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ON THE IMPLEMENTATION OF A SMOOTHED PERTURBATION ANALYSIS ESTIMATOR FOR A SINGLE-SERVER QUEUE WITH MULTIPLE VACATIONS

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ABSTRACT

We consider to apply the perturbation analysis (PA) to a single-server queue with multiple server vacations. A major difficulty in the implementation of PA estimator for such queueing systems is that the introduced perturbations are propagated and accumulated continuously without any resetting. This fact may lead to the divergence of PA estimates even if the limiting distribution exists. We show that it is possible to construct a sequence of points on the observed sample path such that the perturbations are accumulated only between the two adjacent points. The key idea lies in constructing a perturbed path which is not on the same sample as the observed nominal path but is identical in probability law.

1 INTRODUCTION

Perturbation analysis (PA) was proposed as an effective technique for sensitivity estimation of stochastic discrete event systems and has been developed increasingly for about the last two decades (see e.g. Ho and Cao 1991, Glasserman 1991, Fu and Hu 1997 and references therein). In particular, the discovery of commuting condition in the infinitesimal perturbation analysis (IPA) and the generalization of smoothed perturbation analysis (SPA) have made the theory of PA richer and expanded the applicable class of systems (Glasserman 1991, Fu and Hu 1997). In this paper, we apply the PA to a single-server queue with multiple server vacations, where the server often takes vacations and stops for random durations. Such queueing systems with vacations often arise as models of many computer, communication and production systems, and an extensive survey is presented in (Doshi 1986).

A major difficulty in applying the PA to queueing systems with multiple vacations is that, since the server does not take any idle period but is always either in service or in vacation, the effects of introduced perturbations are generally propagated and accumulated continuously without resetting. This contrasts with the systems without vacations where the perturbations are accumulated only during each busy period. This difficulty may lead to the divergence of the PA estimates along with the length of observed sample path even if the limiting distribution exists. Indeed, it is said that, when estimating the derivative of a long-run average performance by PA, the system must experience the states infinitely often, where only one event is possible to occur or, even if more than one events are possible, all of them but one are exponentially distributed (Sections 8.2 and 8.3 in Glasserman 1991). In other words, these system states play a role of absorbing the effect of accumulated perturbations. According to this statement, Miyoshi and Hasegawa (1994) assumed that the vacation lengths are exponentially distributed in a similar model. Now, we tackle the case of non-exponential event-lifetimes and show that it is possible to construct a sequence of points on the observed sample path such that the perturbations are accumulated only between the two adjacent points. The key idea lies in constructing a perturbed path which is not on the same sample as the observed nominal path but is identical in probability law, and to develop a similar argument used to show the coupling of renewal processes in (Asmussen 1987) and (Lindvall 1992). We also consider an alternative problem that the implementation of the SPA estimators generally requires the additional subpaths (Fu and Hu 1997). To this problem, we show that, for some cases, the required subpaths can be constructed from the observed sample path.

The paper is organized as follows: The model considered in the paper is described and some notations are introduced in the next section. The derivative estimator is provided through the SPA technique in Section 3. In Section 4, two problems in the implementation of the SPA estimator are considered: First in 4.1, we treat the problem that the perturbations are accumulated continuously and we show that it is possible to construct a sequence of points such that the perturbations are accumulated only between the two adjacent points. We consider the other problem in 4.2, where we show that, for some cases, the required subpaths can be constructed from the observed path through the sample path analysis.

2 MODEL DESCRIPTION

The model considered in the paper consists of a single server and an unlimited waiting buffer, where the server takes vacations according to some vacation policy (described below). Let $\mathbb{N} = \{1, 2, \dots\}$ and $\{T_n\}_{n\in\mathbb{N}}$ be the sequence of arrival epochs of customers, satisfying $0 = T_1 < T_2 < \cdots$. We assume that distribution function F for customer service times is parameterized by a real number θ in an open interval Θ and we note the service time of the nth customer by $\sigma_n(\theta)$ for $n = 1, 2, \dots$. In applying the PA, the underlying probability space is required to be independent of the parameter θ. To have such a probability space, we define the inverse function of $F(\cdot, \theta)$ on $[0, 1]$ by

$$F^{-1}(u, \theta) = \sup\big\{x \geq 0 : F(x, \theta) \leq u\big\}.$$

Thus, introducing a sequence $\{U_n^d\}_{n\in\mathbb{N}}$ of uniformly distributed random variables on $[0, 1]$, $\sigma_n(\theta)$ is obtained by $F^{-1}(U_n^d, \theta)$. We introduce another sequence $\{U_n^r\}_{n\in\mathbb{N}}$ of uniformly distributed random variables on $[0, 1]$ for the server to make a decision whether serving a customer or taking a vacation after each service completion, which allows us to deal with Bernoulli-type vacation policies. We define the canonical version of probability space $(\Omega_A, \mathcal{F}_A, \mathrm{P}_A)$ such that a sample of Ω_A is a realization of marked point process $\{(T_n, U_n^d, U_n^r)\}_{n\in\mathbb{N}}$. In order to represent the vacation lengths, we use a random sequence $\{V_n\}_{n\in\mathbb{Z}_+}$ of nonnegative real numbers, where each V_n is independent of others and has a common distribution function G for $n \neq 0$. In other words, the process $\{S_n\}_{n\in\mathbb{N}}$ given by $S_n = \sum_{i=0}^{n-1} V_i$ forms a delayed renewal process with inter-renewal distribution G. We assume that $G(0) = 0$ and that $\{V_n\}_{n\in\mathbb{Z}_+}$ is independent of $\{(T_n, U_n^d, U_n^r)\}_{n\in\mathbb{N}}$. Define also the canonical probability space $(\Omega_V, \mathcal{F}_V, \mathrm{P}_V)$ such that a realization of $\{V_n\}_{n\in\mathbb{Z}_+}$ is a sample of Ω_V. We work on the product probability space $(\Omega, \mathcal{F}, \mathrm{P}) = (\Omega_A \times \Omega_V, \mathcal{F}_A \otimes \mathcal{F}_V, \mathrm{P}_A \times \mathrm{P}_V)$.

A state of the system is described by a couple $(x, y) \in \mathcal{S} = \mathbb{Z}_+ \times \overline{\mathbb{Z}}_+ \setminus \{(0, y) : y \geq 1\}$, where $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{+\infty\}$. The first element of (x, y) represents the number of customers in the system (including one in service if any), and a positive value of y is the possible number of customers (including one in service) served until the server goes out to the next vacation, whereas $y = 0$ means that the server is in vacation. The value of y plays a role of the counter and reduces by one at each service completion. The exclusion of $\{(0, y) : y \geq 1\}$ represents that the server is in vacation whenever the system is empty. We assume without a loss of generality that the initial state at time $0-$ is $(0, 0)$ and the remaining vacation time is V_0. Symbols a, d and v are used to represent the events, the occurrences of which may change the system state, corresponding to an arrival and a departure of a customer, and a vacation termination of the server, respectively. For each $(x, y) \in \mathcal{S}$, the set of possible events in state (x, y) is given by

$$\mathcal{E}(x, y) = \begin{cases} \{a, d\} & \text{if } y > 0; \\ \{a, v\} & \text{if } y = 0. \end{cases}$$

We consider a mixed type vacation policy as follows: There are constants $\alpha, \beta \in \overline{\mathbb{Z}}_+$ $(\alpha < \beta)$ and, when the server returns from a vacation and finds more than α customers waiting in the queue, the value of y is set up at β. If the server finds less than or equal to α customers in the queue, it takes another vacation immediately and repeats this manner (multiple vacations). In addition, there is a constant $p \in [0, 1]$ and, after each service completion, the server takes a vacation with probability p even if the system state is (x, y) with $x > 0$ and $y > 0$. Since the occurrence of any event can change the system state, we regard each event as a mapping taking values on $\mathcal{S}$. Noting that only the departure event d determines the next state randomly, we have a, v: $\mathcal{S} \to \mathcal{S}$ and d: $\mathcal{S} \times [0, 1] \to \mathcal{S}$ such as

$$\begin{aligned} a(x, y) &= (x + 1, y), \\ v(x, 0) &= \begin{cases} (x, \beta) & \text{if } x > \alpha; \\ (x, 0) & \text{if } x \leq \alpha, \end{cases} \\ d((x, y), u) &= \begin{cases} (x - 1, y - 1) \\ \quad \text{if } x \geq 2 \text{ and } u > p; \\ (x - 1, 0) \\ \quad \text{if } x = 1, \text{ or } x \geq 2 \text{ and } u \leq p. \end{cases} \end{aligned} \tag{1}$$

We can see that, when $\alpha = 0$ and $\beta = +\infty$, we have the ordinary Bernoulli vacation policy, and further when $p = 0$ is added, we obtain the exhaustive policy. On the other hand, when $\alpha = 0$ and either $\beta = 1$ or $p = 1$, we have the pure limited policy. We assume that customers are served in the first-come, first-served (FCFS) order and that each service is continued without interruption once it starts. Note that once given a vacation policy, a sample path of the system is uniquely determined by $(\omega_A, \omega_V) \in \Omega$ for each $\theta \in \Theta$.

The performance measure of our interest is the average mean sojourn time over the first m customers given by

$$J_m(\theta) = \mathrm{E}[L_m(\theta)] = \frac{1}{m} \sum_{n=1}^{m} \mathrm{E}[W_n(\theta)],$$

where $W_n(\theta)$ denotes the sojourn time of the nth customer. In the following sections, we intend to give the PA estimator for $\mathrm{d}J_m(\theta)/\mathrm{d}\theta$ provided that it exists.

3 APPLICATION OF PERTURBATION ANALYSIS

In applying the PA, we compare the two sample paths, one of which is determined by $(\omega_A, \omega_V, \theta)$ and the other is by $(\omega_A, \omega_V, \theta + \Delta\theta)$, and consider the limit;

$$\lim_{\Delta\theta\to 0} \frac{\mathrm{E}\big[L_m(\theta+\Delta\theta) - L_m(\theta)\big]}{\Delta\theta} = \frac{1}{m}\sum_{n=1}^{m} \lim_{\Delta\theta\to 0} \mathrm{E}\left[\frac{W_n(\theta+\Delta\theta) - W_n(\theta)}{\Delta\theta}\right]. \qquad (2)$$

We refer to the sample path with θ as the nominal path and that with $\theta + \Delta\theta$ as the perturbed path. Generally, the nominal path is obtained directly by a simulation experiment (or observing the driving system) whereas the perturbed path is constructed from the nominal path by a thought experiment.

The argument in this section essentially follows from (Fu and Hu 1997) and we will proceed without some details. First, we impose the following assumption on the service time distribution, which is conventional in the PA application (see Glasserman 1991):

Assumption 1

1. For any $u \in [0, 1]$, $F^{-1}(u, \cdot)$ is differentiable.
2. For each $\theta \in \Theta$, $F(\cdot, \theta)$ is absolutely continuous with density $\partial_x F(\cdot, \theta)$ which is strictly positive on an open interval $I(\theta)$ and zero elsewhere. F is continuously differentiable on $\bigcup_{\theta\in\Theta} I(\theta) \times \{\theta\}$.

We know from (Suri 1987) and also (Glasserman 1991) that, under Assumption 1, the derivatives $\{\mathrm{d}\sigma_n(\theta)/\mathrm{d}\theta\}_{n\in\mathbb{N}}$ can be calculated in a real experiment without the knowledge of $\{U_n^d\}_{n\in\mathbb{N}}$ by

$$\frac{\mathrm{d}\sigma_n(\theta)}{\mathrm{d}\theta} = -\frac{\partial_\theta F(\sigma_n(\theta), \theta)}{\partial_x F(\sigma_n(\theta), \theta)}, \qquad (3)$$

where $\partial_\theta F(x, \theta) = \partial F(x, \theta)/\partial\theta$. In this paper, we add the following assumption for simplicity:

Assumption 2 *For each $u \in [0, 1]$, $F^{-1}(u, \cdot)$ is nondecreasing on Θ.*

Due to the perturbation in service times, the occurrence of an arrival event may change the order with a service completion or a vacation termination. It is well known that, if such an order change in the event occurrences does not influence the system state, that is, satisfies the commuting condition, then the IPA gives the unbiased derivative estimates for many performance measures (Glasserman 1991). In our model, however, the system state can be changed by such an order change of event occurrences. For example, when only one customer is in the system and the system state is $(1, y)$ with $y \geq 2$, the possible events to occur are an arrival and the departure of customer in service. Suppose that the departure occurs first and then the next customer arrives in the nominal path and that, due to the perturbations, these departure and arrival change their order in the perturbed path. In this case, we can see that, just after the occurrences of these two events, the server is in a vacation in the nominal path, whereas the server is serving the arriving customer (with probability $1 - p$) in the perturbed path. As another example, when the server is in a vacation and α customers are waiting in the queue, the possible events are an arrival and the vacation termination. Suppose that the vacation termination occurs first and then a customer arrives in the nominal path and the order changes in the perturbed path. Just after the occurrences of these two events, the server is in vacation in the nominal path, whereas in service in the perturbed path. Thus, the queueing system with multiple vacations does not satisfy the commuting condition and generally the IPA gives biased estimators.

We apply the SPA technique according to (Fu and Hu 1997). We only consider the right-hand derivative, that is, $\Delta\theta > 0$ in (2). Under Assumption 2, each service time in the perturbed path is larger than or equal to the corresponding one in the nominal path, and an arrival event can change the order with the past departures or vacation terminations. Let $\mathcal{B}_n(\Delta\theta) \in \mathcal{F}_A \otimes \mathcal{F}_V$ be the set of samples in which a sample path does not experience the order change of event occurrences between the kth arrival and other events (departure or vacation termination) for $k = 1, \ldots, n-1$, but does between the nth arrival and other events. Let also $N_A(t)$ be the number of points of $\{T_n\}$ observed in $[0, t]$, and let $D_n(\theta)$ denote the departure epoch of the nth customer. Then, the numerator of the right-hand side of (2) is rewritten as

$$\mathrm{E}\big[\Delta_\theta W_n(\theta)\big] = \mathrm{E}\Big[\Delta_\theta W_n(\theta)\, 1_{\overline{\mathcal{B}}^c_{N_A(D_n)}(\Delta\theta)}\Big] + \mathrm{E}\left[\sum_{k=1}^{N_A(D_n)} \Delta_\theta W_n(\theta)\, 1_{\mathcal{B}_k(\Delta\theta)}\right], \qquad (4)$$

where $\Delta_\theta W_n(\theta) = W_n(\theta + \Delta\theta) - W_n(\theta)$, $\overline{\mathcal{B}}_n(\Delta\theta) = \bigcup_{k=1}^{n} \mathcal{B}_k(\Delta\theta)$, and $\mathcal{B}^c$ is the complement of $\mathcal{B}\ (\in \mathcal{F}_A \otimes \mathcal{F}_V)$. In (4) and hereafter, we suppress the dependence on θ of

$D_n(\theta)$ for the simplicity of notation. For the first term of the right-hand side, we have the following:

Lemma 1 *Under Assumption 1 and some additional assumptions,*

$$\lim_{\Delta\theta\to 0} \mathrm{E}\left[\frac{\Delta_\theta W_n(\theta)}{\Delta\theta} 1_{\overline{\mathcal{B}}^c_{N_A(D_n)}(\Delta\theta)}\right] = \mathrm{E}\left[\frac{\mathrm{d}W_n(\theta)}{\mathrm{d}\theta}\right], \tag{5}$$

where

$$\frac{\mathrm{d}W_n(\theta)}{\mathrm{d}\theta} = \sum_{i=1}^{n} \frac{\mathrm{d}\sigma_i(\theta)}{\mathrm{d}\theta}, \tag{6}$$

and each $\mathrm{d}\sigma_i(\theta)/\mathrm{d}\theta$ *is calculated from (3).*

Proof: See (p. 84 of Fu and Hu 1997).

Next, consider the second term of (4) and further impose the following assumption:

Assumption 3 *The vacation length distribution G is absolutely continuous with the continuous density ∂G.*

Let $\big(e_k(\theta), l_k(\theta)\big)$ be the pair on $\{d, v\} \times \mathbb{N}$ specifying the type and index of the event (other than arrival) which occurred just before T_k. For example, $\big(e_k(\theta), l_k(\theta)\big) = (d, l)$ means that the event (other than arrival) just before T_k is the departure of the lth customer. Let $\zeta_k(\theta)$ be the time length to T_k from the time at which $\big(e_k(\theta), l_k(\theta)\big)$ became active, and let also $\eta_k(\theta)$ denote the lifetime of the event which became active at the occurrence of $\big(e_k(\theta), l_k(\theta)\big)$. Then, we have the following:

Lemma 2 *Under Assumptions 1–3 and other additional assumptions,*

$$\begin{aligned}
&\lim_{\Delta\theta\to 0} \mathrm{E}\left[\sum_{k=1}^{N_A(D_n)} \frac{\Delta_\theta W_n(\theta)}{\Delta\theta} 1_{\mathcal{B}_k(\Delta\theta)}\right] \\
&= -\mathrm{E}\Bigg[\sum_{k=2}^{N_A(D_n)} \Bigg\{\Delta W_n\big(\theta; \sigma_{l_k} = \zeta_k(\theta)\big) \\
&\qquad \times \frac{\partial_x F\big(\zeta_k(\theta), \theta\big)}{F\big(\zeta_k(\theta), \theta\big) - F\big((\zeta_k(\theta) - \eta_k(\theta))^+, \theta\big)} \\
&\qquad \times \left(\frac{\mathrm{d}\zeta_k(\theta)}{\mathrm{d}\theta} - \frac{\mathrm{d}\sigma_{l_k}(\theta)}{\mathrm{d}\theta}\Big|_{\zeta_k(\theta)}\right) 1_{\{e_k(\theta)=d\}} \\
&\quad + \Delta W_n\big(\theta; V_{l_k} = \zeta_k(\theta)\big) \\
&\qquad \times \frac{\partial G\big(\zeta_k(\theta)\big)}{G\big(\zeta_k(\theta)\big) - G\big((\zeta_k(\theta) - \eta_k(\theta))^+\big)} \\
&\qquad \times \frac{\mathrm{d}\zeta_k(\theta)}{\mathrm{d}\theta} 1_{\{e_k(\theta)=v\}}\Bigg\}\Bigg],
\end{aligned} \tag{7}$$

where $\mathrm{d}\sigma_i(\theta)/\mathrm{d}\theta\big|_\zeta$ *represents the value of* $\mathrm{d}\sigma_i(\theta)/\mathrm{d}\theta$ *at* $\sigma_i(\theta) = \zeta$, *and*

$$\frac{\mathrm{d}\zeta_k(\theta)}{\mathrm{d}\theta} = -\sum_{i\ge 1} \frac{\mathrm{d}\sigma_i(\theta)}{\mathrm{d}\theta} 1_{(0,T_k)}(D_i(\theta)). \tag{8}$$

Furthermore, for $Z = \sigma_{l_k}$ *or* V_{l_k},

$$\Delta W_n\big(\theta; Z = \zeta\big) = W_n\big(\theta; Z = \zeta+\big) - W_n\big(\theta; Z = \zeta-\big)$$

represents the difference of two $W_n(\theta)$*'s on the nominal paths with the additional condition of* $Z = \zeta+$ *and* $Z = \zeta-$, *respectively.*

Proof: See (pp. 85–92 of Fu and Hu 1997).

From Lemmas 1 and 2, we have the following:

Theorem 1 *Under the same assumptions as those in Lemmas 1 and 2, the unbiased SPA estimator of (2) is given by the sum of the quantities inside the expectations of right-hand side of (5) and (7).*

4 IMPLEMENTATION OF THE ESTIMATOR

4.1 Perturbation Propagations

In the implementation of the SPA estimator in the previous section, a major difficulty is that the effects of perturbations are accumulated continuously without any resetting, that is, under Assumption 2, the absolute values of the derivatives $\mathrm{d}W_n(\theta)/\mathrm{d}\theta$ in (6) and $\mathrm{d}\zeta_k(\theta)/\mathrm{d}\theta$ in (8) increase along with the value of n unless $\mathrm{d}\sigma_i(\theta)/\mathrm{d}\theta = 0$ for every $i \in \mathbb{N}$. This contrasts with the case of queueing systems without vacations where the perturbations are accumulated only during each busy period. Here, we tackle this problem and show that it is possible to construct a point sequence on the observed sample path such that the service time derivatives are accumulated only between the two adjacent points. We use the following lemma presented in (Asmussen 1987: Lemma 2.4, Chap. VI) and also in (Lindvall 1992; Lemma 5.1, Chap. III):

Lemma 3 *Consider a zero-delayed renewal process $\{S_n\}_{n\in\mathbb{N}}$ and suppose that the inter-renewal distribution function G has a continuous density. Then, there exist constants $C \in [0, \infty)$ and $b \in (0, \infty)$ such that the distributions of forward recurrence times $\{B(t)\}$ with $t \ge C$ have a common uniform component on $(0, b)$, that is, there exists a constant $c \in (0, 1)$ such that, for all $t \ge C$,*

$$\mathrm{P}\big(B(t) \in (u, v]\big) \ge c\,\frac{v-u}{b}, \quad 0 < u < v < b. \tag{9}$$

Note that the assumption in the above lemma holds under Assumption 3. In (Asmussen 1987) and (Lindvall 1992), the above lemma is used to show the coupling of renewal processes, while we use this to construct an alternative sample path of the renewal process which corresponds to the sum of vacation lengths on the perturbed path. The constructed path is identical in probability law to the observed one, but is able to absorb the accumulated perturbations.

Let $\{\tilde{S}_n(\theta)\}_{n\in\mathbb{N}}$ be the sequence of epochs at which the vacations are terminated. This sequence is constructed from a given $(\omega_A, \omega_V) \in \Omega_A \times \Omega_V$ for each $\theta \in \Theta$.

Theorem 2 *Under Assumption 3, we can construct a point process $\{R_n(\theta)\}_{n\in\mathbb{N}}$, which is a subsequence of $\{\tilde{S}_n(\theta)\}_{n\in\mathbb{N}}$, such that,*

$$\sum_{i\geq 1} \frac{\mathrm{d}\sigma_i(\theta)}{\mathrm{d}\theta} 1_{(0,t]}(D_i) \stackrel{d}{=} \sum_{i\geq 1} \frac{\mathrm{d}\sigma_i(\theta)}{\mathrm{d}\theta} 1_{(R_-(t,\theta),t]}(D_i), \tag{10}$$

where $\stackrel{d}{=}$ means equality in probability distribution and $R_-(t,\theta) = \max(R_n(\theta) \leq t)$.

Proof: For given $\omega_A = \{(T_n, U_n^d, U_n^r)\}_{n\in\mathbb{N}} \in \Omega_A$ and $\omega_V = \{S_n\}_{n\in\mathbb{N}} \in \Omega_V$, we construct below a sample $\omega_V(\Delta\theta) = \{S_n(\Delta\theta)\}_{n\in\mathbb{N}} \in \Omega_V$ which is identical in probability law to ω_V and satisfies $\omega_V(\Delta\theta) \to \omega_V$ as $\Delta\theta \to 0$ (in the sense of conventional metric on $\mathbb{R}^\infty$, e.g. $d(x,y) = \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|/(1+|x_i - y_i|)$), and moreover satisfies that, for n in a subset of $\mathbb{N}$,

$$\tilde{S}_n(\theta, \omega_A, \omega_V) = \tilde{S}_n(\theta + \Delta\theta, \omega_A, \omega_V(\Delta\theta)). \tag{11}$$

First, let $S_1(\Delta\theta) = S_1$. For $n \geq 2$, $S_n(\Delta\theta)$ is placed recursively as follows: Let $t_0 = 0$ and for $k = 1, 2, \ldots$, we choose a t_k appropriately satisfying

$$S_{N_V(t_k)+1} - t_k < b \quad \text{and} \quad t_k - S_{N_V(t_{k-1})+1} \geq C, \tag{12}$$

where b and C are the same as in Lemma 3, and $N_V(t)$ denotes the number of points of $\{S_n\}_{n\in\mathbb{N}}$ in $(0,t]$. For all n such that $S_n \in (S_{N_V(t_{k-1})+1}, t_k)$, we set $S_n(\Delta\theta) = S_{n-1}(\Delta\theta) + V_{n-1}$. Now, we can regard that the position of $S_{N_V(t_k)+1}$ is determined at t_k by the overshot distribution of $B(t)$. Furthermore, under the condition of (12), we can consider from Lemma 3 as if there exist random variables ξ_k, X_k and Y_k such that $\mathrm{P}(\xi_k = 1) = 1 - \mathrm{P}(\xi_k = 0) = c$, X_k is uniformly distributed on $(0,b)$ and $B(t_k) = \xi_k X_k + (1-\xi_k) Y_k$ has the overshot distribution at $t_k - S_{N_V(t_k-1)+1}$ $(\geq C)$, where c is in (9), and that ξ_k, X_k and Y_k are taken independently of $\{(\xi_i, X_i, Y_i)\}_{i\leq k-1}$. Since X_k is uniformly distributed on $(0,b)$, $B(t_k, \Delta\theta) = \xi_k X_k(\Delta\theta) + (1-\xi_k) Y_k$ with

$$X_k(\Delta\theta) = X_k - \sum_{i\geq 1} \Delta\sigma_i(\theta) 1_{(\tilde{S}_{N_V(t_{k-1})+1}, \tilde{S}_{N_V(t_k)+1})}(D_i(\theta)),$$

is identical in probability law to $B(t_k)$ for a sufficiently small $|\Delta\theta|$. Using this, we put $S_{N_V(t_k)+1}(\Delta\theta)$ by

$$S_{N_V(t_k)+1}(\Delta\theta) = S_{N_V(t_k)}(\Delta\theta) + \left(t_k - S_{N_V(t_k)}\right) + B(t_k, \Delta\theta).$$

The renewal process $\omega_V(\Delta\theta) = \{S_n(\Delta\theta)\}_{n\in\mathbb{N}}$ obtained through the above procedure is identical in probability law to $\{S_n\}_{n\in\mathbb{N}}$, and if $\xi_k = 1$, we have (11) with $n = N_V(t_k)+1$. Hence, we have (10) for $\{R_n(\theta)\} = \{\tilde{S}_{N_V(t_k)+1}(\theta) : \xi_k = 1\}$. Note that $\{R_n(\theta)\}$ is the infinite sequence if $\{\tilde{S}_n(\theta)\}$ is so.

In the practical implementation of the above resetting points $\{R_n(\theta)\}_{n\in\mathbb{N}}$, a problem is the computation of C and c in Lemma 3 (see the proof of Lemma 2.4 in Chap. VI of Asmussen 1987), while the value of b can be chosen relatively easily. However, if we can bear the long resetting intervals, we can take a sufficiently small C by choosing a small value of c. Hence, we can use a simple procedure such that the accumulated perturbation is reset with probability c when an arriving customer finds the system in vacation.

4.2 Coupling of Nominal and Perturbed Paths

Another problem in the implementation of the SPA estimator arises in the calculations of $\Delta W_n(\theta; \sigma_{l_k} = \zeta_k(\theta))$ and $\Delta W_n(\theta; V_{l_k} = \zeta_k(\theta))$ in (7), which generally need additional sample subpaths, that is, the nominal paths with additional conditions of $\sigma_{l_k} = \zeta_k\pm$ or $V_{l_k} = \zeta_k\pm$ (Fu and Hu 1997). We refer to these paths with the modifications as the kth degenerated nominal and perturbed paths, respectively.

Here we show that, for the case of $\alpha = 0$ and $\beta = +\infty$ (pure Bernoulli policy), these degenerated paths can be constructed from the observed sample path (cf. Miyoshi and Hasegawa, 1994). First, note that if the pair of two adjacent occurrences of a departure and an arrival (resp. a vacation termination and an arrival) satisfies the commuting condition (in the local sense), that is, if the system state remains the same after the event order change, then the term $\Delta W_n(\theta; \sigma_{l_k} = \zeta_k)$ (resp. $\Delta W_n(\theta; V_{l_k} = \zeta_k)$) equals to zero and the corresponding summand in (7) vanishes. We call the pair of event occurrences critical if it violates the commuting condition. Checking the event mappings in (1), we know that the critical event pairs are found only in the following situations:

$$\begin{cases} a \circ d((1,y),u) = a(0,0) = (1,0); \\ d \circ a((1,y),u) = d((2,y),u) = \begin{cases} (1, y-1), & u > p; \\ (1,0), & u \leq p, \end{cases} \end{cases}$$

for $y \geq 2$, and

$$\begin{cases} a \circ v(0,0) = a(0,0) = (1,0); \\ r \circ a(0,0) = v(1,0) = (1,+\infty). \end{cases}$$

Since a vacation is postponed due to the event order change in the perturbed path and the arrival sequence is identical in both the degenerated paths, we can construct the kth perturbed path to keep the condition that the total sum of the vacation lengths at any epoch $t \geq T_k$ in the kth nominal path is not smaller than that in the perturbed path. In other words, the number of customers served by $t \geq T_k$ in the kth perturbed path is greater than or equal to that in the kth nominal path.

The construction of the kth perturbed path proceeds as follows: The above condition clearly holds until the time at which the first customer is served after T_k in the kth degenerated nominal path. If there is a vacation after a service in the nominal path, then we put the vacation with the same length after the corresponding service in the perturbed path. If there is no vacation after a service in the nominal path but the server is to take a vacation after the corresponding service in the perturbed path (due to being empty), then we insert there a vacation length which is observed in the kth nominal path just after T_k but postponed in the perturbed path due to the event order change. Then, the total sum of the vacation lengths becomes identical in both the degenerated paths and we can get the coupling of these paths.

5 EXAMPLES AND SIMULATION EXPERIMENTS

This section contains the results of simulation experiments for some examples. The estimates from the proposed method are compared with those from some different methods: If the decomposition formula holds (see Doshi 1986 and Miyazawa 1994), we can obtain the unbiased IPA estimates from the corresponding system without vacations. Furthermore, if the vacation lengths are exponentially distributed, we have two different unbiased SPA estimates based on memoryless property: One is calculated from the alternative model, where there is no vacation even though the system becomes empty but, when a customer arrives to the empty system, the server takes a set-up time with the exponential distribution before his service. The other is from the original model where the perturbation accumulation is reset when a customer arrives to the empty system due to the memoryless property. These PA estimates are also compared with the symmetric finite difference estimates. In addition, the analytical results are used for comparison if available, where all analytical values are computed from the differential of formulas in (Takagi 1991). In the tables below, the exact analytical values and the estimates are referred to as "Exact," "Decomposed," "Set-up," "Memoryless," and "Finite Difference," respectively. The estimates through the proposed method are referred to as "Proposed."

The length of each sample path is in total $1,000,000$ customers served. The table entries other than analytical values are given with 95% confidence intervals, taken from 30 independent replications. Parameter θ, which is the mean service time, is fixed at 1.0. The arrival rate, the mean vacation length and Bernoulli scheduling probability are also fixed at $\lambda = 0.5$, $\mathrm{E}[V_1] = 0.6$ and $p = 1/3$, respectively.

Example 1 ($M/M/1$ with exponential vacation length distribution): The case of $M/M/1$ with exponential vacation length distribution was simulated. The proposed estimates are calculated with the values of $b = \mathrm{E}[V_1]/2$ and $c = 1/(4\,\mathrm{e})$ in (9). The experimental results are given in Table 1 with the analytical values. In this case, the decomposition formula holds, and we can use the memoryless property of vacation lengths. Hence, the proposed method seems to show the worst performance. However, note that the on-line calculation of estimates from the original model with vacations is possible only in "Memoryless" and "Proposed."

Table 1: Estimates for $M/M/1$ with Exponential Vacation Length Distribution

	$\mathrm{E}[W(\theta)]$	$\mathrm{dE}[W(\theta)]/\mathrm{d}\theta$
Exact	3.250 —	5.8125 —
Decomposed (IPA)	—	5.815 ± 0.015
Set-up (SPA)	3.251 ± 0.003	5.822 ± 0.015
Memoryless (SPA)	3.251 ± 0.002	5.827 ± 0.011
Proposed (SPA)	3.251 ± 0.002	5.593 ± 0.283
Finite Difference	—	5.742 ± 0.174

Example 2 ($E_2/E_2/1$ with uniform vacation length ditribution): The case of $GI/GI/1$ with uniform vacation length distribution was simulated, where the interarrival and service times are both of the second Erlang distributions. The proposed estimates are calculated with the values of $b = \mathrm{E}[V_1]$ and $c = 0.06$ in (9). The exact analysis is not available and the estimation results are given in Table 2. In this case, the decomposition formula holds again, but the memoryless property of vacation lengths is not available. Thus, the on-line calculation of unbiased estimates is possible only in the proposed method.

6 CONCLUDING REMARKS

In this work, we have considered the implementation of PA estimator for queueing systems with multiple vacations, where a difficulty lies in that the perturbations are propagated continuously without resetting. We have proposed the construction of point sequence on the observed sample path

Table 2: Estimates for $E_2/E_2/1$ with Uniform Vacation Length Distribution

	E$[W(\theta)]$	dE$[W(\theta)]/\mathrm{d}\theta$
Decomposed (IPA)	—	3.256 ± 0.005
Proposed (SPA)	2.097 ± 0.001	3.247 ± 0.119
Finite Difference	—	3.296 ± 0.076

such that the perturbations are accumulated only between the two adjacent points. The proposed method is fairly general, but as in the experimental results, further improvements may be needed as well as the exact calculation of the values of C and c in Lemma 3.

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