Proceedings of the 1998 Winter Simulation Conference
D.J. Medeiros, E.F. Watson, J.S. Carson and M.S. Manivannan, eds.

ACCELERATED SIMULATION FOR PRICING ASIAN OPTIONS

Felisa J. Vázquez-Abad

Department of Computer Science and Operations Research
University of Montreal
Montréal, Québec H3C 3J7, CANADA

Daniel Dufresne

Department of Economics
University of Melbourne
Melbourne, AUSTRALIA

ABSTRACT

When pricing options via Monte Carlo simulations, precision can be improved either by performing longer simulations, or by reducing the variance of the estimators. In this paper, two methods for variance reduction are combined, the control variable and the change of measure (or likelihood) methods. We specifically consider Asian options, and show that a change of measure can very significantly improve the precision when the option is deeply out of the money, which is the harder estimation problem. We also show that the simulation method itself can be used to find the best change of measure. This is done by incorporating an updating rule, based on an estimate of the gradient of the variance. The paper includes simulation results.

1 INTRODUCTION

Options are financial instruments traded on organized markets as well as over the counter. Options are also embedded in other financial instruments, such as bonds; for instance, the possibility for the seller of a bond to buy it back at some stated price is an option. While there is a great variety of options, only a few explicit valuation formulas are known. For a description of option pricing the reader is referred to the book by Hull (1993) or to Boyle et al. (1998).

Monte Carlo simulation is one of the preferred pricing tools when no explicit formula is available. In this paper, we introduce an apparently new method for improving the efficiency of Monte Carlo simulation, based on a change of measure; the method may also be used in conjunction with a control variable. We focus on Asian (or average) options, but it will be seen that the methodology can be applied to other types of options. Further extensions of the method will be examined in subsequent publications.

After a brief description of the probem of valuing Asian options in Section 2, Section 3 shows how control variables and change of measure may be combined. Then Section 4 introduces Infinitesimal Perturbation Analysis (IPA) in order to find the optimal change of measure at the same time the simulation is performed.

2 PRICING ASIAN OPTIONS

There are two assets: the risk-free asset e^{rt} and the risky asset S, with

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

for $0 \le t \le T$, where (1) r, $S_0 > 0$, μ and $\sigma > 0$ are constants and (2) W is standard Brownian motion. Let $\{\mathcal{F}_t; 0 \le t \le T\}$ be the augmented filtration generated by W. A European contingent claim is a non-negative $\mathcal{F}_T$-measurable random variable, such that $\mathsf{E}^* P_T$ is finite. Here the expectation E^* is with respect to the equivalent measure P^* under which discounted asset prices $\{e^{-rt} S_t\}$ form a martingale (the so-called risk-neutral measure, see Harrison and Pliska 1981). This model is arbitrage-free and, moreover, complete, meaning that (1) no strategies exist that permit risk-less profits, (2) any European type contingent claim with a time T maturity can be exactly replicated, and (3) the price at time t of such a contingent claim, with payoff P_T at time T is

$$P_t = e^{-r(T-t)} \mathsf{E}^*(P_T \,|\, \mathcal{F}_t), \quad 0 \le t \le T,$$

In the sequel no more reference will be made to μ, as it has effectively disappeared from the problem, and P^* (resp. E^*) will be denoted P (resp. E).

Asian (or average) options have payoffs which depend on the average value of the risky asset at some specified time points in $[0, T]$. In the literature, two cases have been considered: (1) continuous averaging, see for example Geman and Yor 1993; (2) discrete-averaging at equally spaced time points (Turnbull and Wakeman 1991). In either case no explicit formula for the distribution of the average exists yet, though Geman and Yor derived the Laplace transform of call option prices (with respect to time, not exercise price), which can be inverted numerically (see Fu, Madan and Wang 1997). We describe below how to use simulation to produce discrete averages; if the number of averaging points is large enough, we obtain an approximation for the value of options on a continuous average.

Suppose the payoff of the option is a function of

$$A = \frac{1}{N}\sum_{i=1}^{N} S_{ih} \qquad \text{where} \qquad h = \frac{T}{N}. \tag{1}$$

Consider a call option on the indicated average, with exercise price K, that is to say, the payoff at maturity is $P_T = (A-K)_+$ (other types of options on the average, such as puts, can be treated in the same way). From the previous discussion, the problem reduces to the estimation of

$$\theta = \mathsf{E}\left[(A-K)_+\right]. \tag{2}$$

We will suppose that we wish to value the option at time 0. For dates between 0 and T, it is straightforward to transform the problem so it becomes equivalent to the valuation at time 0 (see Geman and Yor 1993 for details). From our assumptions, and the fact that $\{e^{-rt}S_t\}$ is a martingale, the log-returns $\{X_i\}$, with

$$X_i = \log\left(\frac{S_{ih}}{S_{(i-1)h}}\right),$$

are independent with common $\mathcal{N}((r-\sigma^2/2)h, \sigma^2 h)$ distribution.

From now on r is a given risk-free interest rate, and the problem considered is the estimation of $\mathsf{E}(A-K)$ for that particular value of r. However, our method requires that the risk-free rate be varied in the simulations. This is why we use the following notation, where u represents any risk-free rate:

$$\begin{aligned}
X_i^u &= (u-\sigma^2/2)h + \sigma\sqrt{h}Z_i, \quad 1 \le i \le N \\
\tilde{X}^u &= (X_1^u, \ldots, X_N^u) \\
\tilde{Z} &= (Z_1, \ldots, Z_N) \\
B_i &= Z_1 + \cdots + Z_i, \quad 1 \le i \le N, \quad B_0 = 0 \\
S_{ih}^u &= S_{(i-1)h}^u \exp(X_i^u), \quad 1 \le i \le N \qquad (3) \\
A_u &= a(\tilde{X}^u) = \frac{1}{N}\sum_{i=1}^{N} S_{ih}^u \qquad (4) \\
r_\sigma &= r - \sigma^2/2, \quad u_\sigma = u - \sigma^2/2,
\end{aligned}$$

where the $\{Z_i\}$ are independent with common distribution $\mathcal{N}(0,1)$.

The "naive" Monte Carlo estimation of the option is peformed by generating independent random variables $\{Z_i\} \sim \mathcal{N}(0,1)$ to obtain the sample mean for $(A_r - K)_+$. This estimator is unbiased, and the Central Limit Theorem yields confidence intervals for its precision. However, it is well known now (Broadie and Glasserman 1996; Boyle, Broadie, and Glasserman 1997; Fu, Madan and Wang 1997) that accuracy of Monte Carlo simulations can be improved by using control variables or changes of measure.

3 CONTROL VARIABLES PLUS CHANGE OF MEASURE

3.1 The Estimators

The method of the control variable (see Bratley, Fox, and Schrage 1997; Ross 1997) has been applied to the pricing of Asian options by Boyle, Broadie, and Glasserman (1997) among others, using the geometric average as control variable. Let

$$G_u = \left(\prod_{i=1}^{N} S_{ih}^u\right)^{\frac{1}{N}}, \tag{5}$$

denote the geometric average, and let

$$Y_1^u = (A_u - K)_+, \quad Y_2^u = (G_u - K)_+.$$

The controlled estimator

$$D_1 = Y_1^r + \alpha(\mathsf{E}\,Y_2^r - Y_2^r) \tag{6}$$

is an unbiased estimator of θ for any constant α. In particular, as shown in Ross (1997), the variance $\mathsf{Var}\,\hat{\theta}$ is minimized when

$$\alpha = \frac{\mathsf{Cov}(Y_1, Y_2)}{\mathsf{Var}\,Y_2}.$$

It is well known (Hull 1993) that

$$\mathsf{E}\left[(G_u - K)_+\right] = e^{c+s^2/2}\Phi(d_1) - K\Phi(d_2) \tag{7}$$

where:

$$\begin{aligned}
c &= \log S_0 + mh\frac{(N+1)}{2} \\
s^2 &= \sigma^2 h \frac{N+1}{(2N+1)(6N)} \\
d_1 &= \frac{c + s^2 - \log K}{s}, \quad d_2 = d_1 - s.
\end{aligned}$$

Straightforward calculations also lead to

$$\begin{aligned}
\mathsf{Var}[(G_u - K)_+] &= e^{2(c+s^2)}\Phi(d_3) - 2K\mathsf{E}[Y_2^u] \\
&- K^2\Phi(d_2) - (\mathsf{E}[Y_2^u])^2, \qquad (8)
\end{aligned}$$

where $d_3 = d_1 + s$. Since the covariance between Y_1^u and Y_2^u is unknown, the optimal value of α is generally replaced by its usual estimation $\hat{\alpha}$, also obtained from Monte Carlo simulation. In Boyle, Broadie, and Glasserman (1997), Fu, Madan, and Wang (1997), Lemieux (1996), a constant coefficient $\alpha = 1$ was used. In this paper, we use the estimated optimal value $\hat{\alpha}$, as explained in Bratley, Fox, and Schrage (1987) and in Ross (1997).

Another approach that can sometimes improve the precision of Monte Carlo simulation is the change of measure method, or "likelihood ratio method". In this particular case, a well-known formula says that for any measurable $f : \mathbb{R}_N \to \mathbb{R}$ and any $v \in \mathbb{R}$

$$\mathsf{E} f(\tilde{Z}) = \mathsf{E}\, e^{-\frac{Nv^2}{2} - vB_N} f(\tilde{Z} + v).$$

(if one side of the equation exists, then the other exists as well and the two are equal). For the valuation of Asian options, we consider $f(\tilde{Z}) = a(\tilde{X}^u)$, $v = (u-r)\sqrt{h}/\sigma$, and define

$$\begin{aligned} L_u &= \exp\left\{ -\frac{N}{2}\left[\frac{(u-r)\sqrt{h}}{\sigma}\right]^2 - \frac{(u-r)\sqrt{h}}{\sigma} B_N \right\} \\ &= \exp\left\{ \frac{u_\sigma^2 - r_\sigma^2}{2\sigma^2} T - \frac{u-r}{\sigma^2} \sum_{i=1}^{N} X_i^r \right\} \\ &= e^{\frac{u_\sigma^2 - r_\sigma^2}{2\sigma^2} T} \left(\frac{S_N^r}{S_0}\right)^{-\frac{u-r}{\sigma^2}}. \end{aligned}$$

We thus have

$$\begin{aligned} \mathsf{E}\left[(A_r - K)_+\right] &= \mathsf{E}\left[(a(\tilde{X}^r) - K)_+\right] \\ &= \mathsf{E}\left[L_u(a(\tilde{X}^r + (u-r)h) - K)_+\right] \\ &= \mathsf{E}\left[L_u(A_u - K)_+\right]. \end{aligned}$$

Hence, the likelihood ratio L_u changes the risk-neutral rate from u to r.

A call option is "out of the money" (at time 0) if $S_0 < K$; the more an Asian option is out of the money, the larger $\mathsf{P}[A_u < K]$. Suppose the option to be valued is out of the money. Here is an intuitive interpretation of the advantages of the likelihood ratio method. The larger the drift u, the larger the probability that the option ends up in the money at maturity ($S_T^u > K$), and the smaller the number of samples required to estimate the value of $\mathsf{E}\left[L_u(A_u - K)_+\right]$. Changing u changes the way the values of $(A_u - K)_+$ are weighted, so that the expectation remains the same; this is achieved by multiplying by L_u. The variance of the estimator is

$$\mathsf{Var}[L_u(A-K)_+] = \mathsf{E}\, L_u^2 (A-K)_+^2 - \theta^2, \qquad (9)$$

which varies with u. The estimation of θ becomes easier if this variance is reduced. If we can choose u such that $\mathsf{P}(L_u < 1, A_u > K)$ is large, then we may hope that the resulting variance does not hinder the gain in computational effort (see L'Ecuyer 1994).The typical situation observed is that the variance of the estimator $L_u(A_u - K)_+$ has a minimum for some value of u (see below). Unfortunately we cannot solve for the optimal u analytically.

In order to apply the change of measure method, we define the estimator

$$D_2 = L_u(A_u - K)_+. \qquad (10)$$

We can add a control variable to this estimator, which yields

$$D_3 = L_u Y_1^u + \alpha(\mathsf{E} Y_2^u - Y_2^u) \qquad (11)$$

where now $Y_2^r = (G_r - K)_+$ is estimated from (5) and (3) in parallel to Y_1^u. This means that we use common random numbers (CRN) to try to increase the correlation between Y_1^u and Y_2^r. The coefficient $\hat{\alpha}$ that we use is the estimated optimal one.

Finally, we consider applying a change of measure to the controlled estimator as well, which yields:

$$D_4 = L_u Y_1^u + \alpha(\mathsf{E} Y_2^r - L_u Y_2^u) \qquad (12)$$

where, again, the coefficient α is estimated for the optimal variance reduction. Since this quantity is not available analytically, we estimated it.

3.2 Simulation Results

We show in Table 1 the results of experiments using $r = 0.05, \sigma^2 = 0.2, S_0 = 50, T = 1.0$ and $M = 10000$ replications. The efficiency of the estimators is defined as in L"Ecuyer (1994), namely the inverse of the product of the CPU time and the variance of the estimator. Since our simulations are rather short, all of the experiments reported in Table 1 took the same 5 seconds of CPU time to run. We show the estimators in order of decreasing variance (in all but one case: when $K = 30$ and so the option is deep in the money). At the bottom, we have included the estimated value of α that minimizes the variance of D_4.

Remark. Longer simulations could show differences in CPU time, the naive being of course the fastest method, followed by D_2, D_1 and then D_3 and D_4, which have the same computational effort.

We obtained the same pattern of results for other parameter values, namely that the estimators D_1 and D_4 do better than D_2 and D_3, and that D_4 appears to be better than D_3. While both D_3 and D_4 work better as S_0/K decreases, D_4 is consistently better than the rest of the estimators.

Table 1: Comparison of the Methods

$r = 0.05, \sigma^2 = 0.2, S_0 = 50, T = 1.0$ and $M = 10,000$					
Estimators					
Method	K= 30	K= 45	K=50	K= 55	K = 75
Naive	20.46 $\pm$ 0.26	8.45 $\pm$ 0.216	5.80 $\pm$ 0.189	3.83 $\pm$ 0.160	0.630 $\pm$ 0.068
D_2	20.34 $\pm$ 0.137	8.32 $\pm$ 0.115	5.66 $\pm$ 0.096	3.74 $\pm$ 0.075	0.583 $\pm$ 0.020
D_1	20.31 $\pm$ 0.016	8.28 $\pm$ 0.013	5.64 $\pm$ 0.012	3.72 $\pm$ 0.011	0.585 $\pm$ 0.010
D_3	20.31 $\pm$ 0.014	8.28 $\pm$ 0.011	5.64 $\pm$ 0.010	3.71 $\pm$ 0.010	0.583 $\pm$ 0.009
D_4	20.31 $\pm$ 0.014	8.27 $\pm$ 0.009	5.62 $\pm$ 0.008	3.70 $\pm$ 0.006	0.573 $\pm$ 0.003
Variance					
Method	K= 30	K= 45	K=50	K= 55	K = 75
Naive	176.09	121.70	92.58	66.28	12.04
D_2	49.04	34.59	23.76	14.95	1.07
D_1	0.64	0.42	0.36	0.33	0.25
D_3	0.48	0.28	0.25	0.24	0.23
D_4	0.53	0.207	0.150	0.095	0.028
$\hat{\alpha}^*$	0.998	1.05	1.07	1.10	1.20

To simulate D_3 and D_4 we proceed as follows. The initial values are $B_i = 0, S_0, A_u = 0, G_u = 1$.

Algorithm 1: Simulation at u.

1. For $0 < i \leq N$ do:
   (a) Generate $Z_i \sim \mathcal{N}(0,1)$, set $B_i = B_{i-1} + Z_i$,
   (b) Define $X_i^u = u_\sigma h + \sqrt{\sigma^2 h}\, Z_i$,
   (c) Set $S_{ih}^u = S_{(i-1)h}^u e^{X_i^u}$, $A_u = A_u + S_{ih}^u$, $G_u = G_u * S_i^u$.
2. Calculate $A_u = A_u/N, G_u = \sqrt[N]{G_u}$
3. $L_u = \exp\{-\frac{(u-r)^2}{2\sigma^2}T - \frac{(u_- r)}{\sigma^2}\sqrt{\sigma^2 h} B_N\}$

At the end of this loop, a single trajectory of the process with drift u has been simulated, and D_2, D_3 and D_4 can be computed. Then this simulation is repeated M times to obtain the estimated α and the corresponding confidence interval as usual.

Remark. In our simulations, we have used the accelerated Box-Muller method (see Ross 1997 for the details) where trigonometric functions are not used. At each iteration $i \leq N/2$ we use two independent seeds for our uniform variates and produce two independent samples of $\mathcal{N}(0,1)$ variables Z_{2i-1}, Z_{2i}.

In order to estimate the optimal value of u, CRNs were used: in steps 1(b) and 3 of Algorithm 1, several trajectories were evluated in parallel, each corresponding to a different value of u (functional estimation). Figure 1, Figure 2, and Figure 3 show the estimated variance of D_2, D_3 and D_4 using functional estimation with CRN as described, for 10 values of u. The solid line is for $K = 30, 45$, the short dashed line is for $K = 50$, long dashes are for $K = 55$ and the longer dashes are for $K = 75$. For Figure 2 we used $M = 5,000$ replications and it took 9 seconds (for each value of K) with 10 values of u in the range shown. To produce Figure 1 and Figure 3 we used $M = 10,000$, which took 20 to 28 seconds, for each value of K.

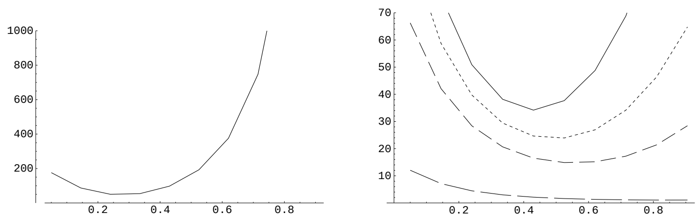

Figure 1: Variance of D_2 (at left is the case $K = 30$)

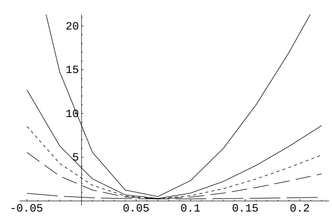

Figure 2: Variance of D_3

To summarize, the estimated optimal values of u used in Table 1 are shown in Table 2, and the best estimator is the one that uses the change of measure in both the

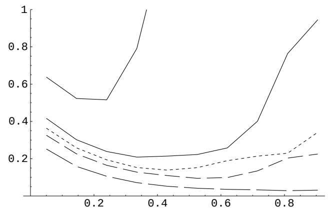

Figure 3: Variance of D_4

Arithmetic Asian Option as well as in the control variable, and the corresponding gain in variance reduction can be considerably high. The problem with this estimator is that in order to determine the optimal value of u, several preliminary simulations must be performed for functional estimation. The rest of the paper deals with this problem.

Table 2: Optimal Values u^* for the Change of Measure

	K=30	K=45	K=50	K=55	K=75
D_2	0.25	0.40	0.50	0.60	0.80
D_3	0.07	0.07	0.07	0.07	0.07
D_4	0.25	0.40	0.50	0.60	0.80

4 SPEEDING UP THE SIMULATION

Knowledge of the optimal parameter u will lead to faster estimation of θ. In the preceeding section, pilot tests had to be performed to estimate the best value u before performing the simulation. Since the optimal u^* and the gain in variance reduction are both problem dependent, it is not clear that such a procedure actually improves the efficiency with respect to D_1. Here we propose the algorithm to adapt and find the optimal value u^* as it estimates the price of the option.

Our method consists in varying the parameter u as the simulation progresses, in the following fashion:

$$u_{n+1} = u_n - \epsilon_n \bar{F}_n(u_n) \tag{13}$$

where $\bar{F}_n(u)$ is an estimator of the derivative with respect to u of the *variance of the estimator* D_4. Such recursions are known as stochastic approximation, or "Robbins-Monroe procedures" (Robbins and Monro 1951) when ϵ_n is a suitably decreasing sequence.

Let u be constrained to some compact interval U, and call $J(u) = \mathsf{Var}(D_4)$. Call $\mathcal{F}_n$ the σ-algebra generated by $\{u_0, \bar{F}_1, \ldots, \bar{F}_{n-1}\}$. we state the following result without proof. The result follows from Fu (1990), Kushner and Yin (1997), Kushner and Vázquez-Abad (1996).

Assumptions:

- $\forall u \in U, \quad \dfrac{\partial}{\partial u} J(\cdot)$ is continuous in u,
- $\sup_{u \in U} \mathsf{E}[\bar{F}_n^2(u)] < K < \infty$,
- $\mathsf{E}[\bar{F}_n(u_n) | \mathcal{F}_n] = \dfrac{\partial}{\partial u} J(u_n) + \beta_n$, where $\displaystyle\sum_{j=n}^{\infty} \|\epsilon_j \beta_j\| < \infty$,
- $\displaystyle\sum_{n=1}^{\infty} \epsilon_n = +\infty, \quad \sum_{n=1}^{\infty} \epsilon_n^2 < \infty$, and
- $J(\cdot)$ is convex and therefore has a unique minimum $u^* \in U$.

Theorem 1 *Under the stated Assumptions, the sequence* $\{u_n\}$ *converges strongly to the optimum:* $u_n \to u^*$ *a.s.*

In order to use Theorem 1 it is necessary to have an estimator of the desired derivative that satisfies the assumptions. Infinitesimal Perturbation Analysis (IPA) can be used, as we proceed to establish. Recall that $D_2 = L_u Y_1$ and notice that:

$$D_4 = D_2 + \alpha(\mathsf{E}(Y_2^r) - L_u Y_2^u)$$

where, as before, $Y_1^u = (A_u - K)_+$ and $Y_2^u = (G_u - K)_+$. Call $F_i(u), i = 2, 4$ the IPA estimators such that:

$$\mathsf{E} F_i(u) = \frac{\partial}{\partial u} \mathsf{Var}_u(D_i) = \frac{\partial}{\partial u} \mathsf{E}[D_i^2].$$

The IPA estimator F_i is defined, as usual (see Glasserman 1991), as the stochastic derivative of D_i^2: if we fix $Z_1, \ldots, Z_N$, the square of D_i is a piecewise differentiable function of u and F_i is its derivative.

Define the following path-dependent quantities:

$$\begin{aligned} l'_u &= -\left(\frac{(u-r)T + \sqrt{\sigma^2 h} B_N}{\sigma^2} \right), \\ A'_u &= \frac{1}{N} \sum_{i=0}^{N} i h S_{ih}^u, \\ G'_u &= \frac{(T+h)}{2} G_u. \end{aligned}$$

Theorem 2 *The IPA estimators F_2 and F_4 are unbiased and are given by:*

$$F_2(u) = 2L_u^2 (l'_u)(Y_1^u)^2 + 2L^2 Y_1^u A'_T, \tag{14}$$

$$\begin{aligned} F_4(u) &= 2L_u^2 (Y_1^u - \alpha Y_2^u)\{l'_u (Y_1^u - \alpha Y_2^u) \\ &+ A'_T \mathbf{1}_{\{Y_1^u > 0\}} - \alpha G'_T \mathbf{1}_{\{Y_2^u > 0\}}\}. \end{aligned} \tag{15}$$

Proof : We shall state the proof for F_2 only, since the proof for F_4 is completely analogous. >From (9),

$$\frac{\partial}{\partial u}\mathsf{Var}[L_u(A_u-K)_+] \quad = \quad \frac{\partial}{\partial u}\mathsf{E}[L_u^2(A_u-K)_+^2].$$

Let $G(u) = L_u^2(A_u-K)_+^2$, and $m \in I$, where I is any compact interval of $I\!R$. Then $\frac{\partial}{\partial u}G(u)$ is given by (14), and is seen to be a continuous function of u. Moreover, its absolute value is uniformly bounded (for $u \in U$) by a variable of the form

$$C_1 e^{C_2 B_N}\left\{A_{\overline{u}}\sum_{i=1}^{N} iS_i^u + C_3(A_{\overline{u}})^2\left[C_4 + C_5 e^{C_6 B_N}\right]\right\}$$

(where $\overline{u}$ and C_1 to C_6 are constants), which has a finite expectation. Observing that from Taylor's Theorem

$$\frac{(G(u+\delta)-G(u))}{\delta} = \frac{\partial}{\partial p}G(p)\Big|_{p=\xi},$$

where ξ is between u and $u+\delta$, we get

$$\frac{\partial}{\partial u}\mathsf{E}\,G(u) = \mathsf{E}\,\frac{\partial}{\partial u}G(u)$$

from the Dominated Convergence Theorem. □

Calculation of both IPA formulas can be done while simulating one path with minimal extra effort: indeed all quantities but A'_u are available at the end of the N readings, and this extra summation adds a negligible computational effort. Table 3 shows the result from simulations performed to estimate the IPA derivatives using $M = 50,000$ replications, which took 31 seconds for each value of u.

Our first simulations used

$$\bar{F}_n(u_n) = \frac{1}{M}\sum_{k=nM+1}^{(n+1)M} F_4(k)$$

where M independent replications were performed at value $u = u_n$ to obtain $F_i(k), D_i(k), k = 1,\ldots,M, i = 2,4$. Then (13) is applied using $\epsilon_n = \epsilon_0/n$. It is straightforward to verify the Assumptions for this case, where $\beta_n = 0$. While we obtained convergence to the correct optimal u, the procedure was very slow. The reason for this is that the values of F_4 are very small: as it should be obvious from Table 3, estimating the derivative is a harder problem for the reduced variance estimator D_4 than is estimating F_2. Yet the two estimators seem to have the same optimal value for u, or at least very close, which happened for other parameter values as well. We therefore accelerated the procedure by driving the stochastic approximation with a convex combination of the two derivatives, or:

$$\begin{aligned}\bar{F}_n(u_n) &= \rho_n\frac{1}{M}\sum_{k=nM+1}^{(n+1)M} F_2(k)\\ &+ (1-\rho_n)\frac{1}{M}\sum_{k=nM+1}^{(n+1)M} F_4(k),\end{aligned}$$

where $\rho_n = \rho_0^n$, so that $\lim_{n\to\infty}\rho_n = 0$. Theorem 1 asserts that $u_n \to u^*$ a.s. still holds, but the convergence is accelerated (we used $\rho = 0.98$).

Summarizing, the accelerated estimation is achieved with:

$$D_5 = \frac{1}{m}\sum_{n=m}^{m}\bar{D}_4(u_n),$$

where $\bar{D}_4(u_n)$ is the sample mean estimator obtained with M replications of Algorithm 1 at value u_n as $\bar{F}_n(u_n)$ is estimated.

Algorithm 2: Accelerated Simulation

1. Choose an initial value $u(0)$
2. For $n = 0,\ldots m$ do:
   (a) Set $u = u(n)$
   (b) For $m = 1,\ldots,M$ do
      i. For $0 < i \le N$ do:
         A. Generate $Z_i \sim \mathcal{N}(0,1)$ and set $B_i = B_i + Z_i$,
         B. Define $X_i^u = u_\sigma h + \sqrt{\sigma^2 h}\,Z_i$,
         C. Set $S_i = S_{i-1}e^{X_i}$, $A_u = A_u + S_i, G_u = G_u * S_i$.
         D. Set $A' = A' + iS_i$.
      ii. Set $A_u = A_u/N, G_u = \sqrt[N]{G_u}, A' = \Delta A'/N$, calculate L, l'
      iii. Update the sample means Y_1, Y_2, F_2 and F_4
   (c) Set $\rho = \rho * \rho_0, \bar{F}_n = \rho F_2 + (1-\rho)F_4$
   (d) Update $u_{n+1} = u_n - (\epsilon_0/n)F_n$

Figure 4 shows a plot of typical trajectories of the values of u_n vs n for our estimator, the solid line for $K = 75, \epsilon_0 = 0.008$, the long dashes for $K = 50, \epsilon_0 = 0.001$ and the short dashes for $K = 30, \epsilon_0 = 0.0001$. Initial values of u were chosen far from the optimum. The update intervals were all of length $M = 500$, with $m = 20$ updates, and the computational effort was 6 seconds for each simulation.

The variance of D_5 is very close to the optimal one in Table 1, since in all cases convergence was achieved within the first three or four iterations of the stochastic approximation.

Derivative Estimation via IPA				
Value of u	$\mathsf{Var}(D_2)$	F_2	$\mathsf{Var}(D_4)$	F_4
0.2	45.32	-175.5 ± 15.7	0.21	-2.08 ± 0.29
0.3	32.01	-93.4 ± 9.2	0.16	-1.13 ± 0.17
0.4	25.44	-38.7 ± 7.3	0.15	-0.28 ± 0.35
0.5	23.69	3.89 ± 8.3	0.17	0.25 ± 0.77
0.6	26.05	45.44 ± 12.0	0.20	0.34 ± 0.53
0.7	32.94	94.88 ± 20.2	0.22	0.36 ± 0.78
0.8	45.80	168.82 ± 41.6	0.49	6.29 ± 21.68

Table 3: $r = 0.05, \sigma^2 = 0.2, S_0 = 50, K = 50, T = 1.0$ and $M = 50,000$

Table 4: Statistical Properties the Self-Optimized Estimator

$r = 0.05, \sigma^2 = 0.2, S_0 = 50, T = 1.0$ and $M = 10,000$					
Estimators					
Method	K= 30	K=50	K = 75	K=90	
D_1	20.31 ± 0.016	5.64 ± 0.012	0.585 ± 0.010	0.1415 ± 0.0085	
D_4	20.31 ± 0.014	5.62 ± 0.008	0.573 ± 0.003	0.1305 ± 0.0018	
D_5	20.31 ± 0.015	5.62 ± 0.008	0.578 ± 0.004	0.1304 ± 0.0017	
Variance					CPU Time
Method	K= 30	K=50	K = 75	K = 90	in seconds
D_1	0.64	0.36	0.25	0.1869	5
D_4	0.53	0.15	0.03	0.0080	31
D_5	0.54	0.18	0.04	0.0081	6

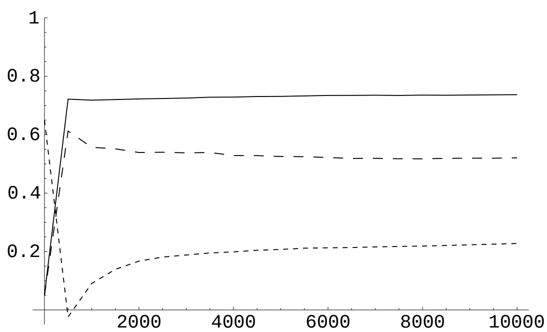

Figure 4: u_n vs n for three different systems.

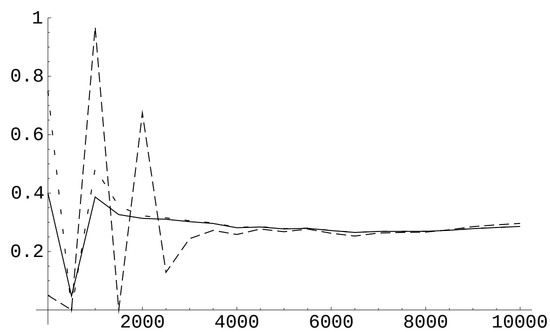

Figure 5: Values of u for $K = 30$ and different μ_0.

Figure 5 gives the results of the algorithm for $K = 30, \epsilon = 0.0005$ with different values of the initial condition.

In practice, it may be difficult to know how to choose the parameters. We suggest an initial guess at $u = r$ and proceed with the updates.

The estimated variances of D_5 are shown in Table 4, as well as those of D_1 and D_4 for ease of comparison. The computational effort is also shown, including the time required for the pilot simulations in order to set-up the estimation of D_4. Using our self-optimized method, we can achieve nearly optimal variance in 6 seconds without previous knowledge of the behaviour or preliminary tests. For the case $K = 90$ (with $\hat{\alpha} = 1.44$, $u^* = 1.16$), which is deeply out of the money, the advantages of the change of measure are more dramatic.

5 CONCLUDING REMARKS

We have presented a Self-Optimized estimator with Accelerated Simulation. It is based on the usual control variable estimator, but changes the measure in the hope of decreasing the variance, which is justified when the option is out of the money. Our method is applicable in conjunction with other methods, such as indirect simulation using the put and call parity, quasi-MonteCarlo methods, and antithetic variables. We are currently extending our results to include other financial instruments besides Asian Options.

The idea of speeding up the simulation by adjusting the parameter of the change of measure can of course be applied to other simulations outside derivative pricing, including simulation of stochastic processes with memory.

ACKNOWLEDGMENTS

Supported in part by grants from NSERC (Canada) and the Australian Research Council.

REFERENCES

Boyle, P., M. Broadie, and P. Glasserman (1997). Monte carlo methods for security pricing.
manuscript.

Boyle, P., S. Cox, D. Dufresne, H. Gerber, H. Mueller, H. Pederson, S. Pliska, M. Sherris, and E. Shiu (1998). *Financial Economics: With Applications to Insurance and Pensions*. Chicago: Society of Actuaries.

Bratley, P., B. Fox, and L. Schrage (1987). *A Guide to Simulation* (Second ed.). New York: Springer-Verlag.

Broadie, M. and P. Glasserman (1996). Estimating security price derivatives using simulation. *Management Science 42*, 269–285.

Fu, M., D. Madan, and T. Wang (1997). Pricing asian options: a comparison of analytical results and monte carlo methods. *Working paper, University of Maryland*.

Fu, M. C. (1990). Convergence of the stochastic approximation algorithm for the GI/G/1 queue using infinitesimal perturbation analysis. *J. of Optimization Theory and Applications* (65), 149–160.

Geman, H. and M. Yor (1993). Bessel processes, asian options and perpetuities. *Mathematical Finance 3*, 349–375.

Glasserman, P. (1991). *Gradient Estimation via Perturbation Analysis*. Boston: Kluwer Academic.

Harrison, J. and S. Pliska (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Proc. Appl. 11*, 215–260.

Hull, J. (1993). *Options, Futures, and Other Derivative Securities* (Second ed.). Englewood Cliffs, New Jersey: Prentice Hall.

Kushner, H. and F. Vázquez-Abad (1996, March). Stochastic approximation methods for systems of interest over an infinite horizon. *SIAM J. on Control and Optimization 34*(2), 712–756.

Kushner, H. and G. Yin (1997). *Stochastic Approximation and Applications*. New York: Springer Verlag.

L'Ecuyer, P. (1994). Efficiency improvement and variance reduction. In *Proceedings of the 1994 Winter Simulation Conference*. IEEE Press.

Lemieux, C. (1996). L'évaluation des options asiatiques. Master's thesis, Université de Montréal.

Robbins, H. and S. Monro (1951). A stochastic approximation method. *Annals of Mathematical Statistics* (22), 400–407.

Ross, S. (1997). *Simulation* (Second ed.). Boston: Academic Press.

Turnbull, S. and L. Wakeman (1991). A quick algorithm for pricing european average options. *The Journal of Financial and Quantitative Analysis 26,* 377–389.

AUTHOR BIOGRAPHIES

FELISA J VÁZQUEZ-ABAD is Professor at the Department of Computer Science and Operations Research at the University of Montreal. She received her B.Sc. degree in Physics in 1983 and her M.Sc. degree in Statistics and OR in 1984 from Universidad Nacional Autónoma de México, and her Ph.D. degree in Applied Mathematics in 1989 from Brown University. Her research interests include stochastic control and simulation with applications in telecommunications, manufacturing, transportation, insurance and finance. She is a member of SIAM and INFORMS.

DANIEL DUFRESNE is currently with the Department of Economics at the University of Melbourne (Australia). His research interests include probability theory, finance and actuarial science.