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A BAYESIAN FRAMEWORK FOR MODELING DEMAND IN SUPPLY CHAIN SIMULATION EXPERIMENTS

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ABSTRACT

In order to postpone production planning based on information obtained close to the time of sale, decision support systems for supply chain management often include demand forecasts based on little historical data and/or subjective information. Particularly, when simulation models for analyzing decisions related to safety inventories, lot sizing or lead times are used, it is convenient to model (daily) demand by considering historical data, as well as information (often subjective) of the near future. This article presents an approach for modeling a random input (e.g., demand) in simulation experiments. Under this approach, the family of distributions proposed for modeling demand should include two types of parameters: the ones that capture information of historical data and the ones that depend on the particular scenario that is to be simulated. The approach is extended to the case where uncertainty on the appropriate family of distributions is present.

1 INTRODUCTION

When evaluating the performance of a supply chain by using a model, incorporating characteristics from the real life system may complicate the task of obtaining analytic solutions for the system's performance measures. For example, when investigating the relationship between safety inventories, chain speed (lead times), and service levels in different points of the chain, demand distributions often change over time, order sizes are not always the same (they may vary depending on the sales' forecast), or there may be uncertainty in lead times because of different order sizes. Nonetheless, these characteristics often may be incorporated in a simulation model, with the purpose of studying the performance of the chain through experimentation with the simulation model. Due to its capacity of modeling complex systems, simulation becomes a powerful tool for evaluating the performance of a supply chain, particularly for analyzing the performance of an inventory policy (see Chopra 2001).

To conduct a simulation experiment, model inputs as well as model outputs (performance measures of the system that is to be analyzed) have to be specified precisely. For example, if one desires to simulate inventory in a supply chain (see Figure 1), often performance measures (model outputs) of the experiment are associated with the service levels and incurred costs. To estimate these performance measures, inventory levels have to be simulated based on inventory policies (particularly safety inventories), lead times distributions, and demand distributions (individual customers demand or by period, for instance, daily). It is convenient to remark that some model inputs may be known (for instance, initial inventory level, review policy, etc.), and others may be defined through its probability distribution. The latter are often referred to as *random inputs* (see Zouaoui and Wilson 2001).

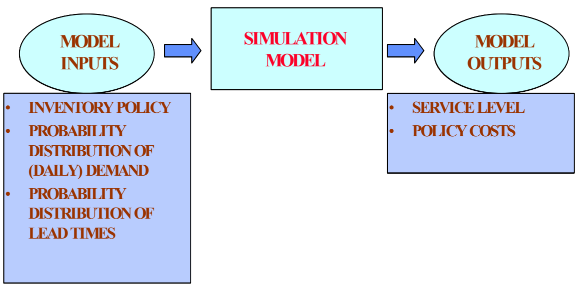

Figure 1: Components of a Simulation Experiment for Analyzing Inventories

The classical approach used to specify the distribution of a random input in a simulation experiment consists in selecting the distribution (and its parameters) that better fits the data. The data are often assumed as independent and identically distributed (i.i.d). To be more precise, parameter estimation is based on the maximum likelihood method and the selected probability distribution is the one that attains the best performance according to a *goodness of fit* measure like the mean squared error (see, for example, Law and Kelton 2000).

Nonetheless, when it is desired to model the demand of products with a short life cycle (fashion or season products), or products whose demand depends on factors associated to a particular scenario (money from banking agencies or other products with a client portfolio), it is not convenient to assume that past observations of the demand come from the same distribution, because the distribution parameters for a subset of observations probably depend on the factors associated to its corresponding scenario. In this case, the classical approach is inadequate for modeling the product's demand, and an alternative methodology that allows the incorporation of the specific characteristics of the scenario's demand distribution must be found.

This article presents a methodology based on Bayesian Theory (see Berger 1985) to model a random input (e.g., demand) in simulation experiments. Under this approach, the family of distributions proposed for modeling a random input depends on two types of parameters, the first ones give information that is common to past observations of the random input and the second ones give information from the particular scenario that is to be simulated. Then, based on the proposed model, past information, and prior probabilities (interpreted as forecasts) for the parameters, posterior probabilities for the model parameters may be obtained in order to estimate the performance measures from simulation experiments.

The remaining of this paper is organized as follows: §2 describes the notation and proposed methodology, as well as the motivation for developing it, §3 develops a simple example of the proposed methodology with the purpose of illustrating the obtained results, §4 presents an extension to the case where there is uncertainty on the family of distributions that is more appropriate to model the random input, and finally §5 presents conclusions and directions for future research.

2 SIMULATION EXPERIMENTS USING A BAYESIAN FRAMEWORK

The motivation for developing the present methodology comes from the case of a manufacturing company that produces certain article(s) (with a standardized design) to satisfy orders from its customers. At the beginning of every month, the company produces estimates of the monthly demand based on the information provided by customers, considering that certain information may correspond to orders made in advance, though with no specific dates or order sizes. The latter will be specified in the course of the month. Based on the initial estimates of the month's aggregated demand forecasts and in their safety inventory policy, the company establishes a production plan to satisfy the orders for a month. The company is interested in running simulation experiments that allow the estimation of the service level (expected percentage of satisfied demand) for a specific production plan, and a demand scenario that is congruent with the month's aggregated demand forecasts and with the information of the (daily) demand's behavior in past months.

The methodology that follows intends to propose how to specify the distribution of a random input (in this case daily demand), trying to incorporate specific information of the scenario (aggregated demand forecasts), as well as past information. With this purpose, we assume that the probability function (density in the continuous case) of the demand is of the form $p(y;\Theta_1,\Theta_2)$, where Θ_1 (of dimension k_1) is the vector of parameters that contains information from past experiences, and Θ_2 (of dimension k_2) is the vector of parameters that contains information of the particular scenario that is to be simulated.

We assume the existence of historical information (e.g., daily) on the demand for Q periods (e.g., monthly), therefore it is assumed that there are Q samples (mutually independent) $x_i = (x_{i1}, x_{i2}, x_{in_i})$, $i = 1,2,\ldots,Q$, where the observations of sample i are i.i.d., and come from the probability function $p(y;\Theta_1,\Theta_2^i)$ (see Figure 2), where the particular value of the parameter vector Θ_1 is the same for all of the samples (even the ones corresponding to the simulation experiments). The particular value of the parameter vector Θ_2^i depends on the particular scenario that corresponds to the period (month) i. It should be mentioned that, according to Figure 2, $p(y;\Theta_1,\Theta_2^{Q+1})$ denotes the probability function that corresponds to the scenario that is to be simulated.

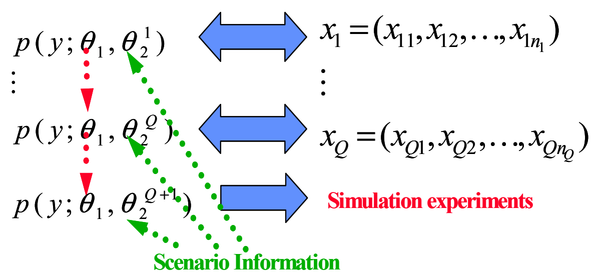

Figure 2: Structure of the Available Historical Information

On the other hand, prior information (forecasts) on the parameters is expressed in the form of a prior density function $p_\Theta(\theta_1,\theta_2^1,\theta_2^2,\ldots\theta_2^{Q+1})$ for the parameters, that (assuming independence) is of the form:

$$
\begin{aligned}
p_\Theta(\theta) &= p_\Theta(\theta_1,\theta_2^1,\theta_2^2,\ldots\theta_2^{Q+1}) \\
&= p_{\Theta_1}(\theta_1)p_{\Theta_2^1}(\theta_2^1)p_{\Theta_2^2}(\theta_2^2)\ldots p_{\Theta_2^{Q+1}}(\theta_2^{Q+1}),
\end{aligned}
\tag{1}
$$

where $\theta=(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1})$ and $\Theta=(\Theta_1,\Theta_2^1,\ldots,\Theta_2^{Q+1})$.

The prior density function $p_{\Theta_2^i}\left(\theta_2^i\right)$ is interpreted as a forecast on the parameter vector Θ_2^i ($i=1,2,\ldots,Q+1$). Note also that a prior density function (prior probability distribution) allows the incorporation of a forecast in a number of ways. For example, one can assume a probability distribution that takes a certain value with probability 1 (appropriate for the case when the parameter is known with certainty), or a uniform distribution in certain interval (when the only available information are the minimum and the maximum values), or any other form that better adapts to the forecast method used in practice.

Using the above notation, an output y (for example, the percentage of satisfied demand) from a simulation experiment, can be conceived as a function of the random numbers u used in the run, and the particular value $\theta^{Q+1}=\left(\theta_1,\theta_2^{Q+1}\right)$ of the random input parameters:

$$y=y\left(u,\theta^{Q+1}\right). \tag{2}$$

As discussed in Zouaoui and Wilson (2001), using a classical (non Bayesian) approach, often the value θ^{Q+1} of the parameters is fixed, and independent replications of the simulation experiment are run in order to estimate performance measures of the form:

$$\eta(\theta^{Q+1})=E\left[y\middle|\Theta^{Q+1}=\theta^{Q+1}\right]=\int y(u,\theta^{Q+1})du\,. \tag{3}$$

On the other hand, under a Bayesian approach, the parameters are random variables, and the performance measures to be estimated are of the form:

$$E\left[y\middle|X=x\right]=\int\eta(\theta^{Q+1})p_{\Theta^{Q+1}|X=x}(\theta^{Q+1})d\theta^{Q+1}, \tag{4}$$

where $x=\left(x_1,x_2,\ldots,x_Q\right)$ is the available historical information, and $p_{\Theta^{Q+1}|X=x}(\theta^{Q+1})$ denotes the posterior density function (given the historical data x) of the scenario parameters, which can be obtained from the prior density function (1) of the parameters, and the historical data x, described as follows.

Because the observations $x_i=\left(x_{i1},x_{i2},\ldots,x_{in_i}\right)$ of each sample $i=1,2,\ldots,Q$ are i.i.d., and the Q samples are mutually independent, given that the vector of parameters Θ takes the value $\theta=(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1})$, the joint probability function of the historical data x is:

$$p_{X|\Theta=\theta}(x)=\prod_{i=1}^{Q}\prod_{j=1}^{n_i}p(x_{ij};\theta_1,\theta_2^i), \tag{5}$$

and then:

$$p_X(x)=\int\prod_{i=1}^{Q}\prod_{j=1}^{n_i}p(x_{ij};\theta_1,\theta_2^i)p_\Theta(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1})d\theta_1\,d\theta_2^1\ldots d\theta_2^{Q+1}, \tag{6}$$

where the prior density function $p_\Theta(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1})$ is defined in equation (1).

According to Bayes' Theorem (see Berger 1985), the (joint) posterior density function of the parameters becomes:

$$p_{\Theta|X=x}(\theta)=\frac{p_\Theta(\theta)p_{X|\Theta=\theta}(x)}{p_X(x)}, \tag{7}$$

where $p_\Theta(\theta)$, $p_{X|\Theta=\theta}(x)$, and $p_X(x)$ are defined in (1), (5) and (6), respectively. Finally, the posterior density function of the parameters $\theta^{Q+1}=\left(\theta_1,\theta_2^{Q+1}\right)$ corresponding to the scenario to be simulated are obtained from:

$$p_{\Theta^{Q+1}|X=x}(\theta^{Q+1})=\int p_{\Theta|X=x}(\theta)d\theta_2^1\ldots d\theta_2^Q, \tag{8}$$

where $p_{\Theta|X=x}(\theta)$ is defined in (7).

Note that the parameter $E\left[y\middle|X=x\right]$ defined in equation (4) is an expectation, so it can be estimated via simulation by using the method of replications, that is, by replicating (n times) the following experiment:

- Generate a vector of parameters $\theta^{Q+1}=\left(\theta_1,\theta_2^{Q+1}\right)$ according to the posterior density function $p_{\Theta^{Q+1}|X=x}(\theta^{Q+1})$ defined in (8).
- Run the simulation model to generate an observation of the output $y=y\left(u,\theta^{Q+1}\right)$.

A consistent estimator for the performance measure $E\left[y\middle|X=x\right]$ is the sample mean of n independent observations from the output $y=y\left(u,\theta^{Q+1}\right)$ of the previous simulation experiment (see Chick 2001).

3 A SIMPLE APPLICATION

With the objective of illustrating the application of the methodology described in the previous section, a simple example is presented. It should be mentioned that the example presented does not intend to identify a real life application, because the proposed model may not consider several factors that could be relevant in real life. On the contrary, the model has been simplified with the goal of illustrating the application of the procedure described in the previous section.

Consider the case of a firm (for example, an airline) that records reservations for a service scheduled on a specific date. Because the service is provided regularly, information of Q past periods providing the service is available; each time the following information is recorded:

$\Theta_2^i =$ number of reservations recorded in period i,

$x_{i1} =$ number of people that used their reservation for period i, $i = 1,2,\ldots,Q$.

Note that in this case $n_1 = n_2 = \ldots = n_Q = 1$. Assuming that the probability that a person makes use of his/her reservation to receive the service is Θ_1 (although it is not known for certain), when θ_2^i reservations are applied for, and $\Theta_1 = \theta_1$, the number of reservations used in period i follows a binomial distribution:

$$p(y;\theta_1,\theta_2^i) = \binom{\theta_2^i}{y}\theta_1^y(1-\theta_1)^{\theta_2^i - y},\ y = 0,1,\ldots,\theta_2^i, \qquad (9)$$

for $i = 1,2,\ldots,Q+1$.

A simulation model has been developed to estimate the cost $y = y(u,\theta^{Q+1})$ (where $\theta^{Q+1} = (\theta_1,\theta_2^{Q+1})$), that is incurred in when an over-booking policy is used. Given that currently there are $\Theta_2^{Q+1} = r_{Q+1}$ reservations, it is desired to use the model to estimate the expected cost $E[y|X = x]$ given information of past periods (particularly, it is known that $\Theta_2^i = r_i, i = 1,2,\ldots Q$). Note that information on the common parameter Θ_1 is present in every past period, while the number of reservations Θ_2^i corresponds to the particular scenario i ($i = 1,2,\ldots Q+1$).

Because the number of reservations is known for each period providing the service, the appropriate prior probability function for the parameters Θ_2^i, $i = 1,2,\ldots Q+1$ is given by:

$$p_{\Theta_2^i}(\theta_2^i) = P[\Theta_2^i = \theta_2^i] = \begin{cases} 1, & \text{if } \theta_2^i = r_i, \\ 0, & \text{otherwise,} \end{cases}$$

$i = 1,\ldots,Q+1$. On the other hand, the prior density function for the common parameter Θ_1 is proposed as a uniform distribution:

$$p_{\Theta_1}(\theta_1) = \frac{1}{q_1 - q_0};\ q_0 < \theta_1 < q_1,$$

where $q_0 > 0, q_1 < 1$, are given constants, so that, according to equation (1):

$$p_\Theta(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1}) = \frac{1}{q_1 - q_0};\ q_0 < \theta_1 < q_1, \theta_2^i = r_i,\ldots,\theta_2^{Q+1} = r_{Q+1}. \qquad (10)$$

From equations (5) and (9), the joint probability function of the available information given the value of the parameters is:

$$p_{X|\Theta=\theta}(x) = \prod_{i=1}^{Q}\binom{\theta_2^i}{x_{i1}}\theta_1^{x_{i1}}(1-\theta_1)^{\theta_2^i - x_{i1}};\ x_{i1} = 0,1,\ldots,\theta_2^i, i = 1,2,\ldots,Q, \qquad (11)$$

and according to equations (6) and (10), the following is obtained:

$$\begin{aligned} p_X(x) &= \int_{q_0}^{q_1}\left[\prod_{i=1}^{Q}\binom{r_i}{x_{i1}}\theta_1^{x_{i1}}(1-\theta_1)^{r_i - x_{i1}}\right]\left(\frac{1}{q_1 - q_0}\right)d\theta_1 \\ &= \left[\int_{q_0}^{q_1}\prod_{i=1}^{Q}\theta_1^{x_{i1}}(1-\theta_1)^{r_i - x_{i1}}d\theta_1\right]\left(\frac{1}{q_1 - q_0}\right)\prod_{i=1}^{Q}\binom{r_i}{x_{i1}} \\ &= K(\alpha,\beta)\left(\frac{1}{q_1 - q_0}\right)\prod_{i=1}^{Q}\binom{r_i}{x_{i1}}, \end{aligned} \qquad (12)$$

where: $K(\alpha,\beta) = \int_{q_0}^{q_1}\theta_1^\alpha(1-\theta_1)^\beta d\theta_1, \alpha = \sum_{i=1}^{Q}x_{i1}, \beta = \sum_{i=1}^{Q}r_i - \sum_{i=1}^{Q}x_{i1}$.

It follows from (7), (10), (11), and (12) that:

$$p_{\Theta|X=x}(\theta_1,\theta_2^1,\ldots,\theta_2^{Q+1}) = \frac{\theta_1^\alpha(1-\theta_1)^\beta}{K(\alpha,\beta)};\ q_0 < \theta_1 < q_1, \theta_2^i = r_i, i = 1,2,\ldots,Q+1,$$

and by considering equation (8):

$$p_{\Theta^{Q+1}|X=x}(\theta_1,\theta_2^{Q+1}) = \frac{\theta_1^\alpha(1-\theta_1)^\beta}{K(\alpha,\beta)};\ q_0 < \theta_1 < q_1, \theta_2^{Q+1} = r_{Q+1}. \qquad (13)$$

This last equation indicates that the parameter Θ_2^{Q+1} takes the value r_{Q+1} with probability 1, and that the common parameter Θ_1 follows a Beta distribution (see Law and Kelton 2000) when $q_0 = 0, q_1 = 1$.

According to the methodology described in the previous section, the expected cost $E[y|X = x]$ given information of past periods can be estimated by simulation, replicating the following experiment:

- Generate a vector of parameters $\theta^{Q+1} = \left(\theta_1, \theta_2^{Q+1}\right)$ according to the posterior distribution $p_{\Theta^{Q+1}|X=x}(\theta^{Q+1})$ defined in (13).
- Run the simulation model and generate an observation of the cost $y = y\left(u, \theta^{Q+1}\right)$.

A consistent estimator for $y = y\left(u, \theta^{Q+1}\right)$ is the sample mean of the observations generated via simulation.

4 UNCERTAINTY IN THE PROPOSED MODEL

It should be mentioned that the methodology described in §2 follows a similar approach as the one developed in Chick (2001) for selecting the distribution of a random input in simulation experiments, by applying the technique named *Bayesian Model Average* (see Draper 1995). The main difference is that in Chick (2001) several models for the same random input are proposed, each model has only one type of parameter, and only one sample is available to obtain the posterior distribution. While in Chick (2001) the emphasis is in incorporating the uncertainty on the appropriate model for the random input, the case described in this article puts more emphasis on the way in which forecasts (may be subjective) for a scenario of interest can be incorporated, assuming the existence of many (Q) samples (each under a different scenario). It is shown in this Section how to apply the Bayesian Model Average technique in order to extend the methodology described in §2 to the case where several models for the random input are considered.

In this case there is uncertainty on the model (family of probability functions) that is more appropriate to represent the random input. In particular, several models m_k, $k = 1,2,\ldots,K$ are proposed with prior probabilities:

$$p_M(m_k) = P[M = m_k], k = 1,2,\ldots,K, \tag{14}$$

which represent an approximation to the goodness of the corresponding model to fit the historical data. For example, under complete uncertainty one can assume equal probabilities $1/K$ for each model (see Chick 1997).

As in §2, samples $x_i = \left(x_{i1}, x_{i2}, x_{in_i}\right)$, $i = 1,2,\ldots,Q$, from the random input are available, and given that $M = m_k$ (the “true” model is m_k), $p_k(y; \Theta_{1k}, \Theta_{2k}^i)$ denotes the probability function (density in the continuous case) corresponding to sample i, where Θ_{1k} (of dimension d_{1k}) is the vector of parameters that contains information from past periods, and Θ_{2k}^i (of dimension d_{2k}) is the vector of parameters that contains information of the particular scenario corresponding to sample i (as in §2, $p_k(y; \Theta_{1k}, \Theta_{2k}^{Q+1})$ corresponds to the scenario to be simulated). Then, given a model and a particular value for the corresponding parameters, the joint probability function for the observations in sample i becomes:

$$p_{X_i|M=m_k,\Theta_k=\theta_k}(x_i) = \prod_{j=1}^{n_i} p_k\left(x_{ij}; \theta_{1k}, \theta_{2k}^i\right),$$

where $x_i = \left(x_{i1}, x_{i2}, \ldots, x_{in_i}\right)$, $\Theta_k = \left(\Theta_{1k}, \Theta_{2k}^1, \ldots, \Theta_{2k}^{Q+1}\right)$, and $\theta_k = \left(\theta_{1k}, \theta_{2k}^1, \ldots, \theta_{2k}^{Q+1}\right)$, $i = 1,2,\ldots,Q, k = 1,2,\ldots,K$, and therefore the joint probability function for the historical information has the form:

$$p_{X|M=m_k,\Theta_k=\theta_k}(x) = \prod_{i=1}^{Q}\prod_{j=1}^{n_i} p_k\left(x_{ij}; \theta_{1k}, \theta_{2k}^i\right), \tag{15}$$

where $x = \left(x_1, x_2, \ldots, x_Q\right)$, $k = 1,2,\ldots,K$.

Given that $M = m_k$, the prior knowledge (forecast) on the corresponding parameters is expressed as a prior density function $p_{\Theta_k|M=m_k}(\theta_k)$ that (assuming independence) has the form:

$$\begin{aligned} p_{\Theta_k|M=m_k}(\theta_k) &= p_{\Theta_k|M=m_k}\left(\theta_{1k}, \theta_{2k}^1, \theta_{2k}^2, \ldots, \theta_{2k}^{Q+1}\right) \\ &= p_{\Theta_{1k}|M=m_k}(\theta_{1k})\, p_{\Theta_{2k}^1|M=m_k}\left(\theta_{2k}^1\right) \ldots p_{\Theta_{2k}^{Q+1}|M=m_k}\left(\theta_{2k}^{Q+1}\right), \end{aligned} \tag{16}$$

$k = 1,2,\ldots,K$.

In this case, an output y of the simulation experiment can be seen as a function of the random numbers u to be generated in the experiment, the model m_k for the random input, and the value $\theta_k^{Q+1} = \left(\theta_{1k}, \theta_{2k}^{Q+1}\right)$ for the particular scenario of interest:

$$y = y\left(u, m_k, \theta_k^{Q+1}\right),$$

under a non Bayesian approach a typical performance measure has the form:

$$\eta(m_k, \theta_k^{Q+1}) = \int y(u, m_k, \theta_k^{Q+1})\, du \quad ,$$

for the Bayesian approach the performance measure to be estimated has the form:

$$E[y|X=x]=\sum_{k=1}^{K} p_{M|X=x}(m_k)\int \eta(m_k,\theta_k^{Q+1})\,p_{\Theta_k^{Q+1}|X=x,M=m_k}(\theta_k^{Q+1})\,d\theta_k^{Q+1}, \tag{17}$$

where $x=(x_1,x_2,\ldots,x_Q)$ is the available historical information, $p_{M|X=x}(m_k)$ is the posterior probability (given the historical information) of model m_k, and $p_{\Theta^{Q+1}|X=x,M=m_k}(\theta_k^{Q+1})$ denotes the posterior probability function (given the historical information and the model m_k) of the parameters corresponding to the scenario to be simulated, which can be obtained from the prior probability functions and the available historical information x.

According to Bayes' Theorem (see Berger 1985), the posterior probabilities $p_{M|X=x}(m_k)$ can be obtained from:

$$p_{M|X=x}(m_k)=\frac{p_M(m_k)\,p_{X|M=m_k}(x)}{\sum_{j=1}^{K} p_M(m_j)\,p_{X|M=m_j}(x)}, \tag{18}$$

$k=1,2,\ldots K$. The prior probabilities $p_M(m_j)$, $j=1,2,\ldots K$ are defined in (14), and the joint probability function $p_{X|M=m_j}(x)$ of the observations given the model $M=m_j$ is obtained from:

$$p_{X|M=m_k}(x)=\int p_{X|M=m_k,\Theta_k=\theta_k}(x_k)\,p_{\Theta_k|M=m_k}(\theta_k)\,d\theta, \tag{19}$$

$k=1,2,\ldots K$, where $p_{X|M=m_k,\Theta_k=\theta_k}(x)$ and $p_{\Theta_k|M=m_k}(\theta_k)$ are defined in (15) and (16), respectively.

On the other hand, given the historical data $X=x$ and the model $M=m_k$. The posterior probability distribution of the parameters $\theta_k^{Q+1}=(\theta_{1k}^{Q+1},\theta_{2k}^{Q+1})$ for the scenario to be simulated becomes:

$$p_{\Theta_k^{Q+1}|X=x,M=m_k}(\theta_k^{Q+1})=\int p_{\Theta_k|X=x,M=m_k}(\theta_k)\,d\theta_{2k}^{1}\ldots d\theta_{2k}^{Q}. \tag{20}$$

The posterior probability $p_{\Theta_k|X=x,M=m_k}(\theta_k)$ can be obtained from:

$$p_{\Theta_k|X=x,M=m_k}(\theta_k)=\frac{p_{\Theta_k|M=m_k}(\theta_k)\,p_{X|M=m_k,\Theta_k=\theta_k}(x)}{p_{X|M=m_k}(x)},$$

$k=1,2,\ldots K$, where $p_{X|M=m_k,\Theta_k=\theta_k}(x)$, $p_{\Theta_k|M=m_k}(\theta_k)$ and $p_{X|M=m_k}(x)$ are defined in (15), (16) and (19), respectively.

In this case, the parameter $E[y|X=x]$ defined in equation (17) can be estimated via simulation by using a similar methodology as in §2, but in this case the model has to be sampled from the probability function defined in (18). To be more precise, $E[y|X=x]$ can be estimated by replicating (n times) the following experiment:

- Generate a model m_k according to the probability distribution $p_{M|X=x}(m_k)$ defined in (18).
- Generate a vector of parameters $\theta_k^{Q+1}=(\theta_{1k}^{Q+1},\theta_{2k}^{Q+1})$ according to the posterior density function $p_{\Theta_k^{Q+1}|X=x,M=m_k}(\theta_k)$ defined in (20).
- Run the simulation model to generate an observation of the output $y=y(u,m_k,\theta_k^{Q+1})$.

A consistent estimator for the performance measure $E[y|X=x]$ defined in (17) is the sample mean of n independent observations from the output $y=y(u,m_k,\theta_k^{Q+1})$ of the previous simulation experiment (see Chick 2001). It is worth mentioning that in this case, alternative methods for the estimation of the performance measure of interest can be proposed (see Zouaoui and Wilson 2001).

5 CONCLUSIONS AND DIRECTIONS FOR FUTURE WORK

As described in §2-3, a Bayesian approach allows the incorporation of information (forecasts) corresponding to a scenario of interest (through a vector of parameters θ_2^{Q+1} and its corresponding prior probability distribution), as well as the information of past periods (through a common parameter θ_1 and its corresponding prior probability distribution).

Given a proposed model for the random input, the fundamental difference between a classical approach and the Bayesian approach proposed in this article is that while in a classical approach the model parameters are fixed, in the Bayesian approach the parameters are sampled according to a posterior probability distribution. The fundamental question is why would it be convenient to apply this new approach. An intuitive answer may be that if there is uncertainty about the parameter values, the classical approach tends to underestimate the variability of the point estimator. Nonetheless, a scientific answer to this question must be sustained in solid evidence, so a more detailed research (maybe empirical) is required to answer this question; to this respect it is suggested to see Chick (2001).

The main motivation for developing the theoretical framework presented in §2 was to incorporate user information for interaction with a simulation model, as well as empirical evidence for decision making on a proposed scenario. Nonetheless, the applicability of a decision support system is intimately linked to its capacity to model a real life system, which depends on a wise election of the scenario parameters θ_2^i, on the choice of the common information from past periods (reflected in the common parameter θ_1), and on a wise selection of the input model $p(y, \Theta_1, \Theta_2)$, which are problem dependent. It is precisely in the area of search for particular models and applications where the greatest potential for research related to this methodology is found.

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