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# ESTIMATING AND SIMULATING POISSON PROCESSES WITH TRENDS OR ASYMMETRIC CYCLIC EFFECTS

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## ABSTRACT

We present a heuristic that provides a nonparametric estimate of the mean-value function of a nonhomogeneous Poisson process having a long-term trend or some cyclic effect(s) that may lack sinusoidal symmetry over the corresponding cycle(s). This heuristic is a multiresolution-based method that allows one to estimate the overall long-term trend of the process at the lowest resolution and then add the details of the process associated with progressively smaller periodic components at progressively higher resolutions. In addition, we present an algorithm for generating realizations of nonhomogeneous Poisson processes with the estimated mean-value function in simulation experiments.

## 1 INTRODUCTION

In this paper we focus on arrival processes, and more particularly, arrival processes that can be classified as nonstationary point processes. These processes have the characteristics that we are able to observe the exact arrival time of each entity, and the arrival intensity changes over time. Under certain assumptions a nonstationary arrival process can be represented as a nonhomogeneous Poisson process (NHPP) (Çinlar, 1975). Using NHPPs, we can accurately represent a large class of arrival processes encountered in practice.

An NHPP $\{N(t) : t \geq 0\}$ given by

$$N(t) = \ \# \text{ of arrivals in } (0, t] \quad \text{for all} \quad t \geq 0$$

is a generalization of the Poisson process in which the instantaneous arrival rate $\lambda(t)$ at time $t$ is a nonnegative integrable function of time. The mean-value function of the NHPP is defined by

$$\mu(t) \equiv E[N(t)] \quad \text{for all} \quad t \geq 0;$$

and the relationship between the rate function and the mean-value function is

$$E[N(t)] = \int_0^t \lambda(z)\, dz \quad \text{for all} \quad t \geq 0.$$

The probabilistic behavior of the NHPP is completely defined by the rate or mean-value functions. The literature in this area discusses both parametric and nonparametric methods for estimating the NHPP rate function. To model arrival processes having several periodic effects or a long-term trend (or both), Kuhl, Wilson, and Johnson (1997) utilized an NHPP whose rate function is of the type (parametric form) exponential-polynomial-trigonometric with multiple periodicities (EPTMP).

In this paper, we propose a heuristic to obtain a nonparametric estimate of these types of processes. In particular, this heuristic focuses on processes having several nested cyclic effects so that each larger cycle (period) includes an integral number of smaller cycles; moreover, although successive cycles of a given length may have different numbers of expected arrivals due to (for example) a long-term trend, all cycles of the same length are similar in the sense that the same cumulative percentage of each cycle's expected arrivals is achieved at the same relative position within the cycle. The benefits of this nonparametric approach include the ability to more accurately model asymmetric periodic components without (a) having to include a large number of trigonometric rate components (as is often required in some parametric approaches); or (b) having to sample from a lengthy historical record (as in trace-driven simulation).

## 2 THE ESTIMATION PROCEDURE

We have developed the heuristic method below in an attempt to represent the periodic behavior and long-term trends exhibited by many nonhomogeneous Poisson processes in such a way that the known arrival patterns can be represented by fewer parameters than are required by an EPTMP-type rate function. This method can be applied to many arrival processes for which the following two assumptions hold:

**Assumption 1** *There are $p$ distinct cycle lengths (periods)*

$$b_1 > b_2 > \ldots > b_p$$

*such that $b_i$ is an integral multiple of $b_{i+1}$ for $i = 1, 2, \ldots, p-1$. Moreover, the time horizon $(0, S]$ is taken such that $S$ is an integral multiple of $b_1$.*

**Assumption 2** *Within each cycle $[(j-1)b_i, jb_i)$ of length $b_i$ $(1 \leq i \leq p)$, the buildup over time $t = (j-1)b_i + s$ of the cumulative percentage $[\mu(t) - \mu((j-1)b_i)] \,/\, [\mu(jb_i) - \mu((j-1)b_i)]$ of the cycle's expected number of arrivals is described by the same function $R_i(s)$ of the time $s$ that has elapsed since the beginning of the cycle so that*

$$R_i(s) \equiv \frac{\mu((j-1)b_i + s) - \mu((j-1)b_i)}{\mu(jb_i) - \mu((j-1)b_i)}$$

*for all $s \in [0, b_i)$ and for $j = 1, 2, \ldots$.*

This method uses a monotonically increasing function to represent each periodic component and the long-term trend. A type of multiresolution analysis is performed similar to that of wavelet multiresolution analysis. At the lowest resolution, Resolution 0, we look at the overall arrival process which may contain long-term trends over time. Then as we increase the level of resolution, more detail is added. The levels of resolution we consider are those that correspond to the periodic components. If there are $p$ periodic components present, Resolution 1 will correspond to the largest period $b_1$ and Resolution $p$ will correspond to the smallest period $b_p$.

For Resolution 0, we take

$$R_0(t) \equiv \mu(t)/\mu(S) \text{ for all } t \in (0, S];$$

and we estimate $R_0(t)$ by fitting an increasing function, $\widehat{R}_0(t)$, to the cumulative fraction of arrivals over the time horizon $(0, S]$. (If no long-term trends exist, a linear function will be fit to the cumulative fraction of arrivals.) For Resolution $i$, $i = 1, 2, \ldots, p$, we fit an increasing function $\widehat{R}_i(s)$ to estimate the distribution of the cumulative percentage $R_i(s)$ of superimposed arrivals for all $s \in [0, b_i)$, the full cycle associated with Resolution $i$. That is, we will fit a curve to the cumulative fraction of arrivals over the period. Note that the fitted curve must have a value of 0 at the start of the period and a value of 1.0 at the end of the period. Also note that only fully observed periods should be used when fitting the curve.

To estimate the mean-value function $\mu(t)$, we must first estimate the functions $R_i(s)$ for all $s \in [0, b_i)$ and $i = 0, 1, \ldots, p$ (where $b_0 \equiv S$). In stage $i = 0$, we fit a monotonically increasing function $\widehat{R}_0(t)$ to the points

$$\left\{[jb_1, N(jb_1)/N(S)]^{\mathrm{T}} : j = 0, 1, \ldots, S/b_1\right\}$$

such that $\widehat{R}_0(0) = 0$ and $\widehat{R}_0(S) = 1$.

To estimate $R_i(\cdot)$ for stage $i$ where $1 \leq i \leq p-1$, we fit a monotonically increasing function $\widehat{R}_i(s)$ to the points

$$\left\{[jb_{i+1}, G_i(jb_{i+1})]^{\mathrm{T}} : j = 0, 1, \ldots, b_i/b_{i+1}\right\},$$

where

$$G_i(s) \equiv \frac{\displaystyle\sum_{\ell=0}^{(S/b_i)-1} N(\ell b_i + s) - N(\ell b_i)}{N(S)} \text{ for } s \in [0, b_i)$$

is the cumulative percentage of arrivals observed up to time $s \in [0, b_i)$, averaged over all observed cycles of length $b_i$; and we require that the function $\widehat{R}_i(s)$ fitted for Resolution $i$ must satisfy $\widehat{R}_i(0) = 0$, $\widehat{R}_i(b_i) = 1$.

To estimate $R_p(\cdot)$ for stage $p$, we let $\tau_1, \ldots, \tau_{N(S)}$ denote the observed $N(S)$ arrival times and define $\eta_i = \tau_i \bmod b_p$. Let $\eta_{(1)} < \eta_{(2)} < \ldots < \eta_{(N(S))}$ be the corresponding ordered event times on the cycle $[0, b_p)$ of the superimposed arrival process. Then we fit a monotonically increasing function $\widehat{R}_p(s)$ to the points

$$\left\{[\eta_{(i)}, i/N(S)]^{\mathrm{T}} : i = 1, \ldots, N(S)\right\}$$

such that $\widehat{R}_p(0) = 0$ and $\widehat{R}_p(b_p) = 1$.

The final estimate $\widehat{\mu}(t)$ of the mean-value function $\mu(t)$ is computed as follows:

$$\widehat{\mu}(t) = N(S)\widehat{Q}_0(t) \quad \text{for} \quad t \in (0, S] \tag{1}$$

where the $\widehat{Q}_i(t)$ for $i = p, p-1, \ldots, 1, 0$ are defined iteratively as follows:

$$\widehat{Q}_p(t) = \widehat{R}_p\left(t - (j_{p,t} - 1)b_p\right), \tag{2}$$

and

$$\begin{aligned}\widehat{Q}_i(t) = {} & \widehat{R}_i\left((j_{i+1,t} - 1)b_{i+1} - (j_{i,t} - 1)b_i\right) \\ & + \widehat{Q}_{i+1}(t)\Big[\widehat{R}_i\left(j_{i+1,t}b_{i+1} - (j_{i,t} - 1)b_i\right) \\ & - \widehat{R}_i\left((j_{i+1,t} - 1)b_{i+1} - (j_{i,t} - 1)b_i\right)\Big],\end{aligned} \tag{3}$$

and $j_{i,t}$ is the unique integer $j$ such that

$$(j-1)b_i \leq t < jb_i.$$

Notice that for $i = 0, 1, \ldots, p-1$, $\widehat{Q}_i(t)$ represents a "refined" estimate of $R_i(t \bmod b_i)$ that depends not only on $\widehat{R}_i(t \bmod b_i)$ but also on $\widehat{Q}_j(t)$ for $j = i+1, \ldots, p$. In particular, (3) shows that each $\widehat{Q}_i(t)$ is the sum of two components: (a) the estimated fraction of arrivals in the current cycle $[(j_{i,t} - 1)b_i, j_{i,t}b_i)$ of length $b_i$ that occur up to the beginning time $(j_{i+1,t} - 1)b_{i+1}$ of the current cycle $[(j_{i+1,t} - 1)b_{i+1}, j_{i+1,t}b_{i+1})$ of length $b_{i+1}$; and (b) the estimated fraction of arrivals in the current cycle of length $b_i$ that also fall in the subinterval $[(j_{i+1,t} - 1)b_{i+1}, t)$ of the current cycle of length $b_{i+1}$. With regard to (b), we see that

$$\begin{aligned}&\widehat{R}_i\left(j_{i+1,t}b_{i+1} - (j_{i,t} - 1)b_i\right) \\ &\quad -\widehat{R}_i\left((j_{i+1,t} - 1)b_{i+1} - (j_{i,t} - 1)b_i\right)\end{aligned}$$

estimates the fraction of arrivals in the current cycle of length $b_i$ that also fall in the current cycle of length $b_{i+1}$; and $\widehat{Q}_{i+1}(t)$ is in trun a "refined" estimate of the fraction of arrivals in the current cycle of length $b_{i+1}$ that occur up to time $t$.

To illustrate how the estimate of the mean-value function $\widehat{\mu}(t)$ is constructed, we will define the following notation. We will denote the mean-value function at Resolution $\ell$ by $\widehat{\mu}_\ell(t)$. This will represent the mean-value function containing the details associated with resolutions up to Resolution $\ell$ and will exclude the details associated with higher resolutions. This amounts to setting $p = \ell$ in (1) such that

$$\widehat{\mu}_\ell(t) = N(S)\widehat{Q}_0(t) \quad \text{for} \quad t \in (0, S] \tag{4}$$

where the $\widehat{Q}_i(t)$ for $i = \ell, \ell-1, \ldots, 1, 0$ are defined iteratively as follows:

$$\widehat{Q}_\ell(t) = \widehat{R}_\ell\left(t - (j_{\ell,t} - 1)b_\ell\right),$$

and

$$\begin{aligned}\widehat{Q}_i(t) = {} & \widehat{R}_i\left((j_{i+1,t} - 1)b_{i+1} - (j_{i,t} - 1)b_i\right) \\ & + \widehat{Q}_{i+1}(t)\Big[\widehat{R}_i\left(j_{i+1,t}b_{i+1} - (j_{i,t} - 1)b_i\right) \\ & - \widehat{R}_i\left((j_{i+1,t} - 1)b_{i+1} - (j_{i,t} - 1)b_i\right)\Big],\end{aligned}$$

So, for $\ell = p$, $\widehat{\mu}_p(t) = \widehat{\mu}(t)$. The following example will illustrate this multiresolution fitting procedure.

## 3 AN EXAMPLE

We apply the above heuristic method to a realization of an NHPP with a low long-term trend and two periodic components. In terms of time units, the periodic components have periods of one week and one day. The time interval we will consider (in days) is $(0, 35]$. The underlying NHPP has the form of the above heuristic, where the cumulative fraction of arrivals for Resolutions 0, 1, and 2, respectively, have the form

$$\begin{aligned}R_0(s) &= 0.0258s + 0.0000647s^2, && 0 < s \leq 35,\\ R_1(s) &= 0.2042s + 0.04426s^2 \\ &\quad - 0.01772s^3 + 0.001449s^4, && 0 < s \leq 7,\\ R_2(s) &= 1.1182s + 0.7636s^2 - 0.8818s^3, && 0 < s \leq 1.\end{aligned}$$

The realization of this arrival process which we will fit is shown in Figure 1. The data for this arrival process was obtained using the variate generation method described in Section 5.

To apply the heuristic method, we begin by fitting curves at each resolution. Since this process contains two periodic components, we will fit curves at: (a) Resolution 0, corresponding to the long-term trend; (b) Resolution 1, corresponding to the component with a period of 1 week, so that $b_1 = 7$ days; and (c) Resolution 2, corresponding to the component with a period of $b_2 = 1$ day. For the construction of the mean-value function, the points of importance at Resolution 0 are those that correspond to the cumulative fraction of arrivals observed at the end of each Resolution 1 period. Therefore, in this example we fit a curve to the observed cumulative fraction of arrivals at the end of each week. This fitted function at Resolution 0, $\widehat{R}_0(s)$, is shown in Figure 2.

The next step in the heuristic method is to fit a curve at Resolution 1 which is associated with a period of 1 week. First, the cumulative fraction of arrivals over each week are superimposed. Then, similar

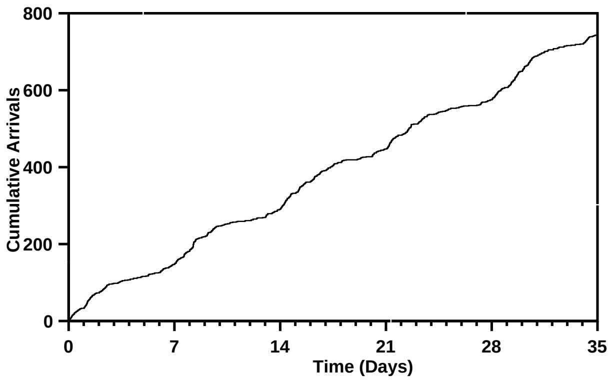


Figure 1: A Realization of the NHPP Having a Linear Trend and Two Periodicities

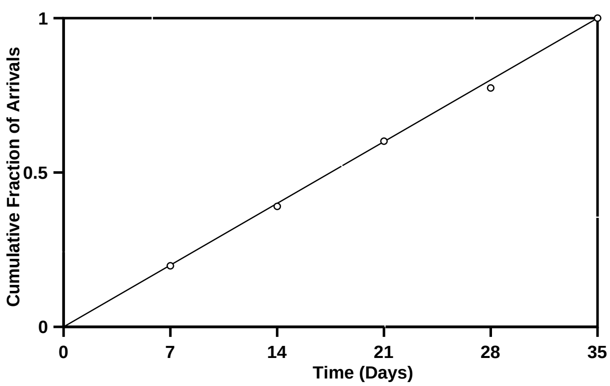


Figure 2: Fitted Function at Resolution 0, $\widehat{R}_0(s)$

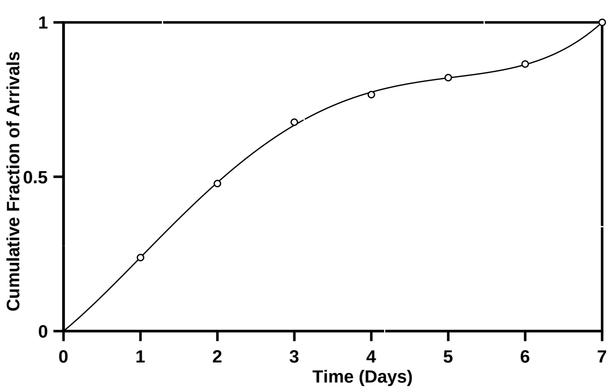


Figure 3: Fitted Function at Resolution 1, $\widehat{R}_1(s)$

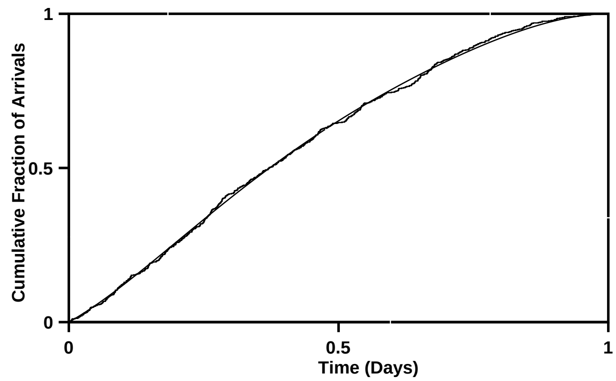


Figure 4: Fitted Function at Resolution 2, $\widehat{R}_2(s)$

to Resolution 0, the points of importance at Resolution 1 for constructing the mean-value function are those that correspond to the cumulative fraction of arrivals observed at the end of each period at Resolution 2. Therefore, in this example we fit a curve over the period of 1 week to the observed cumulative fraction of arrivals at the end of each day of the week. This fitted function at Resolution 1, $\widehat{R}_1(s)$, is shown in Figure 3.

Finally, we fit a curve to Resolution 2 which is associated with the smallest period of 1 day. First, the cumulative number of arrivals over each day are superimposed. Then, a function is fit to these points over a period of one day. The resulting fitted function for Resolution 2, $\widehat{R}_2(s)$, is shown in Figure 4.

After fitting curves at each resolution, we construct the estimated mean-value function. This procedure is presented graphically in Figures 5–7. To show how the mean-value function is constructed, we will use the notation for the mean-value function at Resolution $\ell$ along with a graphical representation of the procedure at each step. For clarity, these Figures 5–7 only show the first 14 days of the 35-day interval. The process of constructing the mean-value function begins by calculating the mean-value function at Resolution 0, $\widehat{\mu}_0(t)$, for $t \in (0, 35]$. That is, we scale the Resolution 0 curve (fit to the cumulative fraction of arrivals over $(0, 35]$) by $N(S)$ (the estimate of the expected number of arrivals in $(0, 35]$). That is, we calculate

$$\widehat{\mu}_0(t) = N(S)\widehat{Q}_0(t) = N(S)\widehat{R}_0(t), \qquad t \in (0, 35].$$

The resulting fitted mean-value function at Resolution 0 is shown in Figure 5.

The next step is to add the details of the weekly periodic component to the mean-value function. This is done by calculating $\widehat{\mu}_1(t)$, $t \in (0, 35]$. Over each

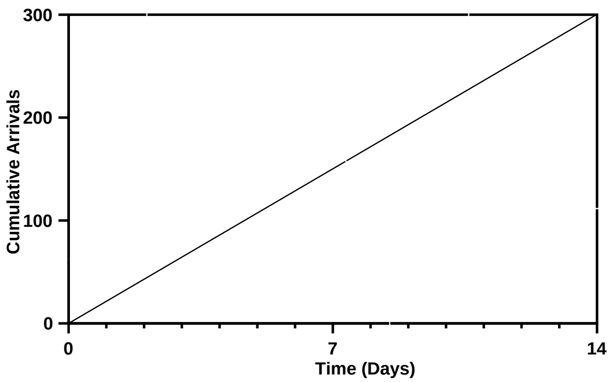


Figure 5: Fitted Mean-Value Function at Resolution 0, $\widehat{\mu}_0(t)$

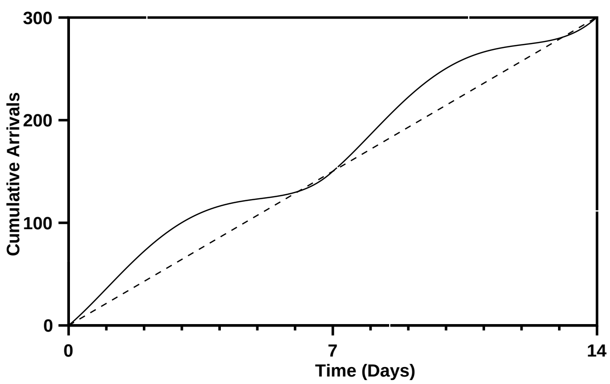


Figure 6: Fitted Mean-Value Function at Resolution 1, $\widehat{\mu}_1(t)$ (Solid Line) Superimposed on $\widehat{\mu}_0(t)$ (Dashed Line)

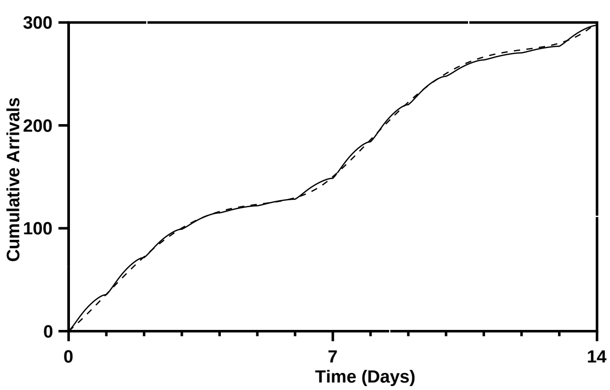


Figure 7: Fitted Mean-Value Function at Resolution 2, $\widehat{\mu}_2(t)$ (Solid Line) Superimposed on $\widehat{\mu}_1(t)$ (Dashed Line)

week, the fitted Resolution 1 curve is scaled by the expected number of arrivals in that week. Here, for the first week from equation (4) we have

$$\begin{aligned}
\widehat{\mu}_1(t) &= N(S)\widehat{Q}_0(t) \\
&= N(S)\widehat{R}_0(7)\widehat{Q}_1(t) \\
&= N(S)\widehat{R}_0(7)\widehat{R}_1(t), \quad t \in (0,7].
\end{aligned}$$

In a similar manner using equation (4), we calculate the mean-value function at Resolution 1 for the subsequent weeks. The resulting mean-value function at Resolution 1 is shown in Figure 6.

Finally, we add the details of the daily periodic component to the mean-value function. To do this, we calculate $\widehat{\mu}_2(t)$, $t \in (0,35]$, the mean-value function at Resolution 2. Over each week, the fitted Resolution 2 curve is scaled by the expected number of arrivals in that day. Here, for the first day from equation (4) we have

$$\begin{aligned}
\widehat{\mu}_2(t) &= N(S)\widehat{Q}_0(t) \\
&= N(S)\widehat{R}_0(7)\widehat{Q}_1(t) \\
&= N(S)\widehat{R}_0(7)\widehat{R}_1(1)\widehat{Q}_2(t) \\
&= N(S)\widehat{R}_0(7)\widehat{R}_1(1)\widehat{R}_2(t), \quad t \in (0,1].
\end{aligned}$$

In a similar manner using equation (4), we calculate the mean-value function at Resolution 2 for the subsequent days. The resulting mean-value function at Resolution 2 is shown in Figure 7. Since only two periodicities are present in the process, the final estimate of the mean-value function $\widehat{\mu}(t)$ is the Resolution 2 estimate $\widehat{\mu}_2(t)$. Figure 8 shows the fitted mean-value function plotted against the observed cumulative number of arrivals. 0 To illustrate the ability of this multiresolution procedure to estimate the underlying mean-value function, $\mu(t)$, at time $t$, for $t \in (0,S]$, from one realization of the process, we generated 100 replications of the NHPP and fit each realization using the procedure. We then constructed a 90% tolerance interval for $\mu(t)$, $t \in (0,35]$, which is plotted in Figure 9. From a visual inspection, the tolerance interval shows that the multiresolution procedure is capable of accurately obtaining an estimate of the underlying mean-value function from one realization of the process.

## 4 AN APPLICATION

In a project sponsored by the United Network for Organ Sharing, Pritsker et al. (1995) applied NHPPs with EPTMP-type rate functions (Kuhl, Wilson, and Johnson, 1997) to model the arrival of liver-transplant patients to transplant centers. In this example, we apply our heuristic method to the arrival process of liver transplant patients to Transplant Center 11 during 1992. The smoothed periodogram of the arrival process indicates two fundamental periodic components with periods of one day and one week. Over the year 1992, there does not appear to be any significant long-term tend. Figures 10 and 11 are histograms of the arrivals of patients by hour-of-the-day and by the day-of-the-week, respectively.

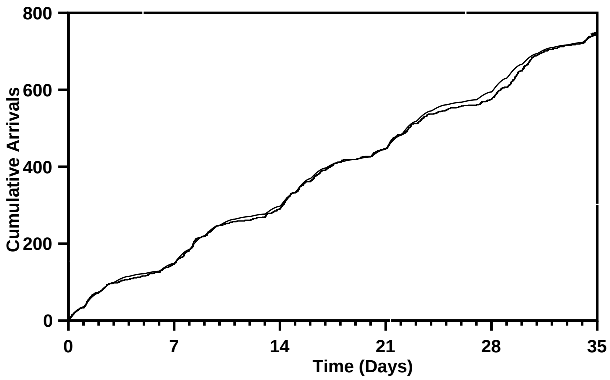


Figure 8: Fitted Mean-Value Function, $\widehat{\mu}(t)$ (Smooth Curve) Superimposed on Observed Cumulative Number of Arrivals (Step Function)

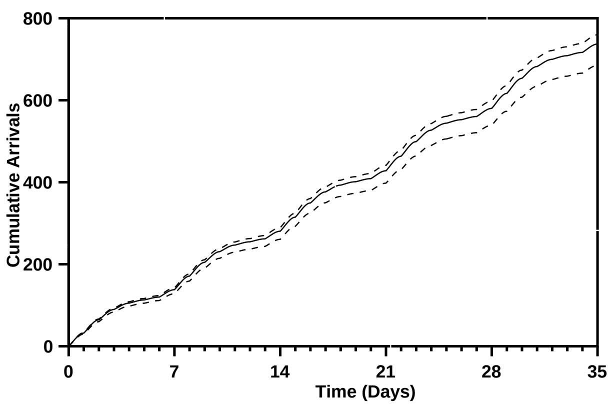


Figure 9: 90% Tolerance Interval for $\mu(t)$, $t \in (0, 35]$

Applying the above method, three resolutions will be required. At Resolution 0, a smooth curve is fit to the cumulative number of arrivals over the interval $(0, 366]$. Since no significant long-term trend is present over this interval, the Resolution 0 curve is a linear function of time and the expected number of arrivals in $(0, 366]$ is estimated by the observed number of arrivals $N(366) = 172$. Thus the fitted mean-value function at Resolution 0 is

$$\widehat{\mu}_0(t) = \frac{172}{366}t \quad \text{for} \quad 0 \leq t \leq S = 366. \tag{5}$$

Figure 12 shows the fitted mean-value function at Resolution 0.

The Resolution 1 curve is fit to the cumulative fraction of arrivals over the period of one week. Fitting a curve to the cumulative fraction of arrivals is the same as fitting a cumulative distribution function to observed data. For this example, a Bézier curve was fit to the cumulative fraction of arrivals at the end of each day. Since the fraction of arrivals occurring each day are the only values needed, the cumulative fraction of arrivals at the end of each day are the only points considered in fitting this curve. The Bézier curve was fit using the software package PRIME (Wagner and Wilson 1996), designed for fitting distribution functions. The Bézier curve was manually fit using nine control points (so that the fitted Bézier function was a ninth-degree polynomial). Figure 13 shows the Resolution 1 Bézier curve fit to the cumulative fraction of arrivals at the end of each day.

The Resolution 2 curve is fit to the cumulative fraction of arrivals over the period of one day. Utilizing PRIME, we fitted a Bézier curve with ten control points (that is, a tenth-degree polynomial) to the cumulative fraction of arrivals using the method of least squares. The Resolution 2 curve is shown in Figure 14.

Given the curves of Resolutions 0, 1, and 2, the mean-value function can be calculated as described above. Figure 15 shows the fitted mean-value function. Since no long-term trend is contained in this model, the arrival rate over each week is the same. The arrival rate over each week is obtained by taking the derivative of the mean-value function. In this case, the mean-value function is just a constant multiplier of the Resolution 2 Bézier curve for each day of the week. Therefore, for each day of the week, the rate function is the constant multiplied by the derivative of the Resolution 2 Bézier curve. Figure 17 shows the rate function for each week of the interval $(0, 366]$. To give some indication of the ability of this heuristic method in comparison to other methods, the same data was fit using an EPTMP rate function. Figures 16 and 18 show the EPTMP mean-value function and rate function, respectively. As can be seen from the graphs of the rate functions, the heuristic method offers more flexibility in modeling asymmetric and multimodal behavior we see in the histograms of the arrivals by the day of the week and hour of the day.

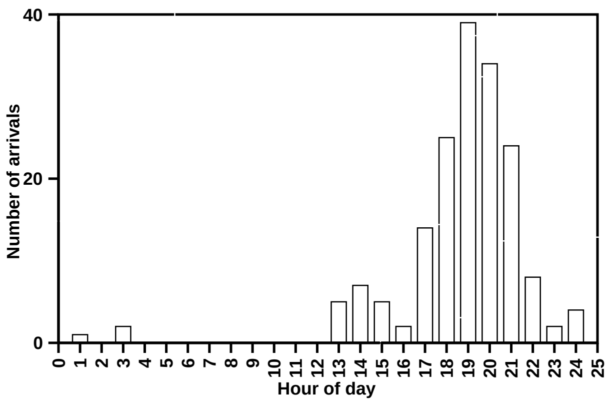


Figure 10: Histogram of the Number of Arrivals to Transplant Center 11 by Hour of the Day during 1992

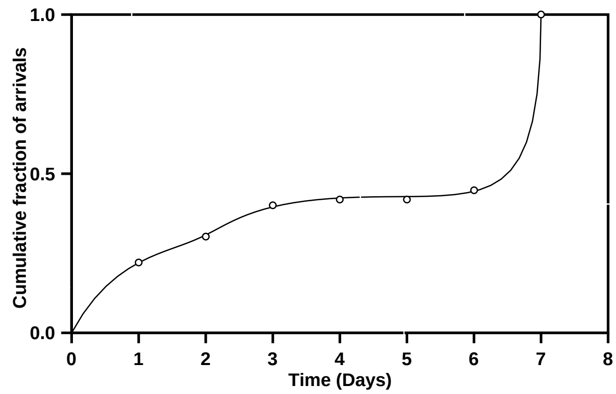


Figure 13: Resolution 1 Function $\widehat{R}_1(\cdot)$ Fitted to the Cumulative Fraction of Arrivals at the End of Each Day of the Week

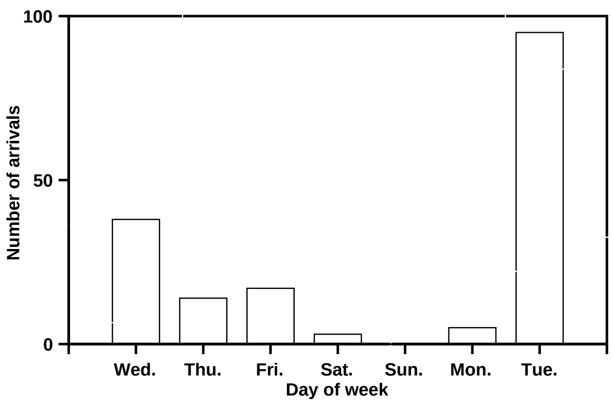


Figure 11: Histogram of the Number of Arrivals to Transplant Center 11 by Day of the Week during 1992

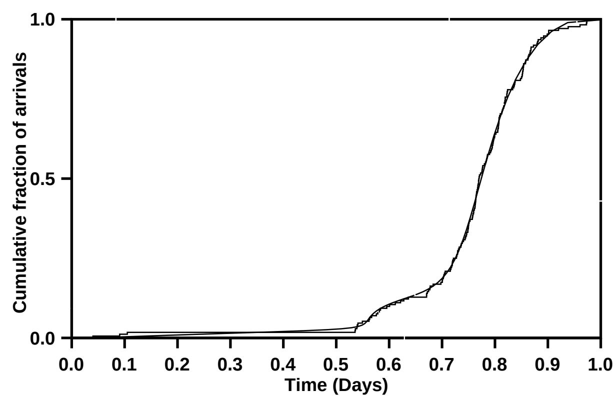


Figure 14: Resolution 2 Curve versus the Cumulative Fraction of Arrivals over the Period of One Day

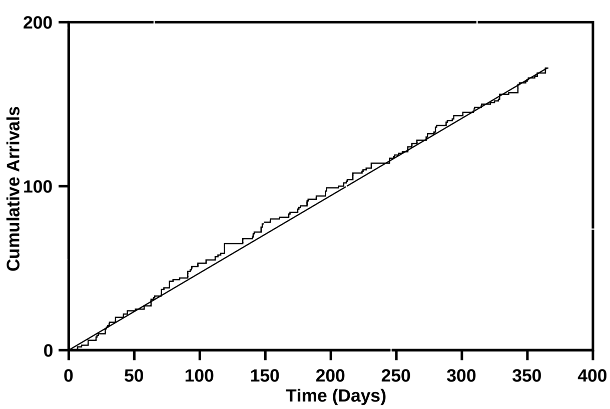


Figure 12: Fitted Mean-Value Function $\widehat{\mu}(\cdot)$ at Resolution 0 versus the Cumulative Number of Arrivals over the Interval $(0, 366]$

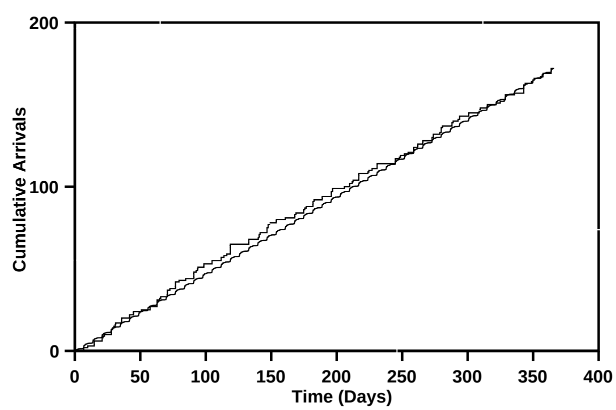


Figure 15: Fitted Mean-Value Function Using Heuristic Method

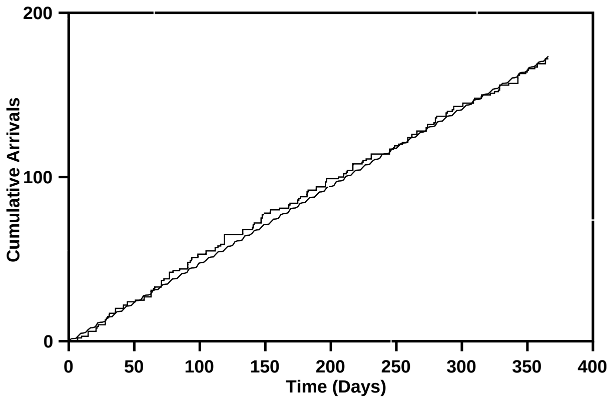


Figure 16: Fitted EPTMP Mean-Value Function

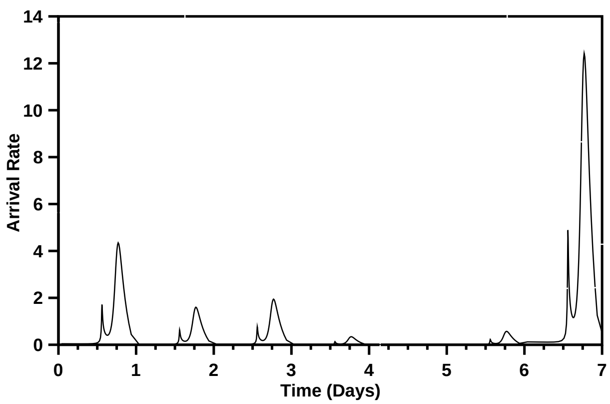


Figure 17: Fitted Weekly Rate Function Using Heuristic Method

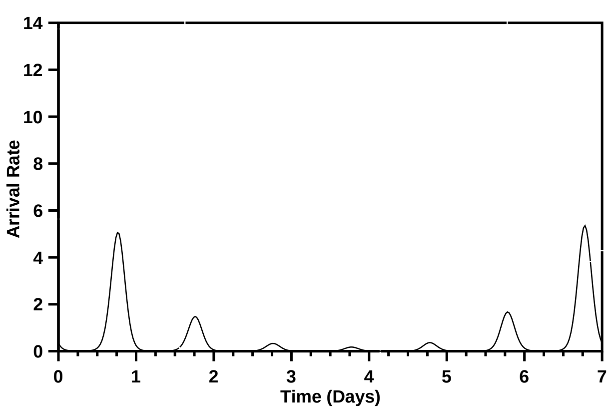


Figure 18: Fitted Weekly EPTMP Rate Function

## 5 VARIATE GENERATION

We propose to use the method of inversion to generate realizations of the fitted NHPP. The following discussion is in terms of the theoretical NHPP. To generate variates from an estimated NHPP, replace the theoretical values with their corresponding estimates from the preceding discussion. For an NHPP having rate function $\lambda(t),\ t \in [0, S]$, the cumulative distribution function of the next event time $\tau_i$ conditioned on the observed value $\tau_{i-1} = t_{i-1}$ of the last event time is given by

$$F_{\tau_i|\tau_{i-1}}(t|t_{i-1}) \equiv \Pr\{\tau_i \le t|\tau_{i-1} = t_{i-1}\}$$
$$= \begin{cases} 1 - \exp\left[-\int_{t_{i-1}}^{t} \lambda(z)\,dz\right], & \text{if } t \ge t_{i-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus to sample $\tau_i$ by inversion given $\tau_{i-1} = t_{i-1}$, generate a random number $U_i$ from the uniform distribution on the unit interval $(0, 1)$ and compute

$$\tau_i = F^{-1}_{\tau_i|\tau_{i-1}}(U_i|t_{i-1}).$$

This amounts to solving for $\tau_i$ in the equation

$$\int_{\tau_{i-1}}^{\tau_i} \lambda(z)\,dz = -\ln(1 - U_i).$$

Taking $\Delta = -\ln(1 - U_i)$, we have

$$\Delta = \int_{\tau_{i-1}}^{\tau_i} \lambda(z)\,dz = \mu(\tau_i) - \mu(\tau_{i-1}).$$

If $\tau_{i-1}$ and $\tau_i$ are in the $j$th cycle of Resolution $p$, where $j$ is an integer such that $(j-1)b_p \le \tau_{i-1}, \tau_i < jb_p$, then we have

$$\Delta = c_j\left[R_p(\tau_i - (j-1)b_p) - \rho_{i-1}\right], \tag{6}$$

where $c_j \equiv \mu(jb_p) - \mu((j-1)b_p)$ and where $\rho_{i-1} \equiv R_p(\tau_{i-1} - (j-1)b_p)$. Solving for $\tau_i$ in (6), we have

$$\tau_i = (j-1)b_p + R_p^{-1}(\rho_{i-1} + \Delta/c_j).$$

Now $\tau_{i-1}$ and $\tau_i$ are in the same Resolution-$p$ cycle $\big[(j-1)b_p, jb_p\big)$ if and only if

$$\Delta < c_j(1 - \rho_{i-1})\,. \tag{7}$$

If condition (7) is not satisfied, then the Resolution-$p$ cycle containing $\tau_i$ has index

$$k(i) = \max\left\{\ell : c_j(1 - \rho_{i-1}) + \sum_{r=j+1}^{\ell-1} c_r < \Delta\right\};$$

and then $\tau_i$ is computed from

$$\tau_i = [k(i)-1]b_p + R_p^{-1}\left[\frac{\Delta - c_j(1-\rho_{i-1}) - \sum_{r=j+1}^{k(i)-1} c_r}{c_{k(i)}}\right].$$

This inversion procedure is implemented in the following algorithm to generate a series of events on the interval $(0, S]$.

1. Initialize $i \leftarrow 1$, $j \leftarrow 0$, and $\rho \leftarrow 0$.
2. Generate $U_i \sim \text{Uniform}(0, 1)$ and take $\rho' \leftarrow \rho$, $\Delta \leftarrow -\ln(1-U_i)$.
3. Take $j \leftarrow j+1$ and calculate
$$Q \leftarrow \Delta - c_j(1-\rho') \tag{8}$$
4. If $Q \leq 0$, then go to step 7; otherwise, take
$$\Delta \leftarrow \Delta - c_j(1-\rho').$$
5. If $\rho' > 0$, then take $\rho' \leftarrow 0$.
6. Go to step 3.
7. Calculate the next event time,
$$\tau_i \leftarrow (j-1)b_p + R_p^{-1}(\rho' + \Delta/c_j)\,. \tag{9}$$
8. If $\tau_i > S$, discard $\tau_i$ and stop; otherwise, take $i \leftarrow i+1$, $\rho \leftarrow \rho' + \Delta/c_j$, and go to step 2.

## 6 CONCLUSION

This heuristic provides a nonparametric estimate of the mean-value function of an NHPP having periodicities or long-term trends. Through this procedure we are able to model asymmetric periodic behavior without having to store all of the observed data. Currently, we are working on establishing the theoretical properties of this procedure as well as its capabilities and limitations in practice.

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