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USING SIMULATION FOR OPTION PRICING

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ABSTRACT

Monte Carlo simulation is a popular method for pricing financial options and other derivative securities because of the availability of powerful workstations and recent advances in applying the tool. The existence of easy-to-use software makes simulation accessible to many users who would otherwise avoid programming the algorithms necessary to value derivative securities. This paper presents examples of option pricing and variance reduction, and demonstrates their implementation with Crystal Ball 2000, a spreadsheet simulation add-in program.

1 INTRODUCTION

A financial *option* is a security that grants its owner the right, but not the obligation, to trade another financial security at specified times in the future for an agreed amount. The financial security that can be traded in the future is called the underlying asset, or simply the *underlying*. An option is an example of a *derivative* security, so named because its value is derived from that of the underlying. The problem of placing a value on an option is made difficult by the assymetric payoff that arises from the option holder's right to trade the underlying in the future if doing so is favorable, but to avoid trading when doing so is unfavorable.

In a modern economy, it is important for firms and households to be able to select an appropriate level of risk in their transactions. This takes place on financial markets, which redistribute risks toward those agents who are willing and able to assume them. Markets for options and other derivatives are essential because agents who anticipate future revenues or payments can ensure a profit above a certain level or insure themselves against a loss above a certain level.

Options allow for hedging against one-sided risk. However, a prerequisite for efficient management of risk is that these derivative securities are priced correctly when they are traded. Nobel laureates Fischer Black, Robert Merton, and Myron Scholes developed in the early 1970s a method to price specific types of options exactly, but their method does not produce exact prices for all types of options. In practice, numerical methods such as simulation are often used to price derivative securities. Simulation is also used for estimating sensitivities, risk analysis, and stress testing portfolios.

The use of Monte Carlo simulation in pricing options was first published by Boyle (1977), but recently the literature in this area has grown rapidly. For example, see the work by Ameur et al. (1999), Boyle et al. (1995 and 1997) Broadie and Glasserman (1996), Caflisch et al. (1997), Fu (1995), Fu and Hu (1995), Fu et al. (1999), Glasserman and Zhao (1999), Grant et al. (1997), Joy et al. (1996), Lemieux and L'Ecuyer (1998), Morokoff (1998), and Vázquez-Abad and Dufresne (1998). This paper describes some of this past work and related Excel files demonstrate how the ideas can be implemented using a spreadsheet simulation add-in package (Crystal Ball 2000). The Excel Files are located on the website `<www2.bschool.ukans.edu/jcharnes/options/wsc00>`.

2 BACKGROUND

The price of the underlying is denoted by S_t, for $0 \le t \le T$, where T is the expiration date of the option. The agreed amount for which the underlying is traded is called the *strike price*, which is denoted by K. There are many different types of options. Some basic types are listed in the next section.

2.1 Types of options

Call. A call option gives its owner the right to *purchase* the underlying for the strike price on the expiration date. The payoff for a call option with strike price K when it is exercised on date t is $(S_t - K)^+$, where $(X)^+ \equiv \max(X, 0)$.

Put. A put option gives its owner the right to *sell* the underlying for the strike price on the expiration date. The payoff for a put option with strike price K when it is exercised on date t is $(K - S_t)^+$.

European. A European option allows the owner to exercise it *only at the termination date*, T. Thus, the owner cannot influence the future cash flows from a Euroepan option with any decision made after purchase.

American. An American option allows the owner to exercise *at any time on or before the termination date*, T. Thus, the owner of an American call (put) option can influence the future cash flows with a decision made after purchase by exercising the option when the price of the underlying is high (low) enough to compel the owner to do so.

Exotic. The payoffs for exotic options depend on more than just the price of the underlying at exercise. Examples of exotics are Asian options, which pay the difference between strike and the average price of the underlying over a specified period; Up-and-in Barrier options, which pay the difference between strike and spot prices at exercise only if the price of the underlying has exceeded some prespecified barrier level; and Down-and-out Barrier options, which pay the difference between strike and spot at exercise only if the price of the underlying has not gone below some prespecified barrier level

New types of options appear frequently. Because they are designed to cover individual circumstances, analytic methods to price new derivative securities are not always available when the securities are developed. However, it is possible to obtain good estimates of the value of most any type of option using simulation and the concept of risk-neutral pricing.

2.2 Risk-neutral pricing

Arbitrage is the purchase of securities on one market for immediate resale on another in order to profit from a price discrepancy. In an efficient market, arbitrage opportunities cannot last for long. As arbitrageurs buy securities in the market with the lower price, the forces of supply and demand cause the price to rise in that market. Similarly, when the arbitrageurs sell the securities in the market with the higher price, the forces of supply and demand cause the price to fall in that market. The combination of the profit motive and nearly instantaneous trading ensures that prices in the two markets will converge quickly if arbitrage opportunities exist.

Using the assumption of no arbitrage, financial economists have shown that the price of a derivative security can be found as the expected value of its discounted payouts when the expected value is taken with respect to a transformation of the original probability measure called the *equivalent martingale* measure or the *risk-neutral* measure. See Duffie (1996), Hull (1997), and Wilmott (1998) for more about risk-neutral pricing.

2.3 Black-Scholes model

The price of a fairly valued European put option is the expected present value of the payoff $E\left[e^{-rT}(K-S_T)^+\right]$, where the expectation is taken under the risk-neutral measure. To compute this expectation, Black and Scholes (1973) modeled the stochastic process generating the price of a non-dividend-paying stock as geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $dW(t)$ represents a Wiener process.

The Black-Scholes price for a European Call option on a non-dividend-paying stock trading at time t is:

$$C_t(S_t, T-t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2), \tag{1}$$

where

$$d_1 = \frac{\log(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{2}$$

$$d_2 = \frac{\log(S/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}, \tag{3}$$

$N(d_i)$ is the cumulative distribution value for a standard normal random variable with value d_i, K is the strike price, r is the risk-free rate of interest, and T is the time of expiration.

The Black-Scholes solution for a European Put option on a non-dividend-paying stock trading at time t is:

$$P_t(S_t, T-t) = -S_t N(-d_1) + K e^{-r(T-t)} N(-d_2), \tag{4}$$

where d_1 and d_2 are given by expressions (2) and (3) above.

Note that the variables appearing in the Black-Scholes equations are the current stock price, time, stock price volatility, and the risk-free rate of interest, all of which are independent of individual risk preferences. This allows for the assumption that all investors are risk neutral, which leads to the solutions above. However, these solutions are valid in all worlds, not just those where investors are risk neutral.

2.4 Using Monte Carlo simulation for determining option prices

In the Black-Scholes world-view, a fair value for an option is the present value of the option payoff at expiration under a risk-neutral random walk for the underlying asset prices.

Therefore the general approach to using simulation to find the price of the option is straightforward:

1. Using the risk-free measure, simulate sample paths of the underlying state variables (e.g., underlying asset prices and interest rates) over the relevant time horizon;
2. Evaluate the discounted cash flows of a security on each sample path, as determined by the structure of the security in question; and
3. Average the discounted cash flows over sample paths.

In effect, this method computes an estimate of a multidimensional integral—the expected value of the discounted payouts over the space of sample paths. The increase in complexity of derivative securities has led to a need to evaluate high-dimensional integrals. Monte Carlo simulation is attractive relative to other numerical techniques because it is flexible, easy to implement and modify, and the error convergence rate is independent of the dimension of the problem.

To simulate stock prices using the Black-Scholes model, generate independent replications of the stock price at time $t + \Delta t$ from the formula

$$S^{(i)}_{t+\Delta t} = S_t \exp\left((r - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z^{(i)}\right), \quad (5)$$

for $i = 1, \ldots, n$,where S_t is the stock price at time t, r is the riskless interest rate, σ is the stock's volatility, and $Z^{(i)}$ is a standard normal random variate.

The Excel files `EuroCall.xls` and `EuroPut.xls` contain simulation models for pricing European call and put options on a stock with current price $S_0 = \$100$, strike price $K = \$100$, and annual volatility $\sigma = 40\%$, in a world with risk-free rate $r = 10\%$. Of course, these are securities for which the Black-Scholes formulas (1) and (4) provide an exact answer, so there is no need to use simulation to price them. However, European options serve the same purpose in financial simulation as the $M/M/1$ model does in queueing simulation—since we know the exact solution, it becomes possible to check the accuracy of our simulation results. In the Excel file `EuroCall.xls`, the European call price estimated by simulation with 10,000 iterations is \$8.12 (with standard error 0.12), while the Black-Scholes price is \$8.09. In `EuroPut.xls`, the European put price estimated by simulation with 10,000 iterations is \$6.11 (0.09), while the Black-Scholes price is \$6.11.

The increased availability of powerful computers and easy-to-use software has enhanced the appeal of simulation to price derivatives. The main drawback of Monte Carlo simulation is that a large number of replications may be required to obtain precise results. However, variance reduction techniques can be applied to sharpen the inferences and reduce the number of replications required.

3 VARIANCE REDUCTION TECHNIQUES

3.1 Antithetic variates

The method of antithetic variates for pricing options is based on the fact that if $Z^{(i)}$ has a standard normal distribution, then so does $-Z^{(i)}$. Therefore, if we replace $Z^{(i)}$ in (5) with $-Z^{(i)}$, we also get a valid sample from the distribution of stock prices at time T. In using antithetic variates, we construct two intermediate estimates, $\theta_1(Z^{(i)})$ and $\theta_2(Z^{(i)})$, then a final estimate, $\theta^{AV} = (\theta_1 + \theta_2)/2$.

In the Excel file `EuroCallAV.xls` the standard error of the estimate of the call price is 0.06, compared to the value 0.12 obtained from the same number of runs specified in §2.4. In `EuroPutAV.xls` the standard error of the estimate of the put price is 0.04, compared to the value 0.09 obtained from the same number of runs specified in §2.4.

3.2 Control variates

The method of control variates replaces the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity whose expectation is known. Kemna and Vorst (1990) use control variates to value Asian options. The unknown quantity of interest is the price, C_a, of a call option whose payoff at expiration is $(A - K)^+$, where A is the arithmetic average of the underlying during the holding period. The related quantity with known expectation is the price, C_g, of an option whose payoff is $(G - K)^+$, where G is the geometric average. Because of the lognormality of the stock price model, an analytic expression is available for C_g, but not for C_a.

The prices are defined as $C_a = \mathrm{E}\left[\hat{C}_a\right]$, and $C_g = \mathrm{E}\left[\hat{C}_g\right]$, where $\hat{C}_a$ and $\hat{C}_g$ are the discounted option payoffs for a single simulated path of the underlying for options that pay off on the arithmetic and geometric means, respectively. Then

$$C_a = C_g + \mathrm{E}\left[\hat{C}_a - \hat{C}_g\right],$$

and an unbiased estimator of C_a is given by

$$\hat{C}^{CV}_a = \hat{C}_a + (C_g - \hat{C}_g).$$

Using C_g as a control variate reduces the estimation error because it "steers" the estimate toward the correct value. See the file `AsianCallCV.xls`, in which the standard error of the estimated price is reduced from 0.0862 without variance reduction applied to 0.0076 when the geometric average is used as a control variate. The use of control variates is well known in simulation, and there are refinements to this technique that can improve the results somewhat (see Boyle et al. 1997).

3.3 Moment Matching

The method of moment matching was introduced by Barraquand and Martineau (1995). Let $Z^{(i)}$, $i = 1, \ldots, n$ denote the standard normal variates used to drive the simulation. Transform these so that the first sample moment matches the first population moment:

$$Z'^{(i)} = Z^{(i)} - \bar{Z},$$

where $\bar{Z} = \sum_{i=1}^{n} Z^{(i)}/n$. Then use $Z'^{(i)}$ to generate each terminal stock price $S_T'^{(i)}$. The first-moment-matched estimator of the call option price is the average of the n values

$$e^{-rT}\left(S_T'^{(i)} - K\right)^+.$$

To match the first two moments, generate each terminal stock price $S_T'^{(i)}$ using the transformation

$$Z'^{(i)} = \left(Z^{(i)} - \bar{Z}\right)\frac{\sigma_Z}{S_Z},$$

where S_Z is the sample standard deviation of the generated values $Z^{(i)}$.

Boyle et al. (1997) take this idea a step further by matching the first two moments of the terminal stock prices as

$$S_T'^{(i)} = \left(S_T^{(i)} - \overline{S_T}\right)\frac{\sigma_{S_T}}{S_{S_T}} + \mu_{S_T}, \tag{6}$$

where $\mu_{S_T} = S_0 e^{rT}$, $\overline{S_T} = \sum_{i=1}^{n} S_T/n$,

$$\sigma_{S_T} = S_0\sqrt{e^{2rT}(e^{\sigma^2 T} - 1)},$$

and

$$S_{S_T} = \sum_{i=1}^{n}\left(S_T'^{(i)} - \overline{S_T}\right)/(n-1).$$

The file `EuroCallMM.xls` demonstrates the reduction in variance obtained through moment matching using (6). Because the random inputs are not independent, this file estimates the standard error by using batches of 100 runs of the simulation. The estimated standard error is the standard deviation of the output distribution. In this file, the standard error without variance reduction is 1.23, while the standard error with moment matching is 0.22. Boyle et al. (1997) show even greater reductions for some inputs that they consider, but also show that whenever a moment is known, it is better to use it as a control variate than for moment matching.

3.4 Latin hypercube sampling

Latin hypercube sampling (LHS) is a restructuring of the simulation method in an attempt to improve the efficiency of the estimation procedure by reducing the estimation error for a fixed computing budget. In LHS, the components of the random-number input vector $\mathbf{U}_{LHS}^{(i)} \equiv (U_1^{(i)}, \ldots, U_d^{(i)})$ are generated according to the relation (see Avramidis and Wilson 1995):

$$U_j^{(i)} = \frac{\pi_j(i) - 1 + U_{ij}^\star}{k} \quad \text{for} \quad \begin{cases} i = 1, \ldots, k, \\ j = 1, \ldots, d, \end{cases}$$

where the $\pi_1(\cdot), \ldots, \pi_d(\cdot)$ are permutations of the integers $\{1, \ldots, k\}$ that are randomly sampled with replacement from the set of $k!$ such permutations, with $\pi_j(i)$ denoting the ith element in the jth randomly sampled permutation; and $\{U_{ij}^\star : j = 1, \ldots, d; i = 1, \ldots, k\}$ are random numbers computed independently of each other and of the permutations $\pi_1(\cdot), \ldots, \pi_d(\cdot)$.

Introduced by McKay et al. (1979), Latin hypercube sampling has been studied by Stein (1987), Owen (1998), and Avramidis and Wilson (1995). Avramidis and Wilson (1996) show that LHS estimates have mean square errors of less than 40% of Monte Carlo estimates of the median response for stochastic activity networks.

The file `EuroCallLHS.xls` contains a comparison of LHS and Monte Carlo for pricing a European call option.

3.5 Importance sampling

Importance sampling is often used to make rare events less rare. For example, consider a down-and-in barrier call option that is far from the barrier. This call option has $S_0 = 95$, $\sigma = .15$, $r = .05$, $K = 90$, and barrier $H = 85$, with payoff $(S_T - K)^+$ only if $S_t < H$ for some time t between 0 and T. This option will pay off at time T infrequently, because to be in the money the stock price must fall below the barrier, then rise above the strike price during the period 0 to T. The time to expiration is $T = .25$, and the barrier is monitored at discrete times $n\Delta t$, $n = 0, 1, \ldots, m = 50$, with $\Delta t = T/m$. Following Boyle et al. (1997), set the barrier $H = S_0 e^{-b}$ and the strike at $K = S_0 e^{-c}$, with $b, c > 0$. A down-and-in call pays $S_T - K$ at time T if $S_T > K$ and $S_{n\Delta t} < H$ for some $n = 1, 2, \ldots, m$. Write the price of the underlying at monitoring instants as $S_{n\Delta t} = S_0 e^{U_n}$, where $U_n = \sum_{i=1}^{n} X_i$ with the X_i i.i.d. normal having mean $(r - \frac{1}{2}\sigma^2)\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$. Let τ be the first time that U_n drops below $-b$. Then the probability of a payout is $P(\tau < m, U_m > c)$. If b and C are large, this probability is small, and most simulation runs return zero. Importance sampling can increase this probability and get more information from each run.

With no variance reduction, the price of the down-and-in call is $e^{-rT}\,\mathrm{E}_r\left[\mathbf{1}_{\{\tau<m\}}(S_T-K)^+\right]$. With importance sampling, the price is $e^{-rT}\,\mathrm{E}_\mu\left[L\mathbf{1}_{\{\tau<m\}}(S_T-K)^+\right]$, where

$$\begin{aligned} L &= \exp\left(-(\theta_1-\theta_2)U_\tau - \theta_2 U_m + m\psi(\theta_2)\right), \\ \theta_i &= (\mu_i - r + \sigma^2/2)/\sigma^2 \text{ for } i=1,2, \\ \psi(\theta) &= (r-\sigma^2/2)\Delta t\theta + \sigma^2/2\Delta t\theta^2, \end{aligned}$$

$\mu_1 = -(2b+c)/T$, and $\mu_2 = (2b+c)/T$.

The intuition behind this estimator is that the drift μ_1 is set to a negative value to drive the asset price to the barrier, then the drift μ_2 is set to a positive value to drive the asset above the strike price. Importance sampling has its greatest advantages when the current price of the underlying is far from the barrier. The file `BarrierCallIS.xls` shows the standard error to be reduced by an order of magnitude for a Down-and-In call option on a stock currently trading at \$95, with barrier $H = \$85$, and strike price $K = \$90$.

3.6 Conditional Monte Carlo

This technique reduces variance because it does part of the integration analytically, which leaves less to be done by Monte Carlo. Conditional Monte Carlo is used by Hull and White (1987) to price options with stochastic volatilities. Their model has an asset price and its variability evolving as

$$\begin{aligned} dS_t &= rS_t\,dt + vS_t\,dW_{1t}, \text{ and} \\ dv_t^2 &= \alpha v_t^2\,dt + \xi v_t^2\,dW_{2t}, \end{aligned}$$

where dW_{1t} and dW_{2t} represent independent Wiener processes.

To price a standard European call on this asset using Monte Carlo, simulate sample paths of v_t^2 and S_t up to time T and average $(S_T-K)^+$ over all paths. To price the call with conditional Monte Carlo, note that the asset price S_t may be treated as having a time varying but deterministic volatility equal to the average squared volatility over the path. Thus, conditional on the volatility path, the option can be priced by the Black-Scholes formula

$$e^{-rT}E\left[(S_T-K)^+|v_t, 0\le t\le T\right].$$

This expression is evaluated as (1) with $\overline{V}$ replacing σ^2, where $\overline{V}$ is the average squared volatility over the path.

The file `EuroCallCMC` demonstrates the use of conditional Monte Carlo. In it, the estimated price and standard error for straightforward Monte Carlo are 0.0303 and 0.0014, respectively, while the estimated price and standard error for conditional Monte Carlo are 0.0295 and 0.0001, respectively.

3.7 Quasi-Monte Carlo Simulation

The pseudo-random numbers used in Monte Carlo simulation are generated to fill the interval $[0,1)$ in a sequence that passes statistical tests of randomness. *Quasi-random* numbers are generated so that the interval $[0,1)$ is filled in a more uniform sequence than it is by pseudo-random number generators. Such sequences are also known as *low–discrepancy* sequences (See Niederreiter 1988, and Niederreiter and Spanier 1998 for more about low-discrepancy sequences).

Quasi-Monte Carlo simulation is used to price options in a manner similar to Monte Carlo simulation, except that quasi-random numbers are used instead of pseudo-random numbers. Quasi-Monte Carlo simulation shows great promise for option pricing, but presents a problem in that elementary statistical theory cannot be used to compute error bounds as is done in Monte Carlo simulation. The file `EuroCallQMC.xls` demonstrates the use of quasi-random numbers and compares results to Monte Carlo using pseudo-random numbers.

4 PRICING AMERICAN OPTIONS

An American put option grants its holder the right, but not the obligation, to sell shares of a common stock for the exercise price, X, at or before time T. The Black-Scholes expressions (1) and (4) yield approximations for the values of an American call and put option, but in practice numerical techniques are used to obtain closer approximations of options that can be exercised at times in addition to time T.

The fair value of an American put option is the discounted expected value of its future cash flows. The cash flows arise because the put can be exercised at the next instant, dt, or the following instant, $2dt$, if not previously exercised, …, *ad infinitum*. In practice, American options are approximated by securities that can be exercised at only a finite number of opportunities, k, before expiration at time T. These types of financial instrument are called *Bermudan options*. By choosing k large enough, the computed value of a Bermudan option will be practically equal to the value of an American option.

Geske and Johnson (1984) develop a numerical approximation for the value of an American option based on extrapolating values for Bermudan options having small numbers (1, 2, and 3) of exercise opportunities. Their results are exact in the limit as the number of exercise opportunities goes to infinity. Broadie and Glasserman (1997) use simulation to price American options by generating two estimators, one biased high and one biased low, both asymptotically unbiased and converging to the true price. Avramidis and Hyden (1999) discuss ways to improve the Broadie and Glasserman estimates. Longstaff and Schwartz (1998) provide an alternate method for pricing American options.

The early exercise feature of American options makes their valuation more difficult because the optimal exercise policy must be estimated as part of the valuation. This *free–boundary* aspect leads some to conclude that Monte Carlo simulation is not suitable for valuing American options (e.g., Hull 1997). However, research in this area is continuing.

The file `BermuPutAV.xls` contains an example of valuing an Bermudan put option with initial stock price $S_0 = 100$, risk-free rate $r = 0.05$, dividend yield $\delta = 0.10$, time to expiration $T = 1.0$, volatility $\sigma = 0.2$, strike price $K = 100$, and two early-exercise opportunities at times $T/3$ and $2T/3$. From Broadie and Glasserman (1997), the true value of this option is 5.726.

The spreadsheet illustrates a method to price this option using simulation and an optimization approach due to Glover (1977 and 1997). This method uses tabu search to identify an optimal policy, then a final set of iterations to estimate the value of the option under the identified policy. The estimated price for the option described above is 5.7264 with standard error 0.0102.

5 CONCLUSION

Interest in use of Monte Carlo methods for option pricing is increasing because of the flexibility of the method in handling complex financial instruments. Further, the use of variance reduction techniques along with the greater power of today's workstations has reduced the execution time required for achieving acceptable precision. Monte Carlo simulation will continue to gain appeal as financial instruments become more complex, workstations become faster, and simulation software is adopted by more users.

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