means that no upper bound on the number of observations required to make a selection with probability P^* can be pre-determined and hence sequential sampling must be permitted. Sequential sampling allows for collecting observations from each system in multiple data-collection stages.

A benefit that the odds-ratio formulation provides is the ability to model the Bernoulli selection problem as a random walk in the case where

$$\frac{p_K(1-p_j)}{p_j(1-p_K)} = \theta, \quad \text{for } j = 1, 2, \ldots, K-1$$

which is referred to as the slippage configuration (SC). In this random walk model, let $X_{i,n}$ be the n^{th} Bernoulli observation from system i. Let $\{S_{ij}(n); n = 0, 1, \ldots\}$ represent the state-change process between systems i and j, such that

$$S_{ij}(n) = \sum_{l=1}^{N_{ij}(n)} (X_{i,l} - X_{j,l}),$$

with $N_{ij}(n)$ being the number of Bernoulli trials required to attain n state changes between systems i and j. Thus,

$$N_{ij}(n) = \min\{m : \sum_{l=1}^{m} I(X_{i,l} \neq X_{j,l}) = n\},$$

where I is the indicator function. Then, we can show that for any $j \neq K$ when the SC applies

$$\begin{aligned} &\Pr\{S_{Kj}(n+1) = b \mid S_{Kj}(n) = a\} \\ &\quad = \begin{cases} \theta/(\theta+1), & b = a+1 \\ 1/(\theta+1), & b = a-1 \\ 0, & o.w. \end{cases} \end{aligned} \tag{5}$$

and $\{S_{Kj}(n); n = 0, 1, \ldots\}$ is a discrete-time Markov chain. Notice that the transition probabilities depend only on θ and not the specific values of the success probabilities.

1.3 Previous Work

Bernoulli selection procedures that utilize an odds-ratio indifference zone have been developed by Bechhofer, Kiefer, and Sobel (1968) and Paulson (1994). These two procedures are referred to as *BKS* and *BP*, respectively.

Procedure *BKS* is derived from basic random walk results. This procedure is sequential and non-eliminating. The advantage of *BKS* compared to *BP* is that it lowers the expected number of data collection stages, but this is at the cost of increasing the total number of observations required to make a selection. In a simulation context, a stage could

be the number of calls made to run the simulation model if each call results in an independent observation from each of the remaining K Bernoulli systems. So, for example, if $K = 5$, *BKS* may call the simulator 100 times obtaining 100 observations from each of the 5 systems for a total of 500 observations. On the other hand *BP* may call the simulator 50 times obtaining 50 observations from each of the 5 systems, and then eliminate 3 of the 5 systems. Next, *BP* may call the simulator 100 times obtaining observations from only the 2 remaining systems. Thus, *BP* would have made 150 calls to the simulator but only (5)(50)+(2)(100) = 450 total observations.

There are applications in which minimizing the number of data collection stages, rather than the total number of observations, should be the primary objective. In such applications, the cost of obtaining individual observations is low relative to the cost of setting up the experiment at each stage. This is typically not the case in simulation experiments, nor is it the objective of our proposed procedure, so we do not consider *BKS* further.

Procedure *BP* is most comparable to our proposed procedure in that it is a sequential, eliminating procedure, which, relative to other Bernoulli selection procedures, effectively reduces the expected total number of observations required to make a correct selection. Paulson (1994) developed this procedure for the more general problem of selecting the best from a group of K Koopman-Darmois populations. All samples are assumed to be independent. The frequency function of $X_{i,n}$ is of the form

$$f_i(x) = \exp[x\beta_i + g(\beta_i) + h(x)],$$

which is subject to the restriction that $g(\beta_i)$ has a derivative $dg/d\beta_i$ that is decreasing in β_i, which is satisfied in the Bernoulli selection problem where

$$\beta_i = \ln\left(\frac{p_i}{1-p_i}\right)$$

and

$$g(\beta_i) = -\ln(1+e^{\beta_i}) = \ln(1-p_i).$$

Furthermore, the absolute indifference-zone formulation

$$\ln\left(\frac{p_K}{1-p_K}\right) - \ln\left(\frac{p_{K-1}}{1-p_{K-1}}\right) \geq \delta,$$

can be re-written as

$$\ln\left(\frac{p_K(1-p_{K-1})}{p_{K-1}(1-p_K)}\right) \geq \delta,$$

leading to the odds-ratio indifference zone

$$\frac{p_K(1-p_{K-1})}{p_{K-1}(1-p_K)} \geq \theta.$$

with $\theta = e^{\delta}$. For the *BP* selection procedure see Bechhofer, Santner, and Goldsman (1995).

Next, we present a new Bernoulli selection procedure, which is followed by an empirical analysis comparing our procedure to *BP*. Empirical results indicate that our procedure is more efficient than *BP* in terms of the expected total number of observations required to make a selection in cases where the number of systems is small ($K \leq 3$) or the minimal difference between systems worth detecting is large ($\theta \geq 2$) and $K \leq 5$.

2 PROCEDURE *BWN*

Our proposed procedure, *BWN*, is a sequential, eliminating procedure that utilizes an odds-ratio indifference zone to select the best of K systems in a Bernoulli-response experiment with PCS $\geq P^*$. Conceptually, *BWN* works by making pairwise comparisons between each inferior system and the unknown best system. In developing *BWN*, we first notice that

$$\text{PCS} \geq \Pr\{\pi_K \text{ eliminates } \{\pi_1, \pi_2, \ldots, \pi_{K-1}\}\}. \quad (6)$$

This inequality becomes more conservative as K increases because the more systems being compared, the higher the probability that any of these systems may eliminate each other, rather than being eliminated by the best. This inequality also becomes more conservative as θ decreases because we are indicating that we wish to detect very small differences between systems, and this increases the probability that any of the inferior systems will eliminate each other because they are all close.

We use the Bonferroni inequality to break up the overall PCS requirement into pairwise comparisons between each of the $K-1$ inferior systems and the best. Applying the Bonferroni inequality, we can show that

$$\begin{aligned}&\Pr\{\pi_K \text{ eliminates } \{\pi_1, \pi_2, \ldots, \pi_{K-1}\}\}\\ &\geq \quad 1 - \sum_{i=1}^{K-1} \Pr\{\pi_i \text{ eliminates } \pi_K\},\end{aligned}$$

which implies that overall the

$$\text{PCS} \geq 1 - \sum_{i=1}^{K-1} \Pr\{\pi_i \text{ eliminates } \pi_K\}. \quad (7)$$

Thus, we require

$$\text{PCS}_i \geq 1 - \frac{1 - P^*}{K - 1}$$

for each pairwise comparison $i = 1, 2, \ldots, K - 1$. The required PCS for each pairwise comparison is denoted P_{req}, where

$$P_{req} = 1 - \frac{1 - P^*}{K - 1}. \tag{8}$$

The Bonferroni inequality also becomes more conservative as K increases. The effect of both the Bonferroni inequality and the inequality used in Equation (6) becoming overly conservative is that the procedure is less efficient in terms of the expected number of observations required to make a correct selection.

We model the state-change process $\{S_{Kj}(n)\}$ for $j = 1, 2, \ldots, K - 1$ as a random walk with symmetric upper and lower absorbtion boundaries $\pm B_0$ as shown in see Figure 1. This is referred to as a "gambler's ruin" (GR) problem (see Ross 2000) with state space $S = \{-B_0, -B_0 + 1, \ldots, B_0\}$ and state transitions given by Equation (5) except that $-B_0$ and B_0 are now absorbing states. Since we use the Bonferroni inequality to break the problem into paired comparisons, and since we assume the worst case (the SC) for each pair, we only need to consider a single state-change process, say $\{S_{K1}(n)\}$.

In the GR, the PCS is equivalent to the probability of being absorbed in state B_0, which is

$$\text{PCS} = 1 - \frac{1}{1 + \theta^{B_0}}. \tag{9}$$

Therefore, given K, θ, and P^*, we can choose integer B_0 such that

$$\frac{1 - P^*}{K - 1} \geq \frac{1}{1 + \theta^{B_0}},$$

or

$$B_0 \geq \frac{\ln(1 - P^*) - \ln(P^* + (K - 2))}{\ln(1/\theta)}. \tag{10}$$

The motivation for modelling the Bernoulli selection problem as a GR is that when $K = 2$, this model is superior to BP, which is the current best sequential, eliminating Bernoulli selection procedure in terms of expected number of observations required to make a correct selection.

Proposition 1: The GR approach is more efficient than BP for the case $K = 2$, and the difference between them increases as P^* decreases.

Proof: We are considering the case where $K = 2$, which greatly simplifies both procedures. In the GR the stage at which the procedure terminates is the smallest n such that

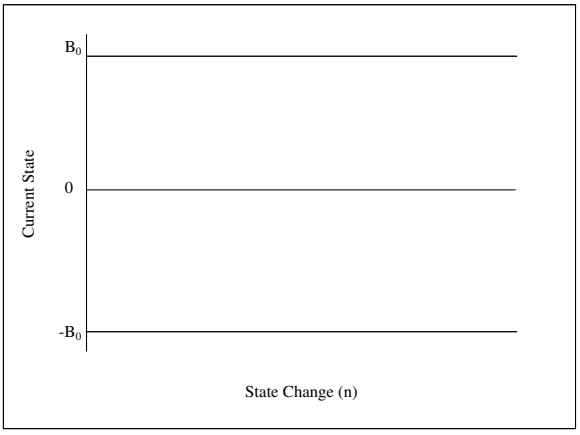

Figure 1: Region Defined by Gambler's Ruin

$$|S_{21}(n)| \geq +B$$

where

$$\begin{aligned} B_0 &= \frac{\ln(1 - P^*) - \ln(P^*)}{\ln(1/\theta)} \\ &= \frac{\ln(1 - P^*) - \ln(P^*)}{-\ln(\theta)}. \end{aligned}$$

Therefore, we stop as soon as

$$|S_{21}(n)| \geq \frac{-\ln(1 - P^*) + \ln(P^*)}{\ln(\theta)}.$$

The BP elimination rule stops as soon as

$$\theta^{|S_{21}(n)|} \geq \frac{1}{1 - P^*}.$$

Taking the natural log of this we get

$$|S_{21}(n)| \ln(\theta) \geq \ln(1) - \ln(1 - P^*) = -\ln(1 - P^*).$$

Therefore we stop as soon as

$$|S_{21}(n)| \geq \frac{-\ln(1 - P^*)}{\ln(\theta)} > \frac{-\ln(1 - P^*) + \ln(P^*)}{\ln(\theta)}.$$

□

The GR is a special case of BWN that occurs only when the right-hand side of Equation (10) happens to be integer. For example, in the case where $\theta = 3.0$, $P^* = 0.90$, and $K = 2$, using Equation (8) we calculate that $P_{req} = 0.90$ and using Equation (10) we calculate that $B_0 \geq 2$. From Equation (9), the PCS is 0.90 when $B_0 = 2$, which is exactly

P_{req}. In this case, *BWN* is equivalent to the GR model we described.

In cases where (10) is not integer, we must be round up to the nearest integer to assure that the PCS $\geq P_{req}$. For example, in the case where $\theta = 3.0$, $P^* = 0.95$, and $K = 2$, we calculate $P_{req} = 0.95$ and $B_0 \geq 2.68$. Hence, to assure that PCS $\geq P_{req}$, we rounded $B_0 = \lceil 2.68 \rceil = 3$. We then calculate that when $B_0 = 3$, PCS $= 0.97$, which is greater than $P_{req} = 0.95$.

The consequence of having to round-up when (10) is not integer is that *BWN* becomes more conservative in terms of the expected number of observations required to make a selection. This inefficiency illustrates the prominent role discreteness plays in the Bernoulli selection problem and, loosely speaking, is due to the fact that the region defined by the $\pm B_0$ is too wide. However, it is possible to reduce the inefficiency caused by rounding by cutting-down the region to help eliminate some of the excess area. *BWN* provides a strategic way for removing the excess area, which was motivated by Kim and Nelson's (2001) procedure KN. The final region defined by *BWN*, illustrated in Figure 2, is comparable in shape to the triangular region of KN, illustrated in Figure 3. In fact, KN can be used for the Bernoulli selection problem, but it is inefficient relative to *BWN* because it was designed for normal populations rather than Bernoulli populations.

The *BWN* region is defined by the the outer-most horizontal absorbing boundaries $\pm B_0$, the outer-most vertical boundary N_0, the series of inner horizontal boundaries $\mathbf{B} = \{\pm B_J, \pm B_{J-1}, \ldots, \pm B_1\}$ and the series of inner vertical boundaries $\mathbf{N} = \{N_1, N_2, \ldots, N_J\}$. Truncating and narrowing the region as the number of state changes increases effectively reduces the expected number of observations required to make a correct selection when (10) is not integer-valued (note that when (10) is integer-valued the region cannot be truncated without the PCS dropping below P_{req}). However, calculating the PCS for this type of region is more complicated than calculating the PCS for a GR model. The trick to calculating the overall PCS for the entire region is to break the region up into rectangles. We define rectangles by the cross products of their horizontal boundaries $[x_a, x_b]$, where $x_a < x_b$, and their vertical boundaries $[y_c, y_d]$, where $y_c < y_d$. For example, consider the *BWN* region defined by $\pm B_0 = 3$, $N_0 = 11$, $\mathbf{B} = \{2, 1\}$, and $\mathbf{N} = \{10, 9\}$ as shown in Figure 2. This region can be broken into rectangles as follows: $R_0 = [0, 9] \times [-3, 3]$, $R_1 = [9, 10] \times [-2, 2]$, and $R_2 = [10, 11] \times [-1, 1]$.

After dividing the region into rectangles, we compute the PCS for the initial rectangle R_0, and then condition on not being absorbed in this rectangle and compute the conditional PCS for the next rectangle R_1. This conditioning argument can be applied iteratively for all rectangles $R_0, R_1, \ldots, R_J$, providing the ability to calculate the overall PCS for any notched region composed of rectangles in series.

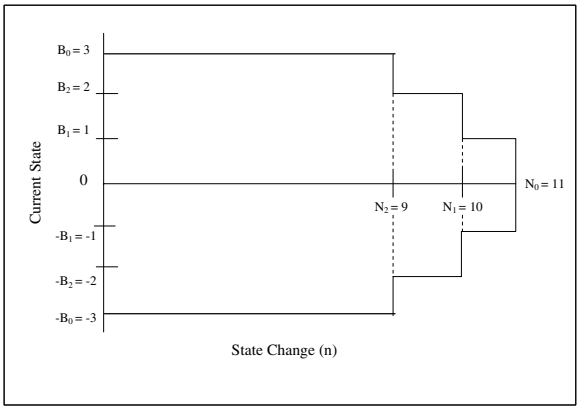

Figure 2: Region Defined *BWN*

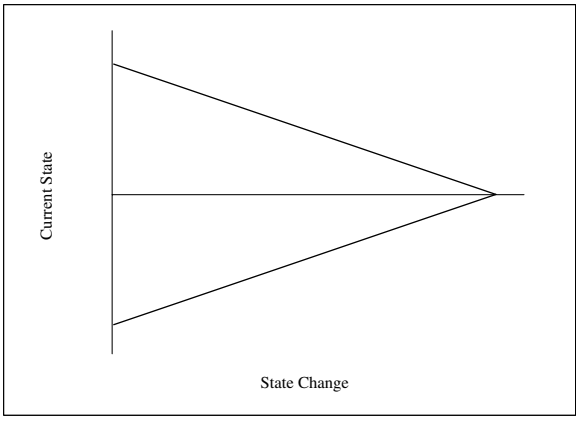

Figure 3: Region Defined by KN

We calculate the PCS for the first rectangle as shown in Figure 4: Given the initial state space $S = \{-B_0, -B_0 + 1, \ldots, B_0\}$ where $\pm B_0$ are absorbing states, state-change process $\{S_{K1}(n), n = 0, 1, \ldots\}$, the initial transient states $Q_0 = \{-B_0 + 1, -B_0 + 2, \ldots, B_0 - 1\}$, the probabilities of starting in each of the initial transient states $q \in Q_0$, and the height of the J^{th} jump B_J, formulate the one-step transition matrix P_0 for the initial rectangle R_0 on S. Let C_0 denote the set of states that, if occupied at state change N_J, indicate we have made a correct selection; $C_0 = \{B_J, \ldots, B_0\}$ in the first rectangle.

The PCS for sub-rectangle R_0, denoted PCS(R_0), corresponds to the probability of being in any state $c \in C_0$ after state change N_J:

$$\text{PCS}(R_0) = \sum_{q \in Q_0} \sum_{c \in C_0} \Pr\{S_{K1}(0) = q\} P_0^{(N_J)}(q, c). \quad (11)$$

To compute the PCS of the next sub-rectangle R_1, we must first compute the probability of starting in each of the transient states $q \in Q_1$ where $Q_1 = \{-B_0 + B_J +$

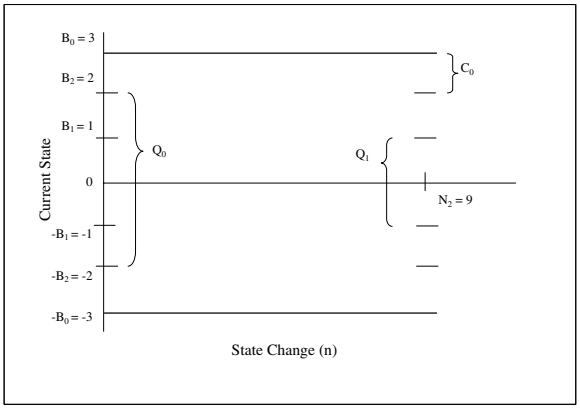

Figure 4: Example Rectangle from BWN Region

$1, \ldots, B_0 - B_J - 1\}$ at state change N_J as follows:

$$\Pr\{S_{K1}(N_J) = q\} = \sum_{i \in Q_0} \Pr\{S_{K1}(0) = i\} P_0^{(N_J)}(i, q). \tag{12}$$

Given the probability of starting in each of the states $q \in Q_1$ and the height of the next jump B_{J-1}, we can then compute the PCS for sub-rectangle R_1 by formulating the one step transition matrix P_1 on the state space $S = \{-B_J, -B_J + 1, \ldots, B_J\}$ where $\pm B_J$ are absorbing states, letting $C_1 = \{B_{J-1}, \ldots, B_J\}$ and using Equation (11).

When the last rectangle is reached, a correct selection is said to be made if we reach the upper absorbing boundary B_1 before the lower absorbing boundary $-B_1$ or if we end up in any state above the horizontal axis (i.e., states $1, 2, \ldots, B_1 - 1$) at state change N_0. In the case where the process ends in state 0, then select the best randomly.

2.1 Example PCS Calculation

We calculate the PCS for the first rectangle of the *BWN* region illustrated in Figure 4 as an example. This region is the region specified by our proposed procedure for the case where $K = 2$, $P^* = 0.95$, and $\theta = 3$. For this region there are a total of $J = 3$ jumps, $S = \{-3, -2, -1, 0, 1, 2, 3\}$ where ± 3 are absorbing states, and $Q_0 = \{-2, -1, 0, 1, 2\}$. Applying Equation (5) we formulate the one-step transition matrix for rectangle R_0 on state space S as follows:

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ .75 & 0 & .25 & 0 & 0 & 0 & 0 \\ 0 & .75 & 0 & .25 & 0 & 0 & 0 \\ 0 & 0 & .75 & 0 & .25 & 0 & 0 \\ 0 & 0 & 0 & .75 & 0 & .25 & 0 \\ 0 & 0 & 0 & 0 & .75 & 0 & .25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In this example, the probabilities of starting in each of the initial transient states $q \in Q_0$ are as follows:

$$\Pr\{S_{K1}(0) = q\} = \begin{cases} 1, & \text{if } q = 0 \\ 0, & o.w. \end{cases}$$

Also, in our example $N_3 = 9$, $B_3 = 2$, and $C_0 = \{2, 3\}$. Then, we apply Equation (11) to calculate PCS(R_0) = 0.8677. If we were to continue this process for each rectangle we would find that, applying Equation (12), the transient state probabilities are

$$\Pr\{S_{K1}(9) = q\} = \begin{cases} 0.0751, & \text{if } q = 1 \\ 0.025, & \text{if } q = -1 \\ 0, & o.w. \end{cases}$$

Continuing, we find that PCS(R_1) = 0.0563 and PCS(R_2) = 0.0282 for an overall PCS = 0.9522.

Given the ability to compute the PCS for any number of sub-rectangles in series, the problem then is to determine how many sub-rectangles of what dimensions to divide the outer region. The goal is to get the PCS associated with the region as close to P_{req} as possible by systematically removing pieces of the region to guide it toward a triangular shape. The following algorithm *BWN* is a heuristic approach for doing this; it is heuristic in the sense that we cannot prove that the final region is in any sense optimal, but we can prove that it maintains the required PCS.

2.2 *BWN* Setup

Setup for the *BWN* selection procedure requires the following inputs: $K \geq 2$ Bernoulli systems, desired PCS P^* such that $1/K < P^* < 1$, and odds-ratio indifference zone $\theta > 1$. *BWN* outputs the outer-most horizontal absorbing boundaries $\pm B_0$, the outer-most vertical boundary N_0, the number of inner rectangles J, the series of inner horizontal boundaries $\{\pm B_J, \pm B_{J-1}, \ldots, \pm B_1\}$ for each of the $j = 0, 2, \ldots, J$ rectangles, and the series of inner vertical boundaries $\{N_1, N_2, \ldots, N_J\}$ for each of the $j = 0, 2, \ldots, J$ inner rectangles. There are a total of $J + 1$ rectangles including the initial rectangle $R_0 = [0, N_J] \times [-B_0, B_0]$.

Step 0: Calculate the required PCS, denoted P_{req}, necessary for each pairwise comparison to achieve an overall PCS $\geq P^*$

$$P_{req} = 1 - \frac{1 - P^*}{K - 1}.$$

Step 1: Find the minimum absorbtion boundary B_0, which corresponds to the height of the outer rect-

angle.

$$B_0 = \left(\frac{\ln(1 - P^*) - \ln(P^* + (K-2))}{\ln(1/\theta)} \right).$$

If B_0 is integer-valued, the region is defined by $\{\pm B_0\}$ and $N_0 = \infty$. Stop.
Else set $B_0 = \lceil B_0 \rceil$, initialize $N = B_0$, and go to Step 2.

Step 2: Find the minimum N_0 at which the process can be terminated, which corresponds to the width of the outer rectangle. First, formulate the one-step transition matrix P_0 on $\{-B_0, -B_0 + 1, \ldots, B_0\}$. Then calculate

$$\text{PCS} = \sum_{s=1}^{B_0} P_0^{(N_0)}(0, s) + 0.5 P_0^{(N_0)}(0, 0).$$

If PCS $< P_{req}$, set $N_0 = N_0 + 1$ and repeat Step 2.
Else initialize $j = 1$, $B_j = 1$ and $N_j = N_0 - 1$ and go to Step 3.

Step 3: Find the height of the j^{th} sub-rectangle R_j. First, calculate the overall PCS for the region, which consists of rectangles $R_0 = [0, N_j] \times [-B_0, B_0], \ldots, R_j = [N_1, N_0] \times [-B_1, B_1]$ using Equations (11) and (12).
If PCS $\geq P_{req}$ set $B_j = B_j - 1$ and go to Step 4.
Else set $B_j = B_j + 1$.
If $B_j = B_0$, stop.
Else repeat Step 3.

Step 4: Find the width of the j^{th} sub-rectangle. First, set $N_j = N_j - 1$. Then calculate the overall PCS for the region, which consists of rectangles $R_0 = [0, N_j] \times [-B_0, B_0], \ldots, R_j = [N_1, N_0] \times [-B_1, B_1]$ using Equations (11) and (12).
If PCS $\geq P_{req}$, repeat Step 4.
Else set $N_j = N_j + 1$ go to Step 5.

Step 5: Check to see if we can remove an additional rectangle. If $B_j < B_0$, increment $j = j + 1$, and go to Step 3.
Else set $J = j$ and stop.

Remark: Due to computational issues associated with finite arithmetic it is practical to add a small value ϵ to P_{req} for checking the first inequality in Step 2 of this setup procedure (i.e., PCS $\leq P_{req} + \epsilon$). Matlab code for executing the *BWN* setup can be obtained from <www.iems.northwestern.edu/~nelsonb/BWNSetup.zip>.

2.3 *BWN* Selection

Given $\mathcal{W} = \{1, 2, \ldots, K\}$, the set of all systems in contention for the best, and the outputs from the *BWN* setup, the *BWN* selection procedure finds the system associated with the largest probability of success on any given trial with PCS $\geq P^*$ when the odds ratio is $\geq \theta$.

Step 0: Define a vector $B'(n), n = 0, 1, \ldots, N_0$ such that

$$B'(n) = \begin{cases} B_0, & \text{if } n \leq N_J \\ B_J, & \text{if } N_J < n \leq N_{J-1} \\ B_{J-1}, & \text{if } N_{J-1} < n \leq N_{J-2} \\ \vdots & \vdots \\ B_1, & \text{if } N_1 < n \leq N_0 \end{cases}$$

Step 1: Obtain independent Bernoulli samples from each $\pi_i, i \in \mathcal{W}$. Calculate

$$S_{ij}(n) = \sum_{l=1}^{N_{ij}(n)} (X_{i,l} - X_{j,l}),$$

for all $i, j \in \mathcal{W}$ such that $i \neq j$.
If $S_{ij}(n) = B'(n)$ remove system j from set $\mathcal{W}$.
If $S_{ij}(n) = -B'(n)$ remove system i from set $\mathcal{W}$.
If $|\mathcal{W}| = 1$ stop and choose the only remaining element of $\mathcal{W}$ as the best system.
Otherwise repeat Step 1.

3 EMPIRICAL ANALYSIS

In this section we report on an empirical evaluation of *BWN* as compared to *BP*.

3.1 Experimental Design

We performed simulation experiments to compare *BWN* to *BP*. There were five factors to be considered: (1) the number of systems being compared, K, (2) the PCS, P^*, (3) the odds-ratio indifference zone, θ, (4) choice of p_K, which is the largest probability of success on any given trial, and (5) the configurations of $p_{K-1}, \ldots, p_1$.

We tested the following levels of each of the five factors
Factor 1 (4 levels): $K = 2, 3, 4, 5$
Factor 2 (3 levels): $P^* = 0.90, 0.95, 0.99$
Factor 3 (3 levels): $\theta = 1.5, 2.0, 3.0$
Factor 4 (2 levels): $p_K = 0.85$ and $p_K = 0.35$
Factor 5 (2 levels): Slippage Configuration (SC) where

$$\frac{p_K(1 - p_j)}{p_j(1 - p_K)} = \theta, \quad \text{for } j = 1, 2, \ldots, K - 1,$$

and the Equally-Spaced Configuration (EC) where

$$\frac{p_i(1-p_j)}{p_j(1-p_i)} = \theta, \quad \text{for } i = 2, 3, \ldots, K, \text{ and } j = i-1.$$

We ran a full-factorial experimental design, requiring 144 simulation experiments. Each simulation experiment was replicated 2500 times. The results of the simulation experiments are discussed in the next section.

3.2 Empirical Results

We have included a subset of the experimental results that illustrate the effects of changing K, θ, and P^*. These results are in Table 1. We did not include the experimental results from changing the configuration (EC versus SC) or changing p_K because these effects were relatively insignificant. However, *BWN* performed slightly better in the EC configuration than in the SC configuration.

The main results from the empirical analysis are as follows:

1. *BWN* is always more efficient than *BP* when $K = 2$, and *BWN* is usually at least as good as *BP* for $K = 3$. This is consistent with Proposition 1 where we proved that *BWN* is more efficient than *BP* for $K = 2$.
2. As anticipated, *BWN* performs worse as K increases, which is due to the fact that the inequalities used in Equations (6) and (7) become more conservative as K increases.
3. *BWN* performs better as P^* decreases, which is consistent with Proposition 1. For example, the savings in terms of expected number of observations required to make a correct selection is much greater for $P^* = 0.90$ than it is for $P^* = 0.99$.
4. *BWN* performs better as θ increases. This is expected due to the fact that the inequality used in Equation (6) is less conservative as θ increases, which increases the efficiency of *BWN*. As a result, in cases where $\theta \geq 2$, *BWN* is superior to *BP* for up to $K = 5$.

4 CONCLUSIONS

In this paper we have presented a sequential, eliminating procedure, *BWN*, for selecting the best of K systems in a single-factor Bernoulli-response experiment using an odds-ratio indifference zone. We have shown analytically that *BWN* is superior to the existing procedure *BP* in that it requires fewer observations to identify the best system when only two systems are being considered. Empirical results show that *BWN* is often superior to *BP* for $K = 3$. Furthermore, when $\theta \geq 2$, *BWN* is often superior for up to $K = 5$. However, *BWN* does not scale up for comparing a large number of systems so for $K > 5$ *BP* should be used. When only a few systems are being compared, considerable savings can be obtained by using *BWN*, especially in cases where the minimal practical difference worth detecting, θ, is large.

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Table 1: Estimated Expected Number of Observations to Make a Correct Selection for the SC When $p_K = 0.85$

	$\theta = 1.5$				$\theta = 2.0$				$\theta = 3.0$			
	$P^* = 0.90$		$P^* = 0.99$		$P^* = 0.90$		$P^* = 0.99$		$P^* = 0.90$		$P^* = 0.99$	
K	*BWN*	*BP*	*BWN*	*BP*	*BWN*	*BP*	*BWN*	*BP*	*BWN*	*BP*	*BWN*	*BP*
2	155	169	387	391	53	65	123	126	16	29	47	51
3	374	380	740	737	122	134	232	235	42	45	79	82
4	577	564	1052	1037	178	178	345	344	69	74	125	129
5	809	754	1329	1318	261	251	431	430	95	98	157	160

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