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OPTIMAL ACTIVE MANAGEMENT FEES

Jakša Cvitanić

Departments of Mathematics and Economics
University of Southern California
1042 W Downey Way, DRB 155
Los Angeles, CA 90089-1113, U.S.A.

Lionel Martellini
Fernando Zapatero

FBE, Marshall School of Business
University of Southern California
Los Angeles, CA 90089-1427, U.S.A.

ABSTRACT

We consider the problem of a mutual fund manager that maximizes the present value of expected fees and has to decide the level of fee to impose on the fund. The fee will be paid by a risk averse investor that maximizes expected utility over final wealth. This investor can invest either in an indexed fund or in a managed fund. The manager has superior ability and, as a result of it, the fund offers a higher expected return. However, the investor has incomplete information about the ability of the fund manager. The investor has priors about this ability that are upgraded according to the performance of the fund. At some optimal level, the investor decides to switch from the market portfolio to the mutual fund. Our problem does not have a closed form solution, but we can compute optimal fees, using simulation.

1 INTRODUCTION

In this paper we attempt to establish the optimal fee structure for a fund manager that attempts to maximize the present value of the fees revenue. In order to be able to solve our problem numerically, we consider a fixed and known horizon and assume that when the investor puts money in the actively managed fund, it will be kept until the end of the time horizon.

We consider a setting with two agents, a mutual fund manager and a risk averse investor that has some wealth to allocate in risky securities. There are two types of risky securities: a market portfolio or passive index and the actively managed fund. The fund manager has superior ability, and as a result of it the actively managed fund offers an expected return higher than the market portfolio (see, for example, Brown and Goetzmann 1995 and Carhart 1997 for a review of this topic). The investor, however, has imperfect information about the ability of the mutual fund manager. The investor has priors about the ability of the fund manager and observes the realizations of the mutual fund. The priors are updated accordingly. When the expected utility conditional on past realizations that results from switching to the actively managed fund exceeds the expected utility from staying with the market portfolio, the investor moves the money to the fund. The fund manager sets the fees in such a way that the present value of expected fees is maximized. When the fee level is too high, the investor might not find optimal to switch. Of course, if the level is very low, the switch will happen almost certainly, but the revenue will be low.

2 THE MODEL

There are two assets, the market portfolio, whose price we denote by M^1 and the managed fund, whose price we denote by M^2. Their prices, and therefore their returns, satisfy the following dynamics:

$$\begin{aligned} dM_t^1 &= M_t^1(\mu dt + \sigma^1 dW_t^1) && (1)\\ dM_t^2 &= M_t^2(\alpha dt + \sigma^{21} dW_t^1 + \sigma^{22} dW_t^2) && (2)\end{aligned}$$

where all the parameters are constant and (W^1, W^2) are two independent Brownian motion processes. As a result of it, the assets are partially (but not perfectly) correlated. We assume that

$$\sigma^1 = \left((\sigma^{21})^2 + (\sigma^{22})^2\right)^{1/2} \qquad (3)$$

so that the risk of both assets is equivalent. Although this assumption is not required to solve the problem, it allows us to compare investment in the market portfolio versus investment in the mutual fund on equal grounds. However, expected returns of the two assets differ. In principle, we will focus in the case in which

$$\alpha > \mu \qquad (4)$$

so that the mutual fund has a larger expected return than the market portfolio due to the ability of the fund manager. Clearly, α is a measure of the ability of the manager. We will also consider the case in which $\alpha = \mu$. If the investor is uncertain about the level α, it might still be optimal for the mutual fund manager to charge a fee.

In this setting, there are two agents. First, we have the manager of the mutual fund whose price is described by M^2. We assume that the fund manager has perfect information about all the parameters of the model, including the parameters that characterize the preferences of the investor that we introduce later.

Then, there is an investor that knows $\mu, \sigma^1, \sigma^{21}$ and σ^{22}, but does not know α. That is, the investor knows the parameters that characterize the market portfolio. The investor also knows the risk of the mutual fund, that, for simplicity, we assume is identical to the risk of the market portfolio. However, the investor does not know α, that is, the ability of the fund manager. The investor has beliefs (priors) about the ability of the fund manager and observes the performance of the mutual fund. However, given a return process for the mutual fund, the investor cannot tell whether the return was the result of the ability of the fund manager or due to luck. The investor will observe the realizations of the mutual fund and will upgrade her priors accordingly, using Bayes rule. For an introduction to problems of incomplete information in finance, see Detemple (1986), Dothan and Feldman (1986) and Gennotte (1986). We now introduce some notation. Denote by

$$\hat{\alpha} := E[\alpha|\mathcal{F}_t] \tag{5}$$

the belief of the agent at date t, where $\mathcal{F}_t$ is the filtration generated by the value of the market portfolio and the fund and, therefore, $E[\cdot|\mathcal{F}_t]$ means expected value conditional on observing the history of the fund M^2, as well as the history of the passive fund M^1.

At moment zero, the investor thinks that α belongs to a normal distribution with mean $\hat{\alpha}_0$ and standard deviation S_0. Using Bayes rule, this problem is equivalent to that of an investor who thinks that M^2 satisfies the following dynamics:

$$dM_t^2 = M_t^2(\hat{\alpha}_t dt + \sigma^{21} dW_t^1 + \sigma^{22} dZ_t) \tag{6}$$

where Z is the innovation process,

$$dZ_t = \frac{1}{\sigma^{22}}\left(\frac{dM_t^2}{M_t^2} - \hat{\alpha}_t dt - \sigma^{21} dW_t^1\right) . \tag{7}$$

Note that by the standard filtering theory, the process

$$dZ_t = dW_t^2 + \frac{\alpha - \hat{\alpha}_t}{\sigma^{22}} dt \tag{8}$$

is a Brownian motion process under the same probability measure $\mathcal{P}$, but under the information filtration of the investor, $\mathcal{F}$.

Also $\hat{\alpha}$ and S (the standard deviation of the beliefs of the investor) satisfy the following dynamics,

$$d\hat{\alpha}_t = \frac{S_t}{\sigma^{22}} dZ_t = \frac{S_t}{\sigma^{22}}\left(dW_t^2 + (\alpha - \hat{\alpha}_t)dt\right) \tag{9}$$

$$dS_t = -\frac{S_t}{\sigma^{22}} dt . \tag{10}$$

The parameter S represents the degree of uncertainty of the investor about her own beliefs.

Next, we assume that this investor has the following objective:

$$\max E \log e^{-\rho T} X_T \tag{11}$$

where ρ is some discount factor and X represents the wealth of the investor at moment T. We assume that T is fixed, so that the investor has a fixed horizon.

This investor starts with a level of wealth X_0 that is initially invested in the market portfolio. At any point in time, the investor can decide to transfer wealth to the market portfolio, but will have to pay a dixed fee. For example, suppose that at moment t the investor has a level of wealth X_t that will be switched from the market portfolio to the mutual fund. The level of wealth to be transfered to the mutual fund is,

$$X_t\,(1 - k) \tag{12}$$

where k is the fee level.

The first step is to determine when it is optimal for the investor to switch from the market portfolio to the actively managed fund. Suppose an investor that has attained a level of wealth C. It will be optimal for this investor to switch at the stopping time t at which

$$\begin{aligned} & E_t \log[C(1-k)\exp[(\hat{\alpha}_t - \frac{1}{2}((\sigma^{21})^2 \\ & +(\sigma^{22})^2))(T-t) \\ & +\sigma^{21}(W_T - W_t) + \sigma^{22}(Z_T - Z_t)]] \qquad (13) \\ = \; & E_t \log[C \exp[(\mu - \frac{1}{2}(\sigma^1)^2)(T-t) \\ & +\sigma^1(W_T - W_t)]] \qquad (14) \end{aligned}$$

where E represents expected value with respect to the probability measure $\mathcal{P}$. It is straightforward to solve the previous equality, since (W, Z) are both martingales with respect to $\mathcal{P}$. The previous inequality obtains at the stoping time t at which

$$\log[1-k] + \hat{\alpha}_t(T-t) = \mu(T-t) . \tag{15}$$

The fund manager chooses the fee level k so as to maximize the present value of expected fees,

$$\max_{(k)} Ee^{-r\tau} kX_\tau \qquad (16)$$

where r is the interest rate level and τ represents the moment at which the investor switches money from the market portfolio to the mutual fund.

3 NUMERICAL RESULTS

It appears that an analytic solution to the problem of the mutual fund manager is not feasible because it involves the computation of the optimal stopping time τ. However, for a given value of the parameter k, it is straightforward to compute the value of the objective of the fund manager using Monte Carlo simulation (see Boyle 1977 and Duffie 2001 for an introduction and review of the application of this technique in finance). We simulate a possible path of the Brownian motion processes by discretizing the economy in time intervals of size $\triangle t$ and replacing dW by a pseudo-random process (computer generated) z such that $z \sim \mathcal{N}(0, \triangle t)$. For a given path of the economy (characterized by the realizations of W^1 and W^2) we find the stopping time at which the equality that triggers the switch, obtains. More explicitely, we simulate a path for $\hat{\alpha}$ as described by equation (9), and a path of the wealth of the investor that will be explained by (1). That is, we assume the investor invests all holdings in the indexed portfolio and observes the actively managed fund of (2). We compute the moment t at which it is optimal for the investor to transfer her wealth from the indexed portfolio to the actively managed fund. That happens at the moment in which the equality of (15) obtains. Then we compute the present value of the fee for that moment t and a level of wealth X as in (16). Since the fund manager tries to maximize expected present value and we computed the present value for just one possible path of the economy (that is, the Brownian motion processes that explain the dynamics of all the financial variables of the model), we repeat this exercise for a large number of paths and compute the average, so as to approximate the expected present value for a given fee level k. Obviously, for some of the simulated paths, it might be the case that for the time horizon considered, the equality (15) never arises. In that case, the investor never switches from the indexed portfolio to the actively managed portfolio. The income of the fund manager for that particular realization of the economy is zero, and that is the income we include in the average in order to estimate (15).

In order to determine the optimal k, we construct a one-dimensional grid of the value function of the fund manager: we start with a $k = 0$ and move up in increments of 0.01 until we reach the value $k = 1$ (which is the highest possible fee). We compare our estimates of the value function of the fund manager of (16) and choose the value k that yields the highest value for the fund manager.

We perform two exercises for a large set of parameter values. The results of the exercises are collected in Tables 1 and 2. In the last two columns of Tables 1 and 2 we collect the optimal fee level (k) and the expected present value for the fund manager given that fee level (that we represent by R), as a function of different values for the other parameters of the model. In particular, the parameters over which we perform this comparative statics exercise are: μ, the expected return in the indexed portfolio; α, the parameter that characterizes the "true" ability of the fund manager and is the actual expected return on the actively managed fund, over which the investor is uncertain; $\hat{\alpha}$, that represents the *intial belief* of the investor about the ability of the fund manager, and it is upgraded according to market realizations, as explained by (9); σ, the volatility of both the indexed fund and the actively managed fund from our assumption in (3); finally, S, which represents the initial level of uncertainty of the investor about the ability of the fund manager and it evolves according to the dynamics of (10). To simplify the number of parameters, we assume that $r = \mu$. We assume a total time horizon of three years, $T = 3$. That is, the investor invests right now in the indexed fund, and when (if ever), it is optimal to switch to the managed fund, she will do so. After three years, the investor will cash the value of the fund (either of them). If the investor switches to the actively managed fund, she pays the whole fee k regardless of when the switch takes place. We now analyze the results of each table.

In Table 1 we collect the results for the case in which the investor that switches to the actively managed portfolio invests exactly \$1 in the actively managed portfolio, that is, we assume that $X_t = 1$, regardless of the moment at which the switch takes place. As expected, both the optimal fee and the expected present value of the manager revenue increase with the ability of the manager, α. However, more important than the manager ability is the perception of the investor: the expected present value of revenue and optimal fee are higher when the investor thinks that the manager is of high ability, but the manager is average (that is, $\alpha = \mu$), than when the manager is of high ability but the investor thinks she is average (that is, $\hat{\alpha} = \mu$). With respect to the uncertainty of the investor, it increases both the optimal fee and the expected present value of the revenue. If the investor is very uncertain, it is optimal for the fund manager to charge a high fee because a "lucky streak"will make the investor think the manager is of high quality and she will switch quickly to the managed fund. This happens because an investor who is very uncertain is very sensitive to market realizations and will quickly upgrade her beliefs. With respect to the volatility of the funds, the higher the volatility, the lower the fee the manager can charge. The reason is that the investor will have a more difficult time

Table 1: Optimal Fee Level (k) and Expected Present Value of the Revenue (R) When the Investor Invests $1 in the Mutual Fund at the Optimal Stopping Time. Time Horizon Is 3 Years

μ	α	$\hat{\alpha}$	σ	S	R	k
0.1	0.12	0.1	0.2	0.03	0.005618	0.01
0.1	0.12	0.1	0.2	0.05	0.014845	0.03
0.1	0.12	0.11	0.2	0.03	0.026082	0.03
0.1	0.12	0.12	0.2	0.03	0.05	0.05
0.1	0.13	0.1	0.2	0.03	0.005839	0.01
0.1	0.2	0.1	0.2	0.1	0.065213	0.11
0.1	0.2	0.1	0.2	0.2	0.162164	0.25
0.1	0.2	0.1	0.3	0.1	0.040672	0.07
0.1	0.2	0.2	0.2	0.1	0.25	0.25
0.1	0.2	0.2	0.2	0.2	0.262194	0.33
0.1	0.2	0.2	0.2	0.3	0.314057	0.41
0.1	0.2	0.2	0.3	0.2	0.25	0.25
0.1	0.15	0.1	0.2	0.2	0.055225	0.1
0.1	0.15	0.1	0.2	0.2	0.142326	0.23
0.1	0.15	0.1	0.2	0.05	0.016755	0.03
0.1	0.15	0.1	0.3	0.1	0.03606	0.07
0.1	0.15	0.1	0.15	0.1	0.075009	0.12
0.1	0.1	0.1	0.15	0.1	0.059333	0.11
0.1	0.1	0.1	0.2	0.1	0.046188	0.09
0.1	0.1	0.1	0.2	0.3	0.195521	0.31
0.1	0.1	0.2	0.2	0.1	0.25	0.25
0.1	0.1	0.2	0.2	0.2	0.25	0.25
0.1	0.1	0.2	0.3	0.2	0.25	0.25
0.2	0.3	0.2	0.2	0.2	0.153634	0.25
0.2	0.3	0.3	0.2	0.1	0.25	0.25
0.2	0.3	0.3	0.2	0.3	0.302923	0.42

trying to find out the level of ability of the manager, and will not switch if the fee is high.

In Table 2 we perform a similar exercise, but here we assume that the investor transfers her total holdings in the indexed fund into the actively managed fund. That is, the investor starts with $1 invested in the indexed fund at moment $t = 0$. Then, the value of the initial investment changes according to equation (2). In this case, we simulate the value of the investment as well. As in the case of Table 1, when equation (15) obtains, the investor will find optimal to switch to the In some cases, at some point she might find optimal to switch to the index. As in Table 1, we construct a one-dimensional grid of k on the range $[0, 1]$, and repeat the exercise a larger number of times for each fee value, in order to find the level that maximizes the expected present value of the income of the manager. Overall, the results do not differ greatly from those of Table 1, although the optimal fee tends to be lower than the one we estimated when the wealth to be transfered to the actively managed fund was fixed. The expected present value of the income is also lower, in general.

Table 2: Optimal Fee Level (k) and Expected Present Value of the Revenue (R) When the Investor Transfers Total Value of the Market Index to the Mutual Fund at the Optimal Stopping Time. Time Horizon Is 3 Years. Initial Investment in the Market Index is $1

μ	α	$\hat{\alpha}$	σ	S	R	k
0.1	0.12	0.1	0.2	0.03	0.004858	0.01
0.1	0.12	0.1	0.2	0.05	0.012425	0.03
0.1	0.12	0.11	0.2	0.03	0.02	0.02
0.1	0.12	0.12	0.2	0.03	0.05	0.05
0.1	0.13	0.1	0.2	0.03	0.005058	0.01
0.1	0.2	0.1	0.2	0.1	0.057362	0.1
0.1	0.2	0.1	0.2	0.2	0.136607	0.25
0.1	0.2	0.1	0.3	0.1	0.035333	0.07
0.1	0.2	0.2	0.2	0.1	0.25	0.25
0.1	0.2	0.2	0.2	0.2	0.25	0.25
0.1	0.2	0.2	0.2	0.3	0.25	0.25
0.1	0.2	0.2	0.3	0.2	0.25	0.25
0.1	0.15	0.1	0.2	0.2	0.047489	0.09
0.1	0.15	0.1	0.2	0.2	0.117674	0.23
0.1	0.15	0.1	0.2	0.05	0.014434	0.03
0.1	0.15	0.1	0.3	0.1	0.031413	0.06
0.1	0.15	0.1	0.15	0.1	0.064148	0.12
0.1	0.1	0.1	0.15	0.1	0.049227	0.1
0.1	0.1	0.1	0.2	0.1	0.039299	0.08
0.1	0.1	0.1	0.2	0.3	0.153718	0.34
0.1	0.1	0.2	0.2	0.1	0.25	0.25
0.1	0.1	0.2	0.2	0.2	0.25	0.25
0.1	0.1	0.2	0.3	0.2	0.25	0.25
0.2	0.3	0.2	0.2	0.2	0.136444	0.24
0.2	0.3	0.3	0.2	0.1	0.25	0.25
0.2	0.3	0.3	0.2	0.3	0.25	0.25

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AUTHOR BIOGRAPHIES

JAKSA CVITANIC is a Professor in the Mathematics and Economics Departments of the University of Southern California. He is the Director of the Master in Mathematical Finance. He is Editor of *Finance and Stochastics*.

LIONEL MARTELLINI is an Assistant Professor in the Finance and Business Economics Departments of the University of Southern California. He is an Associate Editor of the *Journal of Alternative Investments*.

FERNANDO ZAPATERO is an Assistant Professor in the Finance and Business Economics Departments of the University of Southern California. He has previously taught at ITAM, UC Berkeley and UT Austin.