Proceedings of the 1994 Winter Simulation Conference
ed. J. D. Tew, S. Manivannan, D. A. Sadowski, and A. F. Seila

CHARACTERIZING A NONSTATIONARY M/G/1 QUEUE USING BODE PLOTS

Mousumi Mitra Hazra

Tektronix, Inc.
P.O. Box 500, MS 50-320
Beaverton, OR 97077

Stephen K. Park

Department of Computer Science,
College of William & Mary,
Williamsburg, Virginia 23187

ABSTRACT

A Frequency Domain simulation Experiment (FDE) is a discrete-event simulation experiment in which selected system parameters are oscillated sinusoidally to induce oscillations in one or more system response statistics of interest. A spectral (Fourier) analysis of these induced oscillations is then performed to gain valuable information about the system. FDEs are based upon an intuitive frequency-invariance assumption—if a particular system response statistic is sensitive to a system parameter, then sinusoidal variation of that system parameter at a fixed frequency will induce similar sinusoidal variations in the system response statistics, at the same frequency. In this paper we provide some theoretical and experimental support for this FDE model assumption as it applies to a non-stationary $M/G/1$ queue with sinusoidally varying arrival rate and fixed service rate. Specifically an analytical expression for the departure rate (system frequency response) is provided which verifies frequency invariance and quantifies the extent to which the induced sinusoidal variation is amplitude modulated and phase shifted relative to the arrival rate.

Traditionally, the frequency response of a system is represented by two plots—amplitude versus frequency and phase angle versus frequency—together known as a *Bode plot.* Although Bode plots are commonly used to characterize the behavior of linear systems, never before have they been used to characterize a nonstationary $M/G/1$ queue. In this paper an FDE method to generate Bode plots for a nonstationary $M/G/1$ queue with sinusoidally varying arrival rate and fixed service rate is presented. The FDE-generated Bode plots are compared with the analytical frequency response expression. In this way we demonstrate that a nonstationary $M/G/1$ queue is characterized as a low-pass filter.

1 INTRODUCTION

FDE's were introduced to discrete event simulation in 1981 by Schruben, et al. (Schruben and Cogliano 1981). Frequency Domain Experiments (FDEs) were first used in discrete-event simulation to perform system parameter sensitivity analysis for factor screening in stochastic system simulations. Since their introduction 13 years ago, significant work has been done to extend the applicability of FDEs to regression analysis (Sanchez and Buss 1987), (Sargent, Som and Schruben 1987), to simulation optimization (Morrice and Schruben 1989), (Schruben 1986) and gradient estimation (Jacobson 1990).

FDEs are based on an intuitive "ω-in/ω-out" or frequency-invariance model *assumption*—if a particular system response statistic is sensitive to a system parameter, then sinusoidal variation of that system parameter at a fixed frequency will induce similar sinusoidal variations in the response statistic, at the same frequency. Spectral (Fourier) analysis of these induced oscillations is then used to characterize and analyze the system. There is, however, a lack of theoretical support for this linear-system-type FDE model assumption. In section 2 some of that theoretical support is provided—an equation is presented which accurately characterizes the extent to which the departure rate from a $M/G/1$ queuing system can be modeled as an amplitude-modulated, phase-shifted version of the oscillated arrival rate. This equation defines the frequency response of the system.

Traditionally, the frequency response of a system is represented by two plots—amplitude versus frequency and phase angle versus frequency—together known as a *Bode plot.* Although Bode plots are commonly used to characterize the behavior of linear systems, they have not been used to characterize a nonstationary $M/G/1$ queue. In section 3, a method of generating Bode plots using the FDE Histogram method (Mitra and Park 1991) is discussed. The

Bode plots generated in this way are compared with the analytical expression for the frequency response. The comparison indicates that the analytical expression for the frequency response of a nonstationary $M/G/1$ queue given in section 2 matches the simulation results, provided the queue does not saturate. The Bode plots support the characterization of a nonstationary $M/G/1$ queue as a low-pass filter.

2 FREQUENCY RESPONSE OF THE $M/G/1$ QUEUE

In this section we provide some mathematical support for the "ω-in/ω-out" or frequency-invariance FDE model assumption as it applies to a $M/G/1$ queue with sinusoidally varying arrival rate

$$\lambda(t) = \lambda_0 + \alpha \cos(2\pi\omega t) \tag{1}$$

and fixed service rate μ. In this equation λ_0 is the nominal arrival rate, α is the amplitude of oscillation, ω is the frequency of oscillation and t is the simulation clock time. The corresponding departure rate (as given by Cohen (Cohen 1969)) from such a system is

$$\xi(t) = \mu\Big(1 - P_0(t)\Big), \tag{2}$$

where $P_0(t)$ is the probability of a free server at time t.

Because the queue is nonstationary, an exact analytical expression for $P_0(t)$ does not exist. However, in (Hazra 1993) an *approximate* expression for $P_0(t)$ was established for the case when the maximum amplitude of oscillation of $\lambda(t)$ is less than μ; i.e., when $\lambda_0 + \alpha < \mu$. That is, in this case

$$\begin{aligned} P_0(t) &= \left(1 - \frac{\lambda_0}{\mu}\right) - \frac{\alpha}{\mu}\frac{1}{\sqrt{1+(\omega/\omega_c)^2}} \times \\ &\quad \cos(2\pi\omega t - \tan^{-1}(\omega/\omega_c)) \end{aligned} \tag{3}$$

where

$$\omega_c = \frac{(\lambda_0 - \mu)^2}{2\pi\mu} = \frac{(1-\rho_0)^2}{2\pi} \tag{4}$$

and $\rho_0 = \lambda_0/\mu$ is the average system utilization. This approximate expression was established for an $M/M/1$ queue; a large amount of experimental study (as discussed in the next section) verified that the expression is valid in the more general context of an $M/G/1$ queue. Moreover, this expression is consistent with recently published light-traffic results for the $M/G/\infty$ queue (Eick, Massey and Whitt 1993).

An equation for the departure rate can then be obtained by substituting equation (3) in equation (2) to yield

$$\xi(t) = \lambda_0 + \alpha\frac{\cos(2\pi\omega t - \tan^{-1}(\omega/\omega_c))}{\sqrt{1+(\omega/\omega_c)^2}} \tag{5}$$

provided $\lambda_0 + \alpha < \mu$. Equation (5) validates the frequency-invariance FDE assumption by demonstrating that, provided the maximum amplitude of oscillation of the arrival rate is less than the service rate, the departure rate is an amplitude modulated, phase-shifted version of the arrival rate. In that sense, the nonstationary $M/G/1$ queue behaves as a linear system.

A linear system is characterized by its frequency response or, equivalently, its frequency transfer function (Liu and Liu 1975). In this case, the $M/G/1$ queue frequency response, $H(\omega)$, is the ratio of the variational part of the departure rate to the variational part of the arrival rate, as a function of ω. That is,

$$H(\omega) = \frac{e^{i\tan^{-1}(\omega/\omega_c)}}{\sqrt{1+(\omega/\omega_c)^2}} \tag{6}$$

where ω_c is given by equation (4).

Figure (1) is a plot of the amplitude, $|H(\omega)|$, versus ω and the phase, $\angle H(\omega)$, versus ω—a system Bode plot for two different values of average utilization. The Bode plots illustrate two things.

- Low frequencies are passed unaltered by the system, but high frequencies are filtered out; i.e., the $M/G/1$ queue behaves as a low-pass filter. Traditional FDE practitioners had intuitively recognized this low-pass filter characteristic and had therefore selected driving frequencies at the low end of the frequency spectrum. The Bode plots provide analytical and visual support for the low-pass filter characterization of the system.

- The system's pass band decreases with an increase in the average utilization of the system. This can be explained using equations (6) and (4). From equation (4) it follows that as the average utilization ρ_0 increases, ω_c decreases (since $\rho_0 < 1$). Equation (6) indicates that when ω_c decreases, $|H(\omega)|$ decreases, thereby resulting in a smaller pass band for higher ρ_0.

3 GENERATING BODE PLOTS USING THE FDE HISTOGRAM METHOD

Bode plots for the $M/G/1$ queue can also be obtained experimentally by performing a FDE. In this paper the FDE Histogram method proposed by Mitra et. al

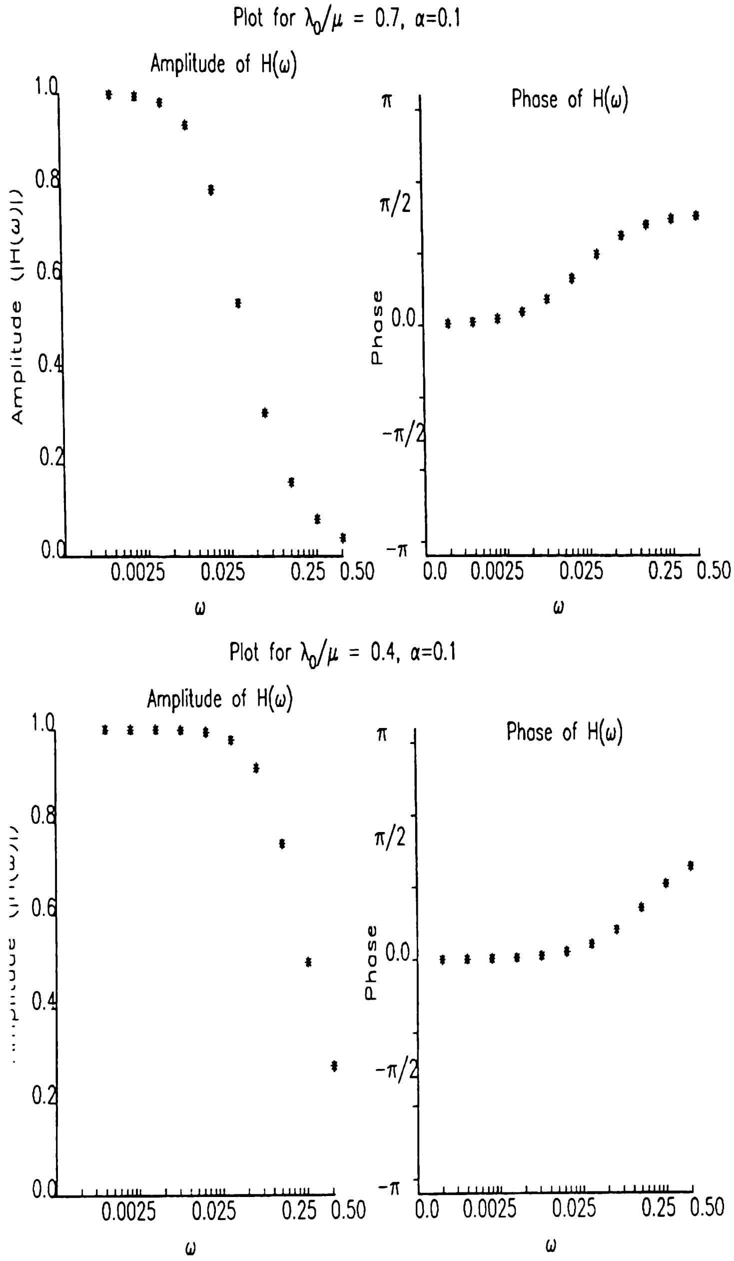

Figure 1: Bode plots for the frequency response of a $M/G/1$ queue using analytical results.

(Mitra and Park 1991) and Hazra (Hazra 93) is used to perform the FDEs. As in the previous section, jobs arrive as a non-stationary Poisson process, with an arrival rate

$$\lambda(t) = \lambda_0 + \alpha \cos(2\pi\omega t) \qquad (7)$$

where λ_0 is the nominal value, α is the amplitude of oscillation and ω is the frequency of oscillation and t is the simulation clock time. The thinning method is used to simulate the arrival process. A fixed service rate μ is used with $\lambda_0 + \alpha < \mu$.

The FDE output is a response sequence of observations corresponding to the departure rate from the system. For each value of ω of interest, the FDE Histogram method is used to build histograms of the number of arrivals to and the number of departures from the system. For each replication, the values of λ_0, α and μ are unaltered. After a selected number of replications, S, an arrival rate sequence is estimated by dividing the number of events in each bin of the arrival histogram by S and the bin width. Similarly, a departure rate sequence is estimated by dividing each bin of the departure histogram by S and the histogram bin width. The discrete Fourier transform (DFT) of the arrival rate sequence and departure rate sequence are then calculated to give the arrival spectrum and the departure spectrum respectively. The ratio of the magnitude of the departure spectrum to that of the arrival spectrum gives $|H(\omega)|$ while the difference between their phase angles give $\angle H(\omega)$.

Bode plots obtained by using the FDE Histogram method for different values of utilization are compared with those given by the frequency response (equation (6)); figure (2) shows the amplitude plots and 3 the phase plots for different values of $\lambda_0/\mu =$ 0.1, 0.3, 0.5, 0.7, with $\alpha = 0.1$ and $\lambda_0 = 1.0$. The figures indicate that the analytical solution for $|H(\omega)|$ given by equation (6) matches the simulation results even for high average utilization, while the analytical solution for $\angle H(\omega)$ matches the simulation results only for small utilization and ω. This is not a major problem, however, since, in FDEs all the important system information is obtained from the amplitude of the response. Like the Bode plots generated by using the analytical solution for $H(\omega)$, the Bode plots generated by using the FDE Histogram method confirms two things.

- The nonstationary $M/G/1$ queue behaves as a low-pass filter, provided the maximum amplitude of oscillation of the arrival rate is less than the service rate.

- The system's pass band decreases with increase in the average system utilization.

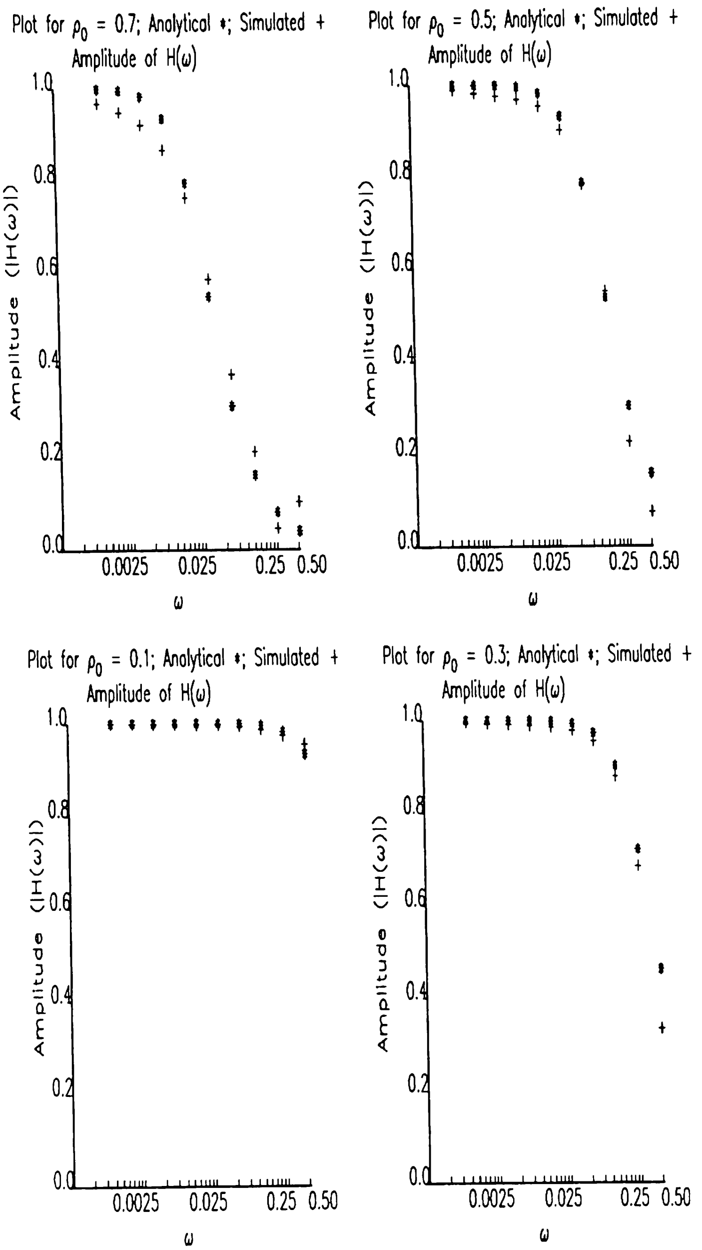

Figure 2: Amplitude plots for the frequency response of a $M/G/1$ queue comparing simulated and analytical results.

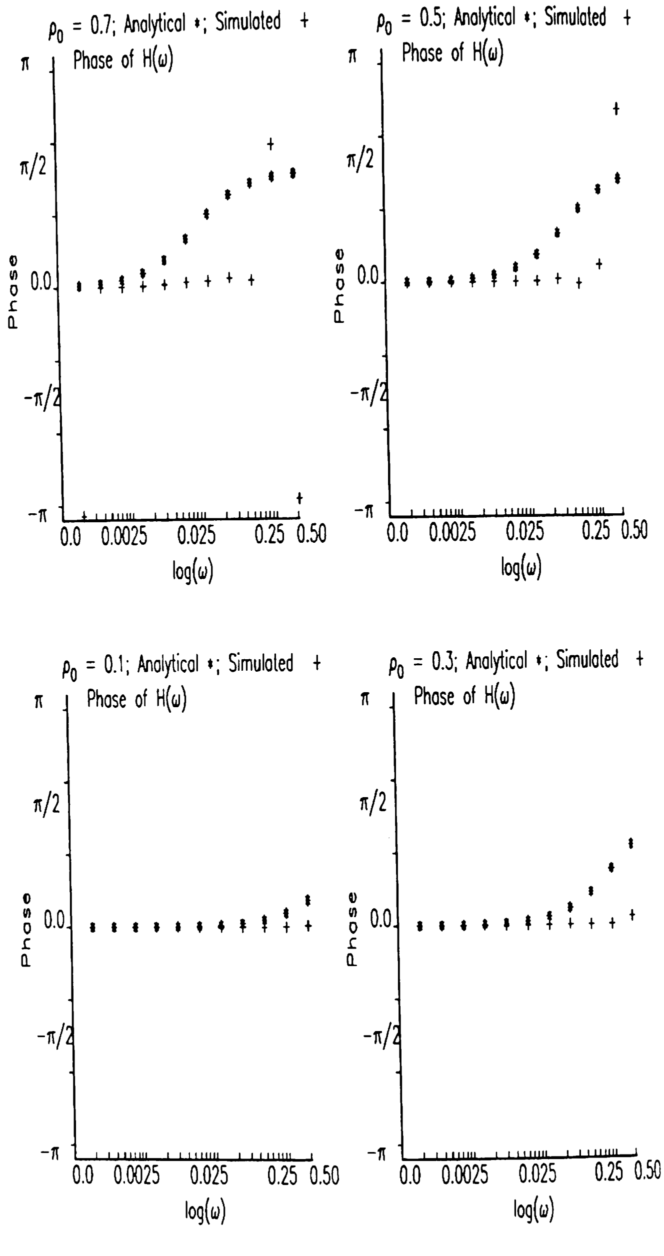

Figure 3: Phase plots for the frequency response of a $M/G/1$ queue comparing simulated and analytical results.

4 CONCLUSIONS

In this paper we have provided some theoretical and experimental support for the "ω-in/ω-out" frequency-invariance FDE assumption for a non-stationary $M/G/1$ queue with sinusoidally varying arrival rate and fixed service rate. Specifically an analytical expression for the frequency response of such a queue was presented. This expression for the frequency response was used to generate system Bode plots. An experimental method of using FDEs to generate Bode plots was also discussed. The experimentally generated Bode plots and the analytical expression for the system frequency response were compared.

Traditionally, Bode plots have been used for characterizing the behavior of linear systems. To the best of our knowledge never before have Bode plots been used to characterize a $M/G/1$ queue. The practical significance of this is fairly profound—it opens up avenues of FDE-based (Bode plot) modeling and (Fourier) analysis of such systems common in a wide variety of applications including communication networks and distributed systems. Analysis tools for such systems today are limited and the potential for FDE-based (Bode plot) analysis is tremendous.

REFERENCES

Cohen, J. W. 1969. *The Single Server Queue.* John Wiley and Sons, Inc., New York, 1969.

Eick, S. G., Massey, W. A. and Whitt, W. 1993 $M_t/G/\infty$ Queues With Sinusoidal Arrival Rates *Management Science*, Volume 39, Number 2, February 1993.

Hazra M. M. 1993. *Fourier Analysis of Frequency Domain Discrete-Event Simulation Experiments.* Ph.D Dissertation, Department of Computer Science, College of William & Mary, Williamsburg, VA, 1993.

Jacobson, S.H. 1990. Input/output Model Extension for Gradient Estimation in the Frequency Domain. *29th. TIMS/ORSA Joint National Meeting, Las Vegas*, 1990.

Liu, C.L. and Liu J. W. S. 1975. *Linear System Analysis.* McGraw-Hill Book Company, 1975.

Mitra, M. and Park, S. K. 1991. Solution to the Indexing Problem of Frequency Domain Simulation Experiments. *1991 Winter Simulation Conference Proceedings*, pages 907–915, 1991.

Morrice, D. J. and Schruben, L. W. 1989. Simulation Sensitivity Analysis Using Frequency Domain Experiments. *1989 Winter Simulation Conference Proceedings*, pages 367–363, 1989.

Sanchez P.J. and Buss, A. H. 1987. A Model for Frequency Domain Experiments. *1987 Winter Simulation Conference Proceedings*, pages 424–427, 1987.

Sargent, R.G., Som, T.K. and Schruben, L.W. 1987. Frequency Domain Modeling of a Feedback Queue. *1987 Winter Simulation Conference Proceedings*, pages 419–423, 1987.

Schruben, L. W. and Cogliano, V. J. 1981. Simulation Sensitivity Analysis: A Frequency Domain Approach. *1981 Winter Simulation Conference Proceedings*, pages 455–459, 1981.

Schruben, L. W. and Cogliano, V. J. 1987. An Experimental Procedure for Simulation Response Surface Model Identification. In *Communications of the ACM*, August 1987, Volume 30, Number 8, pages 716–730.

Schruben, L. W. 1986. Simulation Optimization Using Frequency Domain Methods. *1986 Winter Simulation Conference Proceedings*, pages 366–369, 1986.

AUTHOR BIOGRAPHIES

MOUSUMI MITRA HAZRA Mousumi Mitra Hazra has been a Research Engineer for Tektronix, Inc. since March, 1994. Prior to this, she was a Senior Programmer Associate for the Spacecraft Controls Branch at NASA Langley Research Center. She received a Bachelor's degree in Computer Science and Engineering from Jadavpur University, India in 1987, a Masters degree in Computer Science in 1989 from the College of William and Mary and a doctorate in Computer Science in 1993, also from the College of William and Mary. Her interests include simulation and modeling and linear-system analysis.

Author's present address: Mousumi Mitra Hazra, Tektronix, Inc, P.O. Box 500, MS 50-320, Beaverton, OR 97077 (or, by e-mail, mousumi.hazra@tek.com).

STEPHEN K. PARK Steve Park has been a professor of computer science at the College of William & Mary since 1986 and the chair of that department since 1991. Prior to 1986, he was a member of the research staff at NASA, Langley Research Center. He received a doctorate of applied mathematics from North Carolina State University in 1970.

Dr. Park is a member of several professional societies including the Association of Computing Machinery, the IEEE Computer Society and SPIE. His research interests include: the modeling, simulation and performance analysis of sampled imaging systems; algorithms for the reconstruction and restoration of sampled data; and stochastic methods in discrete-event systems simulation. He has published more than 75 research articles in these and related areas.

Author's present address: Stephen K. Park, Department of Computer Science, College of William and Mary, Williamsburg VA 23187 (or, by e-mail, park@cs.wm.edu).