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HEDGING BEYOND DURATION AND CONVEXITY

Jian Chen

Fannie Mae
3900 Wisconsin Ave. N.W.
Washington, DC 20016, U.S.A.

Michael C. Fu

The Robert H. Smith School of Business
University of Maryland
College Park, MD 20742, U.S.A.

ABSTRACT

Hedging of fixed income securities remains one of the most challenging problems faced by financial institutions. The predominantly used measures of duration and convexity do not completely capture the interest rate risks borne by the holder of these securities. Using historical data for the entire yield curve, we perform a principal components analysis and find that the first four factors capture over 99.99% of the yield curve variation. Incorporating these factors into the pricing of arbitrary fixed income securities via Monte Carlo simulation, we derive perturbation analysis (PA) estimators for the price sensitivities with respect to the factors. Computational results for mortgage-backed securities (MBS) indicate that using these sensitivity measures in hedging provides far more protection against interest risk exposure than the conventional measures of duration and convexity.

1 INTRODUCTION

Despite the abundance of research on identifying the various factors affecting bond prices, e.g. Litterman and Scheikman (1991), Litterman, Scheikman, and Weiss (1991), Knez, Litterman, and Scheikman (1994), Nunes and Webber (1997), there has been little or no research on hedging these factors effectively. Generally people still use duration and convexity to measure the interest risk sensitivity of a fixed income security, which assumes parallel shifts in the yield curve, i.e., only shifts upward and downward in a parallel manner. Chen and Fu (2001) address the need for hedging the different factors affecting the yield curve shape by considering a representation using a Fourier-like harmonic series. However, there is no empirical evidence that such a series provides a good model of the actual yield curve. In this paper, we use historical data to empirically address this question. Based on the assumption of stationary volatility in a short time period, we discompose any yield curve change into a linear combination of these volatility factors, and we are able to derive the hedging measures for these factors.

We then test the accuracy of our hedging strategy on a mortgage-backed security (MBS), which is a security collateralized by residential or commercial mortgage loans, predominantly guaranteed and issued by three major MBS originating agencies: Ginnie Mae, Fannie Mae, and Freddie Mac. The cash flow of an MBS is generally the collected payment from the mortgage borrower, after the deduction of servicing and guaranty fees. However, the cash flows of an MBS are not as stable as that of a government or corporate coupon bond. Because the mortgage borrower has the prepayment option, mainly exercised when moving or refinancing, an MBS investor is actually writing a call option. Furthermore, the mortgage borrower also has the default option, which is likely to be exercised when the property value drops below the mortgage balance, and continuing mortgage payments would not make economical sense. In this case the guarantor is writing the borrower a put option, and the guarantor absorbs the cost. However, the borrower does not always exercise the options whenever it is financially optimal to do so, because there are always non-monetary factors associated with the home, like shelter, sense of stability, etc. And it is also very hard for the borrower to tell whether it is financially optimal to exercise these options because of lack of complete and unbiased information, e.g., they may not be able to obtain an accurate home price, unless they are selling it. And there are also some other fixed/variable costs associated with these options, such as the commission paid to the real estate agent, the cost to initialize another loan, and the negative credit rating impact when the borrower defaults on a mortgage.

All these factors contribute to the complexity of MBS cash flows. In practice, the cash flows are generally projected by complicated prepayment models, which are based on statistical estimation on large historical data sets. Because of the complicated behaviors of the MBS cash flow, due to the complex relationships with the underlying interest rate term structures, and path dependencies in prepayment behaviors, Monte Carlo simulation is generally the only applicable method to price an MBS. An MBS differs from other fixed income securities in the following aspects:

- It has relatively large cash flows far prior to the maturity date, in contrast to zero and coupon bonds.
- Its cash flows are stochastic, affected by prepayment and default behavior.
- There is no single termination event before the maturity, in contrast to callable and default bonds.

All these features make an MBS very difficult to hedge and also make it ideal for our empirical test.

The paper is organized in the following manner. Section 2 presents the principal component analysis used to evaluate the main factors. Section 3 describes the MBS valuation problem, while section 4 presents PA gradient estimators used for hedging the MBS against the factors. Section 5 contains the numerical example. Section 6 concludes the paper.

2 PCA FOR YIELD CURVE SHIFT

The Principal Components Analysis method is generally used to find the explanatory factors that maximize successive contributions to the variance, effectively explaining variations as a diagonal matrix. This method has been used in yield curve analysis for more than 10 years, see Litterman and Scheinkman (1991), Steeley (1990), Carverhill and Strickland (1992). Here we give a brief description of PCA method applied in yield curve analysis:

1. Suppose we have observation of interest rates $r_{t_i}(\tau_j)$ at time t_i, $i=1, 2, \ldots, n+1$, for different tenor dates τ_j.
2. Calculate the difference $d_{i,j} = r_{t_{i+1}}(\tau_j) - r_{t_i}(\tau_j)$, where the $d_{i,j}$ are regarded as observations of a random variable, d_j, that measures the successive variations in the term structure.
3. Find the covariance matrix $\Sigma = \mathrm{cov}(d_1, \ldots, d_k)$. Write $\Sigma = \{\Sigma_{i,j}\}$, where $\Sigma_{i,j} = \mathrm{cov}(d_i, d_j)$.
4. Find an orthogonal matrix P such that $P' = P^{-1}$ and $P\Sigma P' = diag(\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \ldots \geq \lambda_k$.
5. The column vectors of P are the principal components.
6. Using P, each observation of d_j can be discomposed into a linear combination of the principal components. By setting $e_i = p_i' d_j$, where p_i is the i^{th} column of P, we can find e_i, which is the corresponding coefficient for principal component i, $i=1, \ldots, k$. A small change in e_i will cause the term structure to alter by a multiple of p_i along the time horizon.

We use the nominal zero coupon yield from January 1997 to October 2001 as the term structure data. All data were retrieved from Professor McLulloch's web site at the Department of Economics, Ohio State University, at http://econ.ohiostate.edu/jhm/ts/ts.html. For each observation date, interest rates are provided for maturities in monthly increments from the instantaneous rate to the 40-year rate, providing a total of 481 interest rates as principal components. Table 1 lists the eigenvalues and % variance explained by the first ten factors, and Figure 1 graphs the shapes of the first four factors.

Table 1: Statistics for Principal Components

Factor	Eigenvalue	Explained(%)	Cumulative(%)
1	16.38	75.824	75.824
2	4.41	20.432	96.257
3	0.72	3.335	99.592
4	0.087	0.40	99.995
5	0.00088	0.0041	99.999
6	8.67E-05	0.00040	99.9996
7	1.59E-05	7.4E-05	99.99966
8	4.20E-06	1.9E-05	99.99968
9	4.03E-06	1.9E-05	99.99970
10	3.67E-06	1.7E-05	99.99972

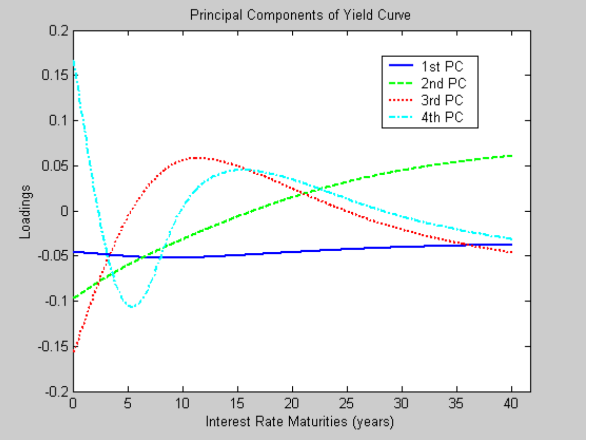

Figure 1: The first four Principal Components

The statistics indicate that the first three factors explain about 99.6% of the yield curve changes, and the first four factors explain about 99.995% of the total variance of yield curve. These results are similar to findings by Litterman and Scheikman (1991), and Nunes and Webber (1997). Figures 2 and 3 plot the matching results with three and four factors, respectively, for a monthly yield curve shift, as well as for an annual shift. The figures indicate that four factors provide a substantially improved match, both for the short term and the long term, over three factors, so in our model we will use four factors. Thus, hedging against these factors will lead to a considerably more stable portfolio, thereby reducing hedging transactions and its associated costs.

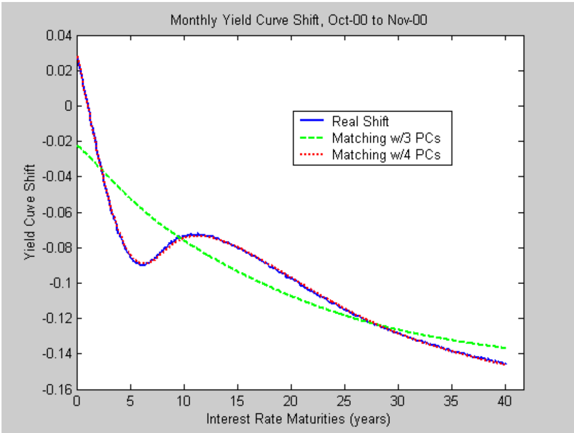

Figure 2: Match Monthly Yield Curve Shift

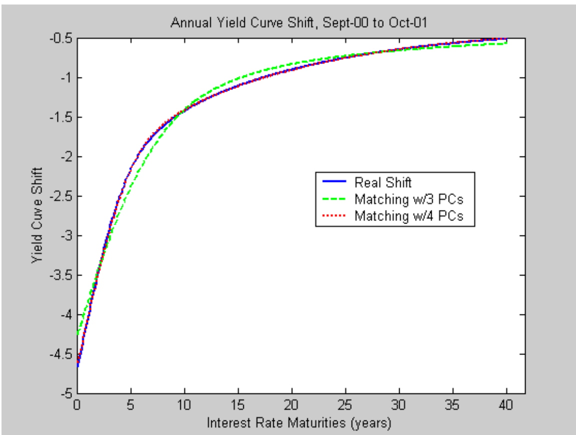

Figure 3: Match Annual Yield Curve Shift

3 MBS VALUATION

Generally the price of any security can be written as the net present value (NPV) of its discounted cash flows under the risk neutral probability measure. Specifying the price of any fixed income security is as follows:

$$P = E^{Q}\left[\sum_{t=0}^{M} PV(t)\right] = E^{Q}\left[\sum_{t=0}^{M} d(t)c(t)\right] \qquad (1)$$

where

P is the price of the security;
Q is the risk neutral probability measure;
$PV(t)$ is the present value for cash flow at time t;
$d(t)$ is the discounting factor for time t;
$c(t)$ is the cash flow at time t;
M is the maturity of the security.

Monte Carlo simulation is a numerical integration technique that is widely used to price derivative securities in the financial industry. See Boyle *et. al.* (1997) for more technical details. Basically, it is used to generate cash flows on many sample paths, so that by the strong law of large numbers, the sample mean taken over all of the paths converges to the desired quantity of interest:

$$P = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} V_i , \qquad (2)$$

where

V_i is the value calculated out in path i., under the risk-neutral probability measure.

The calculation of $d(t)$ is found from the short-term (risk-free) interest rate process:

$$d(t) = d(0,1)d(1,2)d(t-1,t) = \prod_{i=0}^{t-1} \exp(-r(i)\Delta t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\} \qquad (3)$$

where

$d(i, i+1)$ is the discounting factor for the end of period $i+1$ at the end of period i;
$r(i)$ is the short term rate used to generate $d(i, i+1)$, observed at the end of period i;
Δt is the time step in simulation, generally monthly, i.e. Δt= 1 month.

An interest rate model is used to generate the short term-rate $r(i)$; then $d(t)$ is instantly available when the short-term rate path is generated.

For a risk-free zero coupon bond, we know the cash-flows $c(t)$ ahead of time explicitly. For a callable and defaultable coupon bond, we can use an option model to predict what is the best time to recall or default that bond. For an MBS, generating $c(t)$ is more complicated, because the cash flow $c(t)$ for month t, observed at the end of month t, depends not just on the current interest rate, but also on historical prepayment behavior. From Fabozzi (1993), we have the following formula for $c(t)$:

$$\begin{aligned} c(t) &= MP(t) + PP(t) = TPP(t) + IP(t); \\ MP(t) &= SP(t) + IP(t); \\ TPP(t) &= SP(t) + PP(t); \end{aligned} \qquad (4)$$

where

$MP(t)$ is the scheduled mortgage payment for month t;
$TPP(t)$ is the total principal payment for month t;
$IP(t)$ is the Interest payment for month t;
$SP(t)$ is the scheduled principal payment for month t;
$PP(t)$ is the principal prepayment for month t.

These quantities are calculated as follows:

$$
\begin{aligned}
&MP(t) = B(t-1)\left(\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}}\right);\\
&IP(t) = B(t-1)\frac{WAC}{12};\\
&PP(t) = SMM(t)(B(t-1)-SP(t));\\
&B(t) = B(t-1) - TPP(t);\\
&SMM(t) = 1-\sqrt[12]{1-CPR(t)};
\end{aligned}
\qquad (5)
$$

where

B(t) is the principal balance of MBS at end of month *t*;
WAC is the weighted average coupon rate for MBS;
WAM is the weighted average maturity for MBS;
SMM(t) is the single monthly mortality for month *t*, observed at the end of month *t*;
CPR(t) is the conditional prepayment pate for month *t*, observed at the end of month *t*.

In Monte Carlo simulation, along the sample path, the only thing uncertain is *CPR(t)*, and everything else can be calculated out once *CPR(t)* is known. Different prepayment models offer different *CPR(t)*, and it is not our goal to derive or compare prepayment models. Instead, our concern is, given a prepayment model, how can we efficiently estimate the price sensitivities of MBS against parameters of interest? Generally different prepayment models will lead to different sensitivity estimates, so it is at the user's discretion to choose an appropriate prepayment function, as our method for calculating the "Greeks" is universally applicable.

4 DERIVATION OF GENERAL PA ESTIMATORS

If P, the price of the MBS, is a continuous function of the parameter of interest, say θ, we have the following PA estimator by differentiating both sides of (1):

$$
\begin{aligned}
\frac{dP(\theta)}{d\theta} &= E^Q\left[\frac{d\sum_{t=1}^{M} PV(t,\theta)}{d\theta}\right]\\
&= E^Q\left[\sum_{t=1}^{M}\frac{dPV(t,\theta)}{d\theta}\right]
\end{aligned}
\qquad (6)
$$

$$
\frac{d(PV(t,\theta))}{d\theta} = \frac{\partial d(t,\theta)}{\partial\theta}c(t,\theta) + \frac{\partial c(t,\theta)}{\partial\theta}d(t,\theta). \qquad (7)
$$

This reduces the original problem from estimating the gradient of a sum to estimating a sum of gradients. In particular, now we only need to estimate two gradients, $\frac{\partial c(t,\theta)}{\partial\theta}$ and $\frac{\partial d(t,\theta)}{\partial\theta}$, at each time step.

4.1 Gradient Estimator for Discounting Factor

We know that the discounting factor takes the following form from section 2, when the option adjusted spread (OAS) is not considered. For simplification, we write *d(t)* as for *d(t, θ)*:

$$
d(t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}. \qquad (8)
$$

Differentiating *w.r.t.* θ:

$$
\begin{aligned}
\frac{\partial d(t)}{\partial\theta} &= \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial\theta})\Delta t\\
&= d(t)\sum_{i=0}^{t-1}(-\frac{\partial r(i)}{\partial\theta})\Delta t.
\end{aligned}
\qquad (9)
$$

4.2 Gradient Estimator for Cash Flow

To simplify notation, we write *c(t)* for *c(t, θ)*. A simplified expression for *c(t)* is derived from (4) and (5) as follows:

$$
\begin{aligned}
c(t) &= MP(t) + PP(t)\\
&= MP(t) + [B(t-1) - SP(t)]SMM(t)\\
&= MP(t) + \{B(t-1) - [MP(t) - IP(t)]\}SMM(t)\\
&= MP(t)(1-SMM(t)) + B(t-1)(1+\frac{WAC}{12})SMM(t)\\
&= B(t-1)\{A(t)[1-SMM(t)] + g\,SMM(t)\},
\end{aligned}
\qquad (10)
$$

where

$$
\begin{aligned}
A(t) &= \frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}},\\
g &= (1+\frac{WAC}{12}).
\end{aligned}
\qquad (11)
$$

Then we can derive the gradient for *c(t)*, if *WAC* and *t* are independent of θ:

$$
\begin{aligned}
\frac{\partial c(t)}{\partial\theta} &= \frac{\partial B(t-1)}{\partial\theta}\{A(t)[1-SMM(t)] + g\,SMM(t)\}\\
&\quad + \frac{\partial SMM(t)}{\partial\theta}B(t-1)[-A(t)+g].
\end{aligned}
\qquad (12)
$$

This leads to recursive equations for calculation of the above gradient estimator from (5) and (8):

$$\frac{\partial B(t)}{\partial \theta} = \frac{\partial B(t-1)}{\partial \theta} g - \frac{\partial c(t)}{\partial \theta}. \tag{13}$$

We know that the initial balance is not dependent on θ; we have the initial conditions:

$$\begin{aligned} &\frac{\partial B(0)}{\partial \theta} = 0, \\ &\frac{\partial c(1)}{\partial \theta} = \frac{\partial SMM(1)}{\partial \theta} B(0)(-A(1)+g). \end{aligned} \tag{14}$$

Then we can iteratively work out $\frac{\partial c(t)}{\partial \theta}$ for all t. Thus the problem of calculating the gradient estimator of cash flow $c(t)$ is reduced to calculating $\frac{\partial SMM(t)}{\partial \theta}$. From (5), we have

$$\frac{\partial SMM(t)}{\partial \theta} = \frac{1}{12}(1-CPR(t))^{-\frac{11}{12}} \frac{\partial CPR(t)}{\partial \theta}. \tag{15}$$

As discussed earlier, generally $CPR(t)$ is given in the form of a prepayment function, and we are using the following type of prepayment model:

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t), \tag{16}$$

where

RI(t) is refinancing incentive;
AGE(t) is the seasoning multiplier;
MM(t) is the monthly multiplier, which is constant for a certain month;
BM(t) is the burnout multiplier.

From the gradient estimators for cash flow and discounting factor, we can easily get the gradient estimator of *PV(t)* in (7). The last step would be to apply a specific prepayment model and interest rate model to arrive at the actual implemented gradient estimators. To illustrate the procedure, we carry out this exercise in its entirety for one setting in the following section.

5 NUMERICAL EXAMPLE

As discussed in Section 2, any yield curve shift can be decomposed into a linear combination of all the principal components, and we have seen that the first four factors explain 99.995% of the yield curve variation. Here, we estimate the price sensitivities of an MBS w.r.t. these four factors.

The interest rate model we use is a one-factor Hull-White model with the following settings:

$$dr(t) = (\varphi(t) - ar(t))dt + \sigma dB(t), \tag{17}$$

where

B(t) is a standard Brownian motion;
a is the constant mean reverting speed, use 0.1;
σ is the standard deviation, constant, use 0.1;
$\varphi(t)$ is chosen to fit the initial term structure, which is determined by:

$$\varphi(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1-e^{-2at}), \tag{18}$$

where *f(0,t)* is the instantaneous forward rate, which is determined by

$$f(0,t) = t\frac{\partial R(0,t)}{\partial t} + R(0,t). \tag{19}$$

R(0,t) is the continuous compounding interest rate from now to time *t*, i.e. the term structure.

The prepayment model we use, (16), is acquired from <http://www.numerix.com>, with the following components:

$RI(t)=0.28+0.14\tan^{-1}(-8.571+430(WAC-r_{10}(t-1)));$

$AGE(t) = \min(1, \frac{t}{30});$

MM(t)=[0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.1, 1.18, 1.22, 1.23, 0.98], starting from January, ending in December;

$BM(t) = 0.3 + 0.7\frac{B(t-1)}{B(0)};$

$r_{10}(t)$ is the 10-year rate, observed at the end of period *t*, a quantity that is highly correlated with the prevailing 15-year and 30-year fixed mortgage rates.

The MBS we price is a fixed-rate mortgage pool, with a WAC of 6.62 and pool size of $4,000,000.

In order to estimate the accuracy of our PA estimator, we also estimate the gradient via finite differences (FD). Table 2 gives the sensitivities of the MBS price to the principal component factors for each method. The sensitivities measure the percentage change in the price w.r.t. a 1/100 change in the principal components factor coefficient.

From Table 2 we can see that the error is very small, and the 95% confidence intervals are almost the same. Thus, the accuracy of the PA estimator is comparable to that of the FD estimator, but the PA estimator requires over

Table 2: Comparison of PA/FD gradient estimators

PC Factor	1	2	3	4
PA estimators	0.25498%	0.23950%	0.02971%	0.15917%
C.I. of PA	0.01288%	0.01134%	0.02769%	0.02317%
FD estimators	0.25493%	0.23955%	0.02974%	0.15925%
C.I. of FD	0.01289%	0.01135%	0.02770%	0.02317%
Error	0.00005%	-0.00005%	-0.00004%	-0.00008%
Error%	0.0194%	0.0190%	0.1195%	0.0525%

70% less computation time for this four-dimensional gradient. Clearly, for higher dimensions, the efficiency gains using PA will be even greater.

Next we investigate the prediction power for these PC sensitivities against the traditional measures of duration and convexity. From October to November in 2000, the interest rate term structure shift took the form in Figure 2. These changes can be approximated by a linear combination of the first four factors, whose coefficients are determined by $e_i = p_i{}'d_j$,

$[e_1\ e_2\ e_3\ e_4]'$=[2.08941 -0.90018 0.084261 0.303106]'.

So the predicted change in the MBS price would be: $\frac{\Delta P}{P} \cong \sum_{i=1}^{4} e_i g_i = 0.3679\%$, where g_i is the gradient in table 2. By conventional measures like duration and convexity, we have the following approximation:

$$\frac{\Delta P}{P} \cong -\Delta r \cdot duration + \frac{1}{2}\Delta r^2 \cdot convexity\,. \qquad (20)$$

However, it is difficult to define Δr, since no single Δr can summarize the entire yield curve shift. For example, defining the shift as the change in the instantaneous rate is very misleading, since the short rate is increasing while the long-term rate is dropping. So here we define it as the first harmonic series of the Fourier cosine transformation as in Chen and Fu (2001), yielding the value of $\frac{\Delta P}{P} \cong 0.5406\%$. The real percentage change in the MBS price we calculate to be **0.3572%**, so our method provides much better prediction than the duration and convexity measures for this example.

6 CONCLUSION

In this paper, we applied principal components analysis on historical interest rate data to identify the first four factors that explain 99.995% of the variation in the yield curve. We then used perturbation analysis to efficiently estimate MBS price sensitivities w.r.t. these factors. Using these sensitivity measures to predict the MBS price change due to a real scenario yield curve shift leads to significantly greater accuracy than conventional measures like duration and convexity, which implies that our model will also be superior for hedging purposes.

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AUTHOR BIOGRAPHIES

JIAN CHEN `<jchen@rhsmith.umd.edu>` is currently a financial engineer in Fannie Mae. He is also a Ph.D. candidate in the Robert H. Smith School of Business, at the University of Maryland. He received his B.S.E.E. and M.S.E.E. from JiaoTong University, in Xi'an and Shanghai, respectively. His research interests include simulation and mathematical finance, particularly with applications in interest rate modeling and interest rate derivative pricing and hedging.

MICHAEL C. FU `<mfu@rhsmith.umd.edu>` is a Professor in the Robert H. Smith School of Business, with a joint appointment in the Institute for Systems Research and an affiliate appointment in the Department of Electrical and Computer Engineering, all at the University of Maryland. He received degrees in mathematics and EE/CS

from MIT, and a Ph.D. in applied mathematics from Harvard University. His research interests include simulation and applied probability modeling, particularly with applications towards manufacturing systems, inventory control, and financial engineering. He teaches courses in applied probability, stochastic processes, simulation, computational finance, and supply chain/operations management, and in 1995 was awarded the Maryland Business School's Allen J. Krowe Award for Teaching Excellence. He is a member of INFORMS and IEEE. He is currently the Simulation Area Editor of *Operations Research*, and serves on the editorial boards of *Management Science*, *IIE Transactions*, and *Production and Operations Management*. He is co-author (with J.Q. Hu) of the book, *Conditional Monte Carlo: Gradient Estimation and Optimization Applications*, which received the INFORMS College on Simulation Outstanding Publication Award in 1998.