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SOLVING STOCHASTIC MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS USING SIMULATION

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ABSTRACT

Recently, simulation-based methods have been successfully used for solving challenging stochastic optimization problems and equilibrium models. Here we report some of the recent progress we had in broadening the applicability of so-called the sample-path method to include the solution of certain stochastic mathematical programs with equilibrium constraints. We first describe the method and the class of stochastic mathematical programs with complementarity constraints that we are interested in solving and then outline a set of sufficient conditions for its almost-sure convergence. We also illustrate an application of the method to solving a toll pricing problem in transportation networks. These developments also make it possible to solve certain stochastic bilevel optimization problems and Stackelberg games, involving expectations or steady-state functions, using simulation.

1 INTRODUCTION

This paper reports our recent progress in extending the range of problems that can be solved by sample-path methods. This is achieved by analyzing a class of stochastic mathematical programs with equilibrium constraints for which both the objective function and the equilibrium constraints (represented by complementarity constraints) may be stochastic. In this section we briefly outline the class of problems concerned.

Roughly speaking, sample-path methods are developed for solving a problem of optimization or equilibrium, involving a limit function f_∞ for which we do not have an analytical expression. However, we can use simulation to observe functions f_n that converge pointwise to f_∞ as $n \to \infty$ almost surely. In the kind of applications typically encountered, f_∞ could be a steady-state performance measure of a dynamic system or an expected value in a static system. In systems that evolve over time, we simulate the operation of the system for, say, n time units and then compute an appropriate performance measure. In static systems we repeatedly sample instances of the system and compute an average. In both cases, we use the method of common random numbers to observe f_n at different parameter settings. A crucial observation is the following: once we fix an n and a sample point (using common random numbers), f_n becomes a deterministic function. In the simplest form, the sample-path methods then solve the resulting deterministic problem (using f_n with the fixed sample path selected), and take the solution as an estimate of the true solution. Furthermore, in many cases derivatives or directional derivatives of the f_n can be obtained using well-established methods of gradient estimation such as infinitesimal perturbation analysis (IPA), see Ho and Cao (1991) and Glasserman (1991), or automatic differentiation capabilities, see Griewank (2000). Clearly, the success of this basic approach heavily depends on the availability of powerful and efficient deterministic solvers for the underlying deterministic problem type.

It is possible to distinguish between three types of problems. The first involves optimization; in this case we can think of the f_n as extended-real-valued functions: $f_n : \mathbf{R}^k \to \mathbf{R} \cup \{\pm\infty\}$ for $1 \le n \le \infty$, and we are interested in solving

$$\min_x f_\infty(x). \tag{1}$$

This setup also covers optimization problems with deterministic constraints since we can always set $f_\infty(x) = +\infty$ for x that do not satisfy the constraints.

The second problem type is a variational inequality; in this case the f_n would be vector-valued functions: $f_n : \mathbf{R}^k \to \mathbf{R}^k$ for $1 \le n \le \infty$, and our aim would be finding a point $x_0 \in C$, if any exists, satisfying

$$\text{for each } x \in C, \quad \langle x - x_0, f_\infty(x_0)\rangle \geq 0, \tag{2}$$

where $\langle y, z\rangle$ denotes the inner product of y and z, and C is a polyhedral convex subset of $\mathbf{R}^k$. Problem (2) is denoted by $\text{VI}(f_\infty, C)$. It models a very large number of equilibrium phenomena in economics, physics, and operations research; for many examples, see Harker and Pang (1990) and Ferris and Pang (1997). An important class of variational inequalities are complementarity problems; see also Section 2. A well-known area where complementarity is used is nonlinear programming, because the first order necessary conditions for local optimality of a nonlinear program can be stated as a complementarity problem. Note that the availability of very powerful deterministic solvers, for optimization problems, variational inequalities, and equilibrium models, makes the sample-path method an attractive approach.

In the third problem type, the optimization and equilibrium problems are combined together through placing the variational inequalities as a set of constraints into the optimization problem

$$\begin{array}{ll} \min & f_\infty(x, y) \\ \text{s.t.} & (x, y) \in Z \\ & F^\infty(x, y)(v - y) \geq 0 \text{ for all } v \in C(x), \end{array} \tag{3}$$

where $f_\infty : \mathbb{R}^{n+m} \to \mathbb{R}$, $F^\infty : \mathbb{R}^{n+m} \to \mathbb{R}^m$, and $Z \subseteq \mathbb{R}^{n+m}$. In general, problems of type (3) are called mathematical programs with equilibrium constraints (MPEC's); see Luo, Pang, and Ralph (1996) and Outrata, Kocvara, and Zowe (1998). Note that a bilevel optimization problem can be seen as a special case of MPEC in which the mapping $F^\infty(x, \cdot)$ is the partial gradient (with respect to y) of a real-valued, continuously differentiable function. In MPEC's x is called the upper level variable and y is called the lower level variable.

An important class of MPEC's are mathematical programs with complementarity constraints (MPCC's) in which the equilibrium constraints take the special form of complementarity conditions; see next section for further details. In this paper we will deal with such an MPCC problem of the form discussed in Scheel and Scholtes (2000).

More specifically, we are concerned with a class of stochastic MPCC's, in which potentially, all of the defining functions (or some of their components) may represent a limit function that can not be directly observed. Formally, we are interested in the following stochastic mathematical programs with complementarity constraints (SMPCC):

$$\begin{array}{ll} \min & f_\infty(z) \\ \text{s.t.} & \min\{F_{k1}^\infty(z), \ldots, F_{kl}^\infty(z)\} = 0 \quad k = 1, \ldots, m \\ & g_\infty(z) \leq 0 \\ & h_\infty(z) = 0 \\ & z \in \Theta, \end{array} \tag{4}$$

where any of f_∞, F^∞, g_∞, or h_∞ (or some of their components) may be not have analytical expressions available. Here variable z includes the decisions at all levels; the constraint $\min\{F_{k1}^\infty(z), \ldots, F_{kl}^\infty(z)\} = 0$ means that $F_{kj}^\infty(z) \geq 0$ for every $j = 1, \ldots, l$ and $F_{k1}^\infty(z)F_{k2}^\infty(z)\ldots F_{kl}^\infty(z) = 0$. A particular example in which expectations are involved in the objective function and in the constraints is the following:

$$\begin{array}{ll} \min & \mathbb{E}_\omega[f(x, y; \omega)] \\ \text{s.t.} & (x, y) \in Z \\ & y \geq 0, \quad \mathbb{E}_\omega[F(x, y; \omega)] \geq 0 \\ & y\,\mathbb{E}_\omega[F(x, y; \omega)] = 0, \end{array} \tag{5}$$

where $Z = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathbb{E}_\omega[g(x, y; \omega)] \leq 0,\ \mathbb{E}_\omega[h(x, y; \omega)] = 0\}$ and ω denotes the random element in the model.

As an example consider a leader-follower (Stackelberg) type game from mathematical economics. A set of rational players try to select their best strategies in order to maximize their respective utilities. Among these players there is a distinctive player called the leader, whose decision x can impact the strategies y chosen by the other players, i.e., the followers. Given the leader's preference, the other users play a Nash game. The resulting MPEC problem is then selecting the best strategy that maximizes leader's utility, under the constraints of Nash equilibrium among the players.

Suppose that x represents the tax levels which the leader imposes on the followers and that we are interested in the revenues resulting from these taxes in the long run. Then we are trying to find the taxation scheme x of the leader that maximizes the expected revenues $-\mathbb{E}_\omega[f(x, y; \omega)]$. In this case, the leader may relate her decisions to certain activity levels y of the players which satisfy the equilibrium conditions in the long run. In other words, the equilibrium conditions may be required to hold on average and be expressed in terms of the average marginal costs $\mathbb{E}_\omega[F(x, y; \omega)]$. Constraints expressed by $E_\omega[g(x, y; \omega)]$ and $E_\omega[h(x, y; \omega)]$ may model the relations between, for example, average production levels of the followers and the expected demand levels or service level type requirements. Here ω may represent various sources of uncertainty such as market demand for followers' products or randomness in the marginal cost function F itself (for example due to the costs of raw materials or to technological changes). By simulating this system and using gradient estimation techniques or automatic differentiation capabilities incorporated into a modeling language, such as GAMS, we can estimate the performance measures $\mathbb{E}_\omega[f(x, y; \omega)]$, $\mathbb{E}_\omega[F(x, y; \omega)]$, $E_\omega[g(x, y; \omega)]$, and $E_\omega[h(x, y; \omega)]$ as well as their gradients. Using these together with appropriate deterministic techniques we can then compute the optimal taxation scheme of the leader and the equilibrium activity levels of the followers.

Notice that the SMPCC formulation we consider deals with problems in which all decisions, that is, at both upper and lower levels, must hold in the long run or in expectation. From this point of view, the stochastic MPCC under consideration here differs from the type of stochastic MPEC as formulated in Patriksson and Wynter (1999), where the lower level decisions depend on ω and the complementarity (or equilibrium) constraints are required to hold individually for every realization of ω. Lin, Chen, and Fukushima (2003) address the same problem; moreover, they also discuss a different variant, somewhat closer to ours, which makes use of a recourse variable depending on ω and some "adjusted" complementarity constraints are yet required to hold individually, for each ω. Our variant of SMPCC differs from either of those formulations, in that it does not impose individual realization constraints, but rather complementarity constraints at an "average" level. For a more formal and detailed discussion on different stochastic MPCC formulations, we refer to Birbil, Gürkan, and Listeş (2004).

The rest of this paper is organized as follows. In Section 2 we review some background material for MPCC's and provide further details on sample-path methods. We summarize our results on sample-path analysis for SMPCC's in Section 3; this section is based on Birbil, Gürkan, and Listeş (2004). In Section 4 we outline an illustrative application related to toll pricing in transportation networks. Finally, conclusions are in Section 5.

2 MPCC AND SAMPLE-PATH METHODS

In this section we review some background material related to MPCC's and sample-path methods. From the viewpoint of nonlinear programming, the complementarity constraints involving the function F^∞ are problematic, irrespective of the properties of F^∞, since no solution z can be a strictly feasible point. Consequently, the standard Mangasarian-Fromovitz constraint qualification is violated at every feasible point and one needs to deal with the complementarity constraints explicitly. Stochastic MPCC's require additional effort in order to account for the uncertain data. As mentioned earlier, for the SMPCC formulation (4), we propose a simulation-based solution approach using the sample-path method.

The basic case of sample-path optimization, concerning the solution of simulation optimization problems with deterministic constraints, appeared in Plambeck *et al.* (1993, 1996) and was analyzed in Robinson (1996). Plambeck *et al.* (1993, 1996) used infinitesimal perturbation analysis (IPA) for gradient estimation. In the static case, a closely related technique centered around likelihood-ratio methods appeared in Rubinstein and Shapiro (1993) under the name of stochastic counterpart methods. The basic approach (and its variants) is also known as sample average approximation method in the stochastic programming literature; see for example Shapiro and Homem-De-Mello (1998), Kleywegt, Shapiro, and Homem-De-Mello (2001), and Linderoth, Shapiro, and Wright (2002).

In Gürkan, Özge, and Robinson (1996, 1999a) the basic idea of using sample-path information was extended to solving stochastic equilibrium problems. There a framework is presented for stochastic variational inequalities that can model certain equilibrium problems and conditions are provided under which equilibrium points of approximating problems (computed via simulation and deterministic variational inequality solvers) converge almost surely to the solution of the limit problem. Gürkan, Özge, and Robinson (1999a) also contains a numerical application of the derived theory for finding the equilibrium prices of natural gas as well as the equilibrium quantities to produce in the European natural gas market. This work was used further in Gürkan, Özge, and Robinson (1999b) for establishing almost-sure convergence of sample-path methods when dealing with stochastic optimization problems with stochastic constraints.

In order to guarantee the closeness of solutions of the approximating variational inequalities to the solution of the real problem, Gürkan, Özge, and Robinson (1999a) impose a certain functional convergence of the data functions. It is called *continuous convergence* and denoted by $\xrightarrow{\mathcal{C}}$; it is equivalent to uniform convergence on compact sets to a continuous limit. For an elementary treatment of the relationship between different types of functional convergence, see Kall (1986), and for a comprehensive treatment of continuous convergence and related issues, see Rockafellar and Wets (1998). In the sequel we will employ this property as well.

In general the results which provide theoretical support for sample-path methods are based on the sensitivity analysis of the corresponding deterministic problems. This aspect is illustrated in Robinson (1996) in the case of simulation optimization problems with deterministic constraints as well as in Gürkan, Özge, and Robinson (1999a) in the case of stochastic variational inequalities. Following a similar argument, we build our sample-path analysis of stochastic MPCC's on the recent work of Scheel and Scholtes (2000), who set forth important sensitivity results for deterministic MPCC's.

As discussed in Scheel and Scholtes (2000), we work with an open set $\Theta \subseteq \mathbb{R}^{n_0}$ and twice differentiable functions

$f : \Theta \to \mathbb{R}$, $g : \Theta \to \mathbb{R}^p$, $h : \Theta \to \mathbb{R}^q$, and $F : \Theta \to \mathbb{R}^{m\times l}$ with $m \geq 1$, $l \geq 2$, and

$$F(z) = \begin{bmatrix} F_{11}(z) & \dots & F_{1l}(z) \\ \vdots & \ddots & \vdots \\ F_{m1}(z) & \dots & F_{ml}(z) \end{bmatrix}.$$

Given these ingredients, the problem under consideration is the following mathematical program with complementarity constraints (MPCC):

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & \min\{F_{k1}(z), \dots, F_{kl}(z)\} = 0 \quad k = 1, \dots, m \\ & g(z) \leq 0 \\ & h(z) = 0 \\ & z \in \Theta\,. \end{array} \tag{6}$$

If $z = (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $l = 2$, $F_{k1}(x, y) = y_k$, and $G_k(x, y) := F_{k2}(x, y)$, then the constraints $\min\{F_{k1}(z), F_{k2}(z)\} = 0$, $k = 1, ..., m$, represent the parametric nonlinear complementarity problem

$$y \geq 0, \qquad G(x, y) \geq 0, \qquad y\,G(x, y) = 0, \tag{7}$$

with parameter x and variable y. It is well known (see e.g., Harker and Pang (1990)) that problem (7) is equivalent to solving the parametric variational inequality $\text{VI}(G(x, \cdot), \mathbb{R}^{n_2}_+)$ over the positive orthant $\mathbb{R}^{n_2}_+$. Thus, the MPCC's are indeed a very important subclass of the mathematical programs with equilibrium constraints (MPEC's); see Luo, Pang, Ralph (1996).

One can associate with an MPCC the following Lagrangian function:

$$L(z, \Gamma, \lambda, \mu) = f(z) - F(z)\Gamma + g(z)\lambda + h(z)\mu\,, \tag{8}$$

where $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$, and $\Gamma \in \mathbb{R}^{m\times l}$ are the corresponding Lagrange multipliers and $F(z)\Gamma = \sum_i \sum_j F_{ij}(z)\Gamma_{ij}$ is the inner product of the two $m \times l$-matrices.

Scheel and Scholtes (2000) provide a thorough discussion on stationarity concepts for MPCC's and how they relate to the local minima. A basic concept is weak stationarity. A point z is called a *weakly stationary point* for MPCC if there exist multipliers Γ, λ, and μ such that

$$\begin{array}{rcll} \nabla_z L(z, \Gamma, \lambda, \mu) & = & 0 & \\ \min\{F_{k1}(z), \dots, F_{kl}(z)\} & = & 0 & k = 1, \dots, m \\ h(z) & = & 0 & \\ g(z) & \leq & 0 & \\ \lambda & \geq & 0 & \\ g_r(z)\lambda_r & = & 0 & r = 1, \dots, p \\ F_{ki}(z)\Gamma_{ki} & = & 0 & k = 1, \dots, m, \\ & & & i = 1, \dots, l\,. \end{array} \tag{9}$$

Under appropriate assumptions, the multipliers associated with a weakly stationary point can provide valuable information about the local geometry of the problem. Moreover, additional conditions to weak stationarity lead to stronger stationarity concepts for MPCC's, such as C-stationarity and strong stationarity; see Scheel and Scholtes (2000). Under certain constraint qualifications, such stationarity conditions are necessary for local optimality. For conciseness we confine our discussion here to weakly stationary points and refer to Birbil, Gürkan, and Listeş (2004) for further details.

3 STOCHASTIC MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS

We focus in this section on solving the problem SMPCC, in which no explicit description is available, in general, for any of the defining functions f_∞, $g\infty$, $h\infty$, and F^∞. For ease of notation, the ∞ scripts in SMPCC are omitted from now on. Suppose we observe some sequences of functions $\{f_n\}$, $\{g^n\}$, $\{h^n\}$, and $\{F^n\}$ for $n \in \mathbb{N}$, which approximate f, g, h, and F, respectively. Then we are concerned with sufficient conditions under which the solutions of SMPCC can be approximated by the solutions of a sequence of problems of the following type:

$$\begin{array}{ll} & \text{MPCC}_n \\ \min & f_n(z) \\ \text{s.t.} & \min\{F^n_{k1}(z), \dots, F^n_{kl}(z)\} = 0 \quad k = 1, \dots, m \\ & g^n(z) \leq 0 \\ & h^n(z) = 0 \\ & z \in \Theta\,. \end{array}$$

Assuming that the functions $\{f_n\}$, $\{g^n\}$, $\{h^n\}$, and $\{F^n\}$ are twice differentiable, the weak stationarity conditions for MPCC_n are of the form (9), where the true functions are replaced by the approximating functions and their derivatives. As these conditions represent a system which approximates system (9), the strategy could be to try and solve an approximating MPCC_n for n sufficiently large.

Note that an important way of envisioning the approximating setup is to regard the approximating functions as estimates of the true functions obtained from a simulation run of length n. In the context of option pricing, Gürkan, Özge, and Robinson (1996) provide an example in which an unobservable function f_∞ is approximated by a sequence $\{f_n : n \in \mathbb{N}\}$ of step functions. Hence, each f_n has a finite (but large) number of discontinuity points and a zero derivative on the rest of the domain, which makes it extremely difficult to optimize. On the other hand, the authors show that the derivative ∇f_∞ of f_∞ may be approximated by a sequence g_n of nicely behaved (smooth) functions. Clearly, in this example g_n does not coincide with ∇f_n (at the points where the latter is defined). Such examples indicate that in practise it is important to work with assumptions as weak as possible. Therefore, we focus here on a more general context.

We assume that one can observe some sequences of functions $\{g^n\}$, $\{h^n\}$, $\{F^n\}$, $\{a^n\}$, $\{b^n\}$, $\{c^n\}$, and $\{d^n\}$ for $n \in \mathbb{N}$, which approximate g, h, F, ∇f, ∇g, ∇h, and ∇F respectively. In the elaboration of our main result we will use the following notation:

$$J(z) = (\nabla f(z), \nabla g(z), \nabla h(z), \nabla F(z)), \tag{10}$$

$$J^n(z) = (a^n(z), b^n(z), c^n(z), d^n(z)), \tag{11}$$

$$dL^n(z, \Gamma, \lambda, \mu) = a^n(z) - d^n(z)\Gamma + b^n(z)\lambda + c^n(z)\mu. \tag{12}$$

In this setting, we are concerned with when and how well the solutions of (9) can be approximated by the solutions of the sequence of systems of the following type:

$$\begin{array}{rcll} dL^n(z, \Gamma, \lambda, \mu) & = & 0 & \\ \min\{F^n_{k1}(z), \ldots, F^n_{kl}(z)\} & = & 0 & k = 1, \ldots, m \\ h^n(z) & = & 0 & \\ g^n(z) & \le & 0 & \\ \lambda & \ge & 0 & \\ g^n_r(z)\lambda_r & = & 0 & r = 1, \ldots, p \\ F^n_{ki}(z)\Gamma_{ki} & = & 0 & k = 1, \ldots, m, \\ & & & i = 1, \ldots, l. \end{array} \tag{13}$$

Notice that if $b^n = \nabla g^n$, $c^n = \nabla h^n$ and $d^n = \nabla F^n$ for every $n \in \mathbb{N}$ and if moreover, f is approximated by a sequence of functions f_n such that $a^n = \nabla f_n$ for every $n \in \mathbb{N}$, then indeed, the approximating problem (13) represents the weak stationarity conditions for an approximating program MPCC_n of the type above.

The following theorem contains our main result on existence and convergence of approximating solutions.

Theorem 1 *Let Θ be an open set in $\mathbb{R}^{n_0}$. Suppose that f, g, h, and F are functions from Θ to $\mathbb{R}$, $\mathbb{R}^p$, $\mathbb{R}^q$, and $\mathbb{R}^{m\times l}$ respectively, which are twice differentiable and that J is defined as in (10). Let $\bar{z} \in \Theta$, $\bar{\Gamma} \in \mathbb{R}^{m\times l}$, $\bar{\lambda} \in \mathbb{R}^p$, and $\bar{\mu} \in \mathbb{R}^q$. Suppose that $\{J^n \mid n = 1, 2, ...\}$ are random functions defined on Θ, as in (11), $\{dL^n \mid n = 1, 2, ...\}$ are random functions defined as in (12), $\{g^n \mid n = 1, 2, ...\}$ are random functions from Θ to $\mathbb{R}^p$, $\{h^n \mid n = 1, 2, ...\}$ are random functions from Θ to $\mathbb{R}^q$, and $\{F^n \mid n = 1, 2, ...\}$ are random functions from Θ to $\mathbb{R}^{m\times l}$ such that for all $z \in \Theta$ and all finite n the random variables $J^n(z)$, $g^n(z)$, $h^n(z)$, and $F^n(z)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $L(z, \Gamma, \lambda, \mu)$ be defined as in (8) and assume the following:*

1) With probability one, each J^n for $n = 1, 2, ...$ is continuous and $J^n \xrightarrow{\mathcal{C}} J$.

2) With probability one, each g^n for $n = 1, 2, ...$ is continuous and $g^n \xrightarrow{\mathcal{C}} g$.

3) With probability one, each h^n for $n = 1, 2, ...$ is continuous and $h^n \xrightarrow{\mathcal{C}} h$.

4) With probability one, each F^n for $n = 1, 2, ...$ is continuous and $F^n \xrightarrow{\mathcal{C}} F$.

5) $(\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu})$ is a solution of (9) *(that is, a weakly stationary point of the SMPCC).*

6) $\nabla_z L$ has a strong Fréchet derivative $\nabla^2_{zz} L(\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu})$ at the point $(\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu})$ and all the matrices

$$\begin{pmatrix} \nabla^2_{zz} L(\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu}) & -\nabla_z F_I(\bar{z})^\top & \nabla_z g_R(\bar{z})^\top & \nabla_z h(\bar{z})^\top \\ \nabla_z F_I(\bar{z}) & 0 & 0 & 0 \\ -\nabla_z g_R(\bar{z}) & 0 & 0 & 0 \\ \nabla_z h(\bar{z}) & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

$$\begin{array}{c} \textit{with } \{(k,i) \mid \bar{\Gamma}_{ki} \neq 0\} \subseteq I \subseteq \{(k,i) \mid F_{ki}(\bar{z}) = 0\} \\ \textit{and } \forall\, k = 1, ..., m\ \exists\, i \in \{1, ..., l\} : (k,i) \in I\,, \\ \textit{and } \{r \mid \bar{\lambda}_r > 0\} \subseteq R \subseteq \{r \mid g_r(\bar{z}) = 0\}\,, \end{array}$$

have the same nonvanishing determinantal sign.

Then, there exist compact subsets $C_0 \subset \Theta$ containing $\bar{z}$, $U_0 \subset \mathbb{R}^{m\times l}$ containing $\bar{\Gamma}$, $V_0 \subset \mathbb{R}^p$ containing $\bar{\lambda}$, and $W_0 \subset \mathbb{R}^q$ containing $\bar{\mu}$, neighborhoods $Y_1 \subset \Theta$ of $\bar{z}$, $U_1 \subset \mathbb{R}^{m\times l}$ of $\bar{\Gamma}$, $V_1 \subset \mathbb{R}^p$ of $\bar{\lambda}$, and $W_1 \subset \mathbb{R}^q$ of $\bar{\mu}$, a constant $\alpha > 0$ and a set $\Delta \subset \Omega$ of measure zero, with the following properties: for $n = 1, 2, ...$ and $\omega \in \Omega$ let

$$\begin{aligned} \xi_n(\omega) = \sup\{\ & \|(dL^n(\omega, z, \Gamma, \lambda, \mu), (g^n, h^n, F^n)(\omega, z)) \\ & -(\nabla_z L(z, \Gamma, \lambda, \mu), (g, h, F)(z))\| : \\ & (z, \Gamma, \lambda, \mu) \in C_0 \times U_0 \times V_0 \times W_0\,\}, \end{aligned}$$

$$\begin{aligned} Z_n(\omega) = \{\ & (z, \Gamma, \lambda, \mu) \in Y_1 \times U_1 \times V_1 \times W_1 \mid \\ & (z, \Gamma, \lambda, \mu) \textit{ solves } (13) \textit{ associated to } \omega\}. \end{aligned}$$

For each $\omega \notin \Delta$ there is then a finite integer N_ω such that for every $n \geq N_\omega$ the set $Z_n(\omega)$ is a nonempty, compact subset of $B((\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu}), \alpha\, \xi_n(\omega))$.

Although useful, continuous convergence itself does not guarantee neither the existence of approximating solutions nor their convergence. To guarantee these, one needs to impose an additional regularity condition. The particular regularity condition we employ in 6) is the so-called nonvanishing determinantal sign condition, as set forth in Scheel and Scholtes (2000). It is highly technical and we refer the reader to Scheel and Scholtes (2000) and Birbil, Gürkan, and Listeş (2004) for a detailed discussion. In Birbil, Gürkan, and Listeş (2004), we give the proof of Theorem 1 along with a rigorous discussion of the relationship between the nonvanishing determinantal sign and other regularity concepts.

To summarize, Theorem 1 says that under certain niceness conditions, for n sufficiently large (i.e., if we go out long enough on any sample-path), the solution set of (13) becomes nonempty and compact. Furthermore, the distance of every such solution of (13) from the exact solution $(\bar{z}, \bar{\Gamma}, \bar{\lambda}, \bar{\mu})$ of (9) becomes bounded by a constant multiple of the uniform norm of $(dL^n, g^n, h^n, F^n) - (L, g, h, F)$ on a compact set.

By assuming an additional condition at the solution point, similar convergence results to Theorem 1 can be also established for the C-stationary and strong stationary concepts mentioned above. Moreover, if one assumes continuous convergence of the first and second order derivatives of the approximating functions to the first and second order derivatives of the limit functions, the approximating solution set can be guaranteed to be finite. Furthermore, by assuming both continuous convergence of the derivatives and an additional regularity condition at the solution point, the uniqueness of the approximating solutions can be also proven. We refer to Birbil, Gürkan, and Listeş (2004) for technical details on these.

4 AN APPLICATION IN TOLL PRICING

The commuters in a transportation network depart from certain points and try to arrive at their destinations in the most beneficial way. Generally, their benefits are directly related to traversing the shortest or least costly paths. A principle which characterize the steady-state flows in such networks was introduced by Wardrop (1952). In the literature this principle is used to formulate a set of equilibrium conditions, which are in turn, used as part of traffic congestion problems; see Florian and Hearn (1995). For network controllers, one way to deal with the congestion is imposing toll prices on the roads, which can alter the equilibrium conditions. In this regard, the objective of the toll pricing problem is to determine the optimal toll prices so that the congestion is minimized under the user equilibrium conditions and network flow constraints.

The following mathematical description of the toll pricing problem is somewhat standard in the literature and our exposition borrows largely from the model given by Dirkse and Ferris (1997). Consider a transportation network described by a set of nodes $\mathcal{N}$, and a set of arcs $\mathcal{A}$. We denote an arc interchangeably by subscripts $a \in \mathcal{A}$ and $(i, j) \in \mathcal{A}$ with $i, j \in \mathcal{N}$. The set of destination nodes is denoted by $\mathcal{K} \subset \mathcal{N}$. A commodity is associated with a destination node and it is denoted by k. The flow vector of a commodity $k \in \mathcal{K}$ over the network is represented with the variable y^k with the components y_a^k denoting the flow of commodity k on arc a. The cost (or time) of flow on an arc, experienced by a user, is a real-valued function given by c_a. The cost function not only depends on the network flows but also on the nonnegative vector $x = (x_a : a \in \mathcal{A})$ representing the toll prices. We assume that the monetary costs of toll prices are converted, for instance through simple weighting, into time units so that the cost (or time) of a tolled arc a is calculated after incorporating x_a into the cost function c_a. The minimum time to deliver commodity k from node $i \in \mathcal{N}$ is denoted by t_i^k, which for all $i \in \mathcal{N}$ constitute the components of the vector t^k. For each commodity k, there is an associated demand d_i^k at node i and the demand vector for the same commodity over the network is denoted by d^k. In practice the demand is a function of the minimum cost vector t, however for simplicity we assume here that demand is a constant function independent of t.

A standard set of constraints imposes conservation of flow. Denoting the node-arc incidence matrix by A, these constraints can be written as

$$Ay^k = d^k \quad k \in \mathcal{K}.$$

The user equilibrium laws are amenable to a complementarity formulation. They state that if there is a positive flow for a commodity along an arc, then the corresponding time to deliver the commodity should be minimal. This statement can be equivalently formulated by the set of complementarity constraints for all $a = (i, j) \in \mathcal{A}$ and $k \in \mathcal{K}$ as follows:

$$0 \leq c_a(\bar{y}_a, x) + t_j^k - t_i^k \perp y_a^k \geq 0 \quad a = (i, j) \in \mathcal{A},\ k \in \mathcal{D},$$

where $\bar{y}_a := \sum_{l \in \mathcal{K}} y_a^l$, $t_j^k - t_i^k$ represents the minimum time to traverse arc $a = (i, j)$, and $d \perp e$ means $d^\top e = 0$.

An important objective of toll pricing could be reducing the congestion on the network. One way of modeling this is minimizing the total system costs $\sum_{a \in \mathcal{A}} c_a(\bar{y}, x)$, as used by Hearn and Ramana (1998). By expressing complementarity through the nonsmooth min operator, the mathematical model for the toll pricing problem becomes the following:

$$\begin{array}{lll} \min & \sum_{a\in\mathcal{A}} c_a(\bar{y}_a, x) & \\ \text{s.t.} & Ay^k = d^k & k \in \mathcal{K}, \\ \multicolumn{2}{l}{\min\{c_a(\bar{y}_a, x) + t_j^k - t_i^k, y_a^k\} = 0} & a=(i,j)\in\mathcal{A},\ k\in\mathcal{K}, \\ & t^k \geq 0 & k \in \mathcal{K}, \\ & x \geq 0. & \end{array} \quad (15)$$

A cost function commonly used in practice was proposed by Bureau of Public Roads (BPR) in US; see Bureau of Public Roads (1964). Adding now the toll prices to this function leads to

$$c_a(\bar{y}_a, x) = \alpha_a + \beta_a \left(\frac{\bar{y}_a}{\gamma_a}\right)^4 + x_a \quad (16)$$

where α_a and β_a are so-called calibration parameters and γ_a denotes the practical capacity parameter of an arc. The parameters α, β, and γ are usually looked up from tables or found by analyzing the historical data. However, in practice there exist major difficulties in determining their exact values. In general these parameters fluctuate considerably over time and hence, they can be set only within certain tolerances or sampled from appropriate distributions. Incorporating this type of uncertainty into the mathematical model plays an important role for understanding the average behavior of the system. We represent the uncertain element by a random variable ω defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

A traffic controller may try to get insight into the "average" behavior of the system, by assuming that the users will choose their routes based on the most favorable expected costs. Thus, the controller may decide to relate her tolling decisions to some "average" flows over the network. Formally, the situation can be modelled using the following stochastic model with expected equilibrium constraints:

$$\begin{array}{lll} \min & \mathbb{E}_\omega[\sum_{a\in\mathcal{A}} c_a(\bar{y}_a, x; \omega)] & \\ \text{s.t.} & Ay^k = d^k & k \in \mathcal{K}, \\ \multicolumn{2}{l}{\min\{\mathbb{E}_\omega[c_a(\bar{y}_a, x; \omega)] + t_j^k - t_i^k, y_a^k\} = 0} & a=(i,j)\in\mathcal{A}, \\ & & k \in \mathcal{K}, \\ & t^k \geq 0 & k \in \mathcal{K}, \\ & x \geq 0. & \end{array}$$

This model can be analyzed using the following "average"-based interpretation of the equilibrium principle: if the "average" flow on a link is positive, then the expected cost of traversing that link is minimum. Consequently, variable t_i^k represents here the minimum average time for travelling from origin i to destination k and the difference $t_i^k - t_j^k$ represents the minimum average time for traversing arc $a = (i, j)$ (in the case this arc is being used). Similarly, variables y are interpreted as the average network flows to which the toll decisions x are related. Accordingly, d^k represents in this case the average demand the controller expects for commodity (destination) k. The objective is to minimize the total expected costs with the goal of reducing the expected congestion.

We are currently performing numerical experiments in order to understand the effectiveness of the sample-path method for solving the stochastic toll pricing problem. The model is implemented using the GAMS modeling language, and solved using the latest version of the NLPEC package. This is a so-called beta solver which exploits several methodologies for reformulation of MPEC's as nonlinear programs and calls subsequently several off-the-shelf nonlinear programming solvers for their solution; see Ferris, Dirkse, and Meeraus (2002).

Our implementation starts from the deterministic model in `tollmpec.gms` file from the MPECLIB library as explained in Dirkse and Ferris (1997). We modified the original `tollmpec.gms` according to the formulation (15) and the cost functions (16). In the stochastic setting, parameters α, β, and γ are sampled from appropriate distributions. Based on the sampled values an approximating model is then built and solved using NLPEC. Numerical experiments are currently underway.

5 CONCLUSIONS

In this paper we outlined a variant of sample-path method to solve a class of stochastic mathematical programs with equilibrium constraints and a set of sufficient conditions for the almost-sure convergence of the method in this case. This extends the range of applicability of sample-path methods. We also gave an illustrative example of how it can be used to solve a stochastic toll pricing problem. We hope that our analysis turns out being helpful in providing solutions for challenging practical problems arising in the study of complex stochastic systems.

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