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# APPLICATION OF RPA AND THE HARMONIC GRADIENT ESTIMATORS TO A PRIORITY QUEUEING SYSTEM

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## ABSTRACT

We consider a queueing system with $C$ customer classes under a nonpreemptive service discipline. The goal is to find gradient estimators for the stationary average sojourn time per customer of each class under admission control. Due to the service discipline, IPA estimators are not applicable. We present the idea of harmonic gradient (HG) estimation, based on the Fourier decomposition of periodic functions. The canonical estimators can be used to obtain consistent estimators for all the control variables in a single run. However, the large number of values for each parameter required in the estimation can greatly affect the performance. We then describe the implementations of the phantom RPA method. This method requires evaluating, in parallel, the dynamics of as many phantom systems as customers in each busy period. Since this number is random, the implementation of the method can be rather complex. We use the Fourier decomposition ideas to construct a hybrid estimator that we call the phantom HG method. We then give simulation results to compare the performance of the estimators and their complexity.

## 1 PROBLEM FORMULATION

This paper focuses on a single server queue with a non-preemptive service discipline. Customers of class $c$ have priority over customers of class $b > c$, for $b, c = 1\ldots, C$. In our general model, $N_c(t), c = 1,\ldots,C$ are independent renewal processes representing the arrivals of each customer class. The service requirements of customers of each class are represented by a sequence of *i.i.d.* random variables with bounded second moments. We assume that the process operates at values of the arrival rates, $\lambda_c$, and mean service times such that the queueing system is stable. By this we mean that the invariant measure exists, and that it is unique and ergodic.

The customers in the system are labeled $i = 1, 2, \ldots$ according to the order in which they depart from the queue. Let $\tau_i$ be the arrival epoch of customer $i$ and $n(i) = \sum_j I_{\{\tau_j \le \tau_i\}}$. Then customer $i$ is the $n(i)$-th customer that arrives to the system. Clearly, if $i$ starts a busy period, then $n(i) = i$. Indeed, all customers arriving within a busy period also depart by the end of it, although in different order. Call $I_c(n)$ the indicator function that the $n$-th customer that arrives to the system is of class $c = 1,\ldots,C$. We use the notation $\mathcal{I} = \{I_c[n(i)]\}_{c=1\ldots,C;i=1,\ldots,N}$, where $i$ indexes the service completions.

The system operates under parallel admission control strategies, namely, each arrival demanding entry to the system may be accepted or rejected. The rates $\lambda_c$ thus represent the effective arrival rates, after thinning. We assume the simplest model for the admission control strategy, where the decisions of accepting customers of class $c$ are independent Bernoulli variables with parameter $p_c$. Our results can be extended to Markov decision and other strategies (see Vázquez-Abad and Kushner, 1990) via the surrogate estimation approach of Vázquez-Abad and Kushner (1993).

We are interested in estimating the sensitivity of the system's response function $F(\lambda)$, for $\lambda = (\lambda_1, \ldots, \lambda_C)$, with respect to the effective rates $\lambda_c$. Note that since the system is under admission control, strictly speaking, then the sensitivity is not with respect to the arrival rates. That is, the derivative with respect to $\lambda_c$ is defined as the derivative with respect to $p_c$ evaluated at $p_c = 1$.

Call $X_c(\lambda)$ the stationary average sojourn time per customer of class $c$. We shall use as our model example a response function of the form $F(\lambda) = \sum_{b=1}^{C} K_b \lambda_b X_b(\lambda)$, where $K_b, b = 1,\ldots,C$ are constants. We assume knowledge of the rates $\lambda_c$. Since

$$\frac{\partial}{\partial \lambda_c} \lambda_b X_b(\lambda) = \lambda_b \frac{\partial}{\partial \lambda_c} X_b(\lambda) + I_{\{b=c\}} X_b(\lambda),$$

then, in order to estimate $\partial F(\lambda)/\partial \lambda_c$, we can either construct the sensitivities of $\lambda_b X_b(\lambda), b = 1, \ldots, C$ with respect to each $\lambda_c$, or else estimate $X_b(\lambda)$ and its sensitivities with respect to each $\lambda_c$, for $b = 1, \ldots, C$.

Call $X_i$ the sojourn time of the $i$-th customer departing the system. Clearly, the sample average $(\lambda/N)\sum_{i=1}^{N} X_i I_c[n(i)]$ over a fixed number $N$ of (total) customer departures is a consistent estimator of $\lambda_c X_c(\lambda)$. This follows because the ratio of the number of customers of class $c$ to the total number of customers tends to $\lambda_c/\lambda$ almost surely. Also, the ergodicity of the unique invariant measure implies almost sure convergence of the sample sojourn times per class to its stationary average.

Estimation of the desired gradient using a single run simulation presents several problems. First, even if we assume that $\lambda_c$ is a scale parameter of the interarrival times per class, IPA (Ho and Cao, 1991) cannot be applied to this problem (Glasserman, 1991). Other methods such as the likelihood ratio method of Reiman and Weiss (1989) can be applied to the Bernoulli decision variables, but it cannot be used for evaluating the sensitivities at $p = 1$. The SPA method of Gong (1988) can be applied to this problem and it is equivalent to the application of the RPA method of Brémaud and Vázquez-Abad (1992). We shall construct the RPA estimators for this problem, though they may require a significant computational effort. Methods that require perturbing the system, such as the finite difference approach in Kushner and Vázquez-Abad (1994) and the harmonic frequency domain estimators of Jacobson and Schruben (1994) may result in added estimator bias when performed in a single run, by varying the effective arrival rates. Furthermore, these perturbations, while straightforward to implement for simulation experiments, are undesirable for on-line adaptive control.

## 2 FOURIER DECOMPOSITION AND THE HARMONIC GRADIENT

**Fourier Decomposition:** We briefly present the Fourier decomposition for a deterministic, scalar function $X(\lambda)$ of a parameter $\lambda \in I\!R$, whose derivative we want to estimate. We are looking for an expression that relates $\partial X/\partial\lambda$ to the "output" observed from the system. We assume that $X(\lambda)$ is smooth. Under these conditions, let $F(t) = X(\lambda_0 + h(t))$, for $h(t) = a\sin(2\pi t)$. Then $F(t)$ is a periodic function on $[0,1]$ and has the representation $F(t) = F(0) + \sum_{n=1}^{\infty} f_n \sin(n\pi t)$, where $f_n = 2\int_0^1 F(t)\sin(n\pi t)\,dt$. Since $F'(t) = \sum_{n=1}^{\infty}(n\pi f_n)\cos(n\pi t)$, then $f_n = \frac{2}{n\pi}\int_0^1 F'(t)\cos(n\pi t)\,dt$. Using a Taylor series expansion for $F'(t)$, it follows that

$$F'(t) = a2\pi\cos(2\pi t) \times \left[\frac{\partial X}{\partial\lambda}(\lambda_0) + h(t)\frac{\partial^2 X}{\partial\lambda^2}(\lambda_0 + h(t)) + \mathcal{O}(a^2)\right]$$

and therefore,

$$f_2 = a\frac{\partial X}{\partial\lambda}(\lambda_0) + \mathcal{O}(a^3) \tag{1}$$

for $a$ close to 0, since $2\int_0^1 \cos^2(n\pi t)\,dt = 1$, and $\int_0^1 \sin(2\pi t)\cos^2(2\pi t)\,dt = 0$.

Equating the Fourier coefficient $f_2$ of $F(t)$ and (1),

$$\frac{\partial X}{\partial\lambda}(\lambda_0) = \frac{2}{a}\int_0^1 X(\lambda_0 + a\sin(2\pi t))\sin(2\pi t)\,dt + \mathcal{O}(a^2) \tag{2}$$

The last expression allows us to estimate a derivative based on varying the input variable $\lambda$ as a function of time and observing the response from the system. Suppose that we could actually evaluate the steady state average $X(\lambda_0 + h(t))$ from a simulation. Due to the properties of the trigonometric functions, the integral in (2) is equal to

$$D_T(a) = \frac{2}{aT}\sum_{t=0}^{T-1} X(\lambda_0 + a\sin(2\pi\omega t))\sin(2\pi\omega t) \tag{3}$$

for any $T$, where $\omega = 1/T$. It then follows that $D_T(a)$ approximates the desired derivative up to $\mathcal{O}(a^2)$. The form (3) represents a weighted average of centered finite differences with sinusoidally varying step sizes. In order to see this, consider $T = 2M + 1$, for $M$ an integer. Then, since $\sin(2\pi\omega t) = -\sin(2\pi\omega[(2M + 1 - t)]$ for $t = 1, \ldots, M$ and $\sin(0) = 0$, (3) can be rewritten as

$$D_T(a) = \frac{4}{T}\sum_{t=1}^{M}\left(\frac{X(\lambda_0 + h(t)) - X(\lambda_0 - h(t))}{2h(t)}\right)\sin^2(2\pi\omega t)$$

where $(4/T)\sum_{t=1}^{M}\sin^2(2\pi\omega t) = 1$. Note that for $T = 3$, the sum reduces to a single term.

**Harmonic Gradient Estimation:** In frequency domain experiments, the input parameters are varied simultaneously during a simulation run by the expressions $\lambda_c(t) = \lambda_c(0) + a_c\sin(2\pi\omega_c t)$, for $c = 1, \ldots, C; t = 0, \ldots, T$, where $\omega_c$ are the oscillation frequencies, $a_c$ are the oscillation amplitudes, and $T$ is the simulation run length. The oscillation frequencies are typically chosen to be Fourier frequencies (i.e., $\omega_c = h_c/T, h_c \in \{1, \ldots, \lceil T/2\rceil\}$).

The harmonic gradient estimator is defined as

$$\hat{A}_c(a,\omega) = \frac{2}{a_c T} \sum_{t=0}^{T-1} X(\lambda(t)) \sin(2\pi\omega_c t).$$

Assuming that the steady state simulation output process can be locally approximated by a second order Taylor series expansion, Jacobson and Schruben (1994) show that this estimator converges in probability (as $(\omega_c, a_c) \to (0,0)$) to $\partial X(\lambda(0))/\partial\lambda_c$.

If the simulation output process at time index $t$ depends on input parameter values at time indices other than $t$, the oscillation frequencies may be forced close to zero. This is actually the case of the average sojourn time in a queueing system: the arrival rate used to generate the interarrival between the $t$-th and $(t+1)$-st customer of class $c$ is $\lambda_c(0) + a_c \sin(2\pi\omega_c t)$. This affects the observed sojourn time of customers with indices higher than $t$.

A difficult problem of harmonic estimation is the frequency selection problem (see Jacobson, Buss and Schruben, 1991) when gradient estimation ($C > 1$) is desired. This is because we are estimating Fourier coefficients, hence the basis should be chosen so as to extend the analysis in our previous section and be able to estimate the effects of the different control variables without confounding the frequencies.

Finally, a serious problem for the implementation of harmonic estimators to adaptive control is that it requires the system to be perturbed constantly.

## 3 PHANTOM ESTIMATION

We now introduce the concept of a *phantom system.* The notation follows our framework of section 1. The *nominal* system is defined by the input streams of effective arrivals $N_c(t)$, the service time distributions and the service discipline. We will use independence of the input streams and assume that $\lim_{t\to\infty} N_c(t)/t \to \lambda_c$ almost surely.

Let $\eta = \{\eta_n\}_{n=1,2,\ldots}$ be a sequence of binary decision variables. A phantom system is defined by the sequence of interarrival and service requirements, plus a particular sequence $\eta$. The customers for which $\eta_{n(i)} = 0$ are allowed entrance to the phantom system, and those having $\eta_{n(i)} = 1$, called the phantom customers, do not enter the system.

By definition of the phantom system, the effective arrival rates are given by $\tilde{\lambda}_c = \lambda_c p_c$, where $p_c$ is the fraction of customers of class $c$ accepted in the phantom system, that is, $p_c = \lim_{N\to\infty} \sum_{n=1}^{N} I_c(n)(1-\eta_n)/\sum_{n=1}^{N} I_c(n)$.

Let $T_{n(i)} = \tau_i$, $S_{n(i)}$ the service time of customer $i$, whose distribution depends on the class of customer $i$, and $X_i$ its sojourn time. Define $A_i = T_{n(i+1)} - T_{n(i)}$. This quantity may be negative, indicating that customer $i+1$ has higher priority than customer $i$.

In the nominal system, Lindley's equation yields $X_i = [X_{i-1} - A_{i-1}]^+ + S_{n(i)}$, where $[x]^+ = \max(0,x)$.

For the phantom system, the arrival epoch of the $n$-th customer is $T_n$, as it is in the nominal system. Its service time is defined as $\tilde{S}_n = S_n(1-\eta_n)$. Since some of the customers present in the nominal system are no longer present in the phantom system, then the order of service completions is not the same as that of the nominal system, although the order of arrivals is preserved. Let $i$ label the customers in the order of their departures from the phantom system, $\tilde{\tau}_i$ being the arrival time of the $i$-th customer departing the phantom queue. Define $\tilde{n}(i) = \sum_j I_{\{\tilde{\tau}_j > \tilde{\tau}_i\}}$. Then $\tilde{A}_i = T_{\tilde{n}(i+1)} - T_{\tilde{n}(i)}$ represents the difference in arrival times of consecutive departing customers as per the phantom queue. Lindley's equations become

$$\tilde{X}_i(\eta) = [\tilde{X}_{i-1}(\eta) - \tilde{A}_i]^+ + \tilde{S}_{\tilde{n}(i)}, \qquad (4)$$

which represents the actual sojourn time of a customer in the phantom queue, provided that $\eta_{\tilde{n}(i)} = 0$. Otherwise it keeps track of the wait and the interarrivals so that the recursion is valid for customers that follow (see Vázquez-Abad and L'Ecuyer, 1991). In practice or in simulations, each customer is assigned a service time and an arrival epoch. Equation (4) is straightforward to evaluate, if the new order of services can be computed. This is possible from the knowledge of the classes and arrival epochs of each customer. In order to simplify the notation, we shall use $X_i(\eta)$ to denote the sojourn time of the $i$-th customer departing the phantom system determined by $\eta$, so that $X_i(0)$ represents the nominal system, for which $\eta_n = 0, n = 1,2,\ldots$

A useful observation is that the phantom system thus described is *dominated* by the nominal system. By domination we mean that if a busy period finishes in the nominal system, then necessarily the phantom system has finished a busy period. That is, if $i$ starts a busy period in the nominal system, then $n(i) = \tilde{n}(i) = i$, since phantom customers do "enter" the queue in (4) with zero service time. Recall that the sojourn times $X_i(\eta)$ of a customer in the phantom system is only meaningful if $\eta_{\tilde{n}(i)} = 0$.

## 4 THE PHANTOM RPA

We now develop estimators of the desired derivatives, where for ease of notation, the sensitivity parameter $\lambda_1$ is fixed. The extension to other values of $c$ is straightforward. A finite horizon estimator is constructed fixing $N$ as the total number of customer departures. We want to estimate the derivative of the average sojourn time of the customers of class $c$.

We construct a phantom system corresponding to an arrival rate $\lambda_1 - \delta$ by setting $1 - p_1 = P\{\eta_n = 1\} = (\delta/\lambda_1)I_1(n)$. Define the set $E_m \in \{0,1\}^N$ by

$$E_m = \left\{\eta: \sum_{i=1}^{N} \eta_{n(i)}(1 - I_1[n(i)]) = 0, \sum_{i=1}^{N} \eta_{n(i)} = m\right\}$$

so that only customers of class 1 can be phantoms, and there are $m$ of them, for $m = 1, \ldots, N$. Let $N_c = \sum_i I_c[n(i)]$ denote the random number of customers of class $c$ among the first $N$ to complete service. Following the arguments in Brémaud and Vázquez-Abad (1992),

$$P\{\eta \in E_m | N_1\} = \binom{N_1}{m} (1 - p_c)^m p_c^{(N_1 - m)}$$

so that $P\{\eta \in E_1 | \mathcal{I}\} = N_1\delta/\lambda_1$ and $P\{\eta \in E_m | \mathcal{I}\} = \mathcal{O}(\delta^m)$. For small $\delta$, define

$$d_N(\lambda, \delta) = \frac{1}{\delta} E\left\{\frac{1}{N_c}\sum_{i=1}^{N} X_i(0) I_c[n(i)] - \frac{1}{\tilde{N}_c}\sum_{i=1}^{N} X_i(\eta)(1 - \eta_{\tilde{n}(i)}) I_c[\tilde{n}(i)]\right\} \qquad (5)$$

where $\tilde{N}_c = \sum_{i=1}^{N} I_c[\tilde{n}(i)](1 - \eta_{\tilde{n}(i)})$. We do not write down the dependency of $\tilde{N}_c$ on $\eta$. Recall that the indices $i = 1, \ldots, N$ are ordered according to the service completions of the corresponding systems.

We can write the expectation in (5) conditioning on $\eta$ and $\mathcal{I}$ as follows. Call $E_{m,\mathcal{I}}$ the conditional expectation given $\{\eta \in E_m, \mathcal{I}\}$. Then

$$\begin{aligned} E\{X_i(\eta) I_c[\tilde{n}(i)] | \mathcal{I}\} &= \\ &= E\left\{\sum_m E_{m,\mathcal{I}}\{X_i(\eta) I_c[\tilde{n}(i)]\}\right\} \\ &= E\left(\frac{N_1\delta}{\lambda_1}\right) E_{1,\mathcal{I}}\{X_i(\eta) I_c[\tilde{n}(i)]\} + \mathcal{O}(\delta^2) \end{aligned}$$

Now $E_{1,\mathcal{I}}(X_i(\eta) I_c[\tilde{n}(i)]) = E\{(1/N_1) \sum_{j=1}^{N} X_i(j) I_c[\tilde{n}(i)]\, I_1[n(j)] | \mathcal{I}\}$, where $X_i(j), j > 0$ is the sojourn time of the $i$-th customer to complete service in the phantom system determined by $\eta_{\tilde{n}(i)} = I_{\{n(j) = \tilde{n}(i)\}}$ (the index $j$ counts different phantom systems obtained by making each arrival to the nominal system a phantom customer). Let $\bar{\eta}_i(j) = 1 - I_{\{n(j) = \tilde{n}(i)\}}$. Then

$$\begin{aligned} D_N(\lambda) = \lim_{\delta \to 0} d_N(\lambda, \delta) &= \frac{1}{\lambda_1} E\left\{\sum_{j=1}^{N} I_1[n(j)] E\left(\frac{1}{N_c}\sum_{i=1}^{N} X_i(0) I_c[n(i)] \right.\right. \\ &\left.\left. - \frac{1}{\tilde{N}_c}\sum_{i=1}^{N} X_i(j)\bar{\eta}_i(j) I_c[\tilde{n}(i)] \Big| \mathcal{I}\right)\right\} \qquad (6) \end{aligned}$$

is the derivative of the sojourn time of the first customers of class $c$ to complete service by the time of the $N$-th service completion.

Finally, consider the estimator

$$\begin{aligned} y_n(c, N) &= \frac{1}{\lambda_1} \sum_{j=nN+1}^{nN+N} I_1[n(j)] \times \\ &\left(\frac{1}{N_{c,n}(N)} \sum_{i=nN+1}^{nN+N} X_i(0) I_c[n(i)] \right. \\ &\left. - \frac{1}{\tilde{N}_{c,n}(N)} \sum_{i=nN+1}^{nN+N} X_i(j)\bar{\eta}_i(j) I_c[\tilde{n}(i)]\right) \end{aligned}$$

where $N_{c,n}(N) = N_c(nN + N) - N_c(nN + 1) = \sum_{i=nN+1}^{nN+N} I_c[n(i)]$ and $\tilde{N}_{c,n}(N)$ is defined similarly.

The finite horizon derivative in (6) converges to the stationary derivative as $N \to \infty$, since the limits $N \to \infty$ and $\delta \to 0$ are interchangeable in $d_N(\lambda, N)$, provided that the system is stable. Then as $N \to \infty$, $E\{y_0(c, N)\}$ converges to the derivative of the stationary sojourn time of customers of class $c$ with respect to $\lambda_1$. Notice that if $N(n)$ is a random number such that $N(n) \to \infty$ almost surely as $n \to \infty$, then $E\{y_0(c, N(n))\}$ also converges to the same value.

**A "Non-Reset" Estimator:** The estimator $y_n(c, N)$ considers the differences between the nominal and the phantom sojourn times of customers $nN + 1$ to $nN + N$ due to $N_1$ possible phantom systems, given $N$ and $\mathcal{I}$. From the construction of the phantom systems, $X_i(j) = X_i(0)$ for all $i < j$. This follows because if $\tilde{n}(i) \leq n(j)$ then $\tilde{n}(i) = n(i)$, since the nominal and phantom systems coincide. After the ficticious service completion of the phantom customer, services may change their order in the phantom system with respect to that of the nominal. From the domination property, $X_i(j) = X_i(0)$ for all those customers $i$ not belonging to the nominal busy period where $j$ belongs.

We now present a second estimator that does not involve dividing by a random number, and argue that this estimator is more desirable than $y_n(c, N)$. Call $\lambda = \sum_{c=1}^{C} \lambda_c$ the total effective arrival rate, and

$$Y_n^1(c, N) = \frac{\lambda}{\lambda_1 N} \sum_{j=nN+1}^{nN+N}$$

$$\sum_{i=nN+1}^{\alpha(nN+1)} [X_i(0) - X_i(j)\bar{\eta}_i(j)] I_c[n(i)] I_1[n(j)]$$

where $\alpha(i)$ is the index of the last customer in the nominal busy period where customer $i$ belongs. This estimator considers the total differences in the cumulative sojourn times across the whole path, between the nominal and $N$ phantom systems. It can be written as

$$Y_n^1(c,N) = \frac{\lambda}{\lambda_1 N} \sum_{j=nN+1}^{nN+N} I_1[n(j)] \sum_{i=\phi(j)}^{\alpha(j)} [X_i(0) - X_i(j)\bar{\eta}_i(j)] I_c[n(i)] \quad (7)$$

where $\phi(j)$ is the index of the first customer in the busy period where customer $j$ belongs. This follows because customers not belonging to the nominal busy period where the phantom customer belongs have the same sojourn time in the nominal and phantom systems.

It is shown in Vázquez-Abad (1994) that $Y_n^1(c,N)$ is consistent in the average sense of Kushner and Vázquez-Abad (1994) for the derivative of $\lambda_c X_c(\lambda)$, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} E(Y_m^1(c,N)) = \frac{\partial}{\partial \lambda_1}[\lambda_c X_c(\lambda)]. \quad (8)$$

This result is true for any value of $N$. Since we are no longer dividing by a random number, (7) is additive in the sense that $\sum_{m=0}^{n-1} Y_n^1(c,N) = Y_0^1(c,nN)$. In practical optimization problems, property (8) can be used to construct stochastic approximation algorithms that are asymptotically optimal, using a fixed value of $N$ for adaptive control. For the estimator $y_n(c,N)$, a stochastic approximation procedure would require the estimation interval lengths $N$ to increase in order to approximate the optimal performance. This is not practical for adaptive control problems.

Due to the indicator functions $I_1(j)$, the amount of computation required to estimate the gradient of a function is comparable to the amount of computation to estimate the derivative with respect to the total arrival rate. That is, given the utilization factors of the system, the computational effort does not increase with $C$. However, the phantom systems are computed using Lindley's equations and this requires keeping the same number of registers as phantom systems. For each nominal busy period, the number of phantom systems is the same as the (random) number of customers in that busy period. This may become large, when the utilization factor is close to one.

## 5 THE PHANTOM HG METHOD

We now present a hybrid estimation procedure based on (3) using the phantom systems to evaluate, in parallel, the values of the stationary sojourn times $X_c(\lambda_c + a\sin(2\pi\omega t))$. For ease of notation, consider the estimation of the sensitivities with respect to $\lambda_1$. The nominal system has an arrival rate of $\lambda_1(0) + a$, where $a > 0$ is small. We assume that the system is stable for all $\lambda_1 \le \lambda_1(0) + a$. Let $T$ be a fixed integer, $\omega = 1/T$ and define:

$$p_t = \frac{\lambda_1(0) + a\sin(2\pi\omega t)}{\lambda_1(0) + a} \quad \text{for } t = 1, \ldots, T-1$$

**Common Random Numbers:** Let $M$ be an integer and take $T = 2M+1$. A total of $2M$ phantom systems are calculated from the nominal path as follows. For the $n$-th arrival to the nominal system, generate one uniform random variable $u_n$. This random variable is used to determine the $2M$ phantom systems by assigning $\eta_n(t) = I_1(n) I_{\{u_n > p_t\}}, t = 1, \ldots, 2M$.

The phantom systems corresponding to the finite differences (indexed by $t$ and $T-t$, for $t = 1, \ldots, 2M$) use common random variables. The use of common random numbers in the phantom finite difference addresses the problem of variance reduction. For the case $M = 1$, this scheme reduces to the phantom finite difference scheme introduced in Vázquez-Abad (1989).

**Common and Antithetic Random Numbers:** We define the phantom HG estimators exploiting the combined use of common random numbers for the *paired* systems in the finite differences, and antithetic random numbers for *adjacent* systems.

Let $U$ be an integer and let $M = 2U, T = 2M + 1$. Upon arrival of customer $n$ into the nominal system, we generate the uniform variates $\{u_n(t), t = 1, \ldots U\}$. We then define for each $t = 1, \ldots, U$:

$$\begin{aligned}
\eta_n(t) &= I_1(n) I_{\{u_n(t) > p_t\}} \\
\eta_n(T-t) &= I_1(n) I_{\{u_n(t) > p_{(T-t)}\}} \\
\eta_n(U+t) &= I_1(n) I_{\{1-u_n(t) > p_{(U+t)}\}} \\
\eta_n(T-U-t) &= I_1(n) I_{\{1-u_n(t) > p_{(T-U-t)}\}}
\end{aligned}$$

Small values of the amplitude $a$ are desirable for accuracy, but the difference between the corresponding values of $\lambda_1(t)$ and $\lambda_1(M+1-t)$ would be very small. The use of the same random number for the admission variables would result in many busy periods were these systems have exactly the same trajectory. Antithetic random numbers ensure the distinction between these different points; common random numbers are kept for the finite differences.

**The Estimators:** The phantom systems have an effective arrival rate $\lambda_1(t) = \lambda_1(0) + a\sin(2\pi\omega t)$, $\lambda_c(t) = \lambda_c(0), c \neq 1$.

Let $N$ be the total number of customers served within our estimation period for the nominal system. Define $\lambda = \sum_c \lambda_c(0) + a$ to be the total arrival rate. Let $X_i(t)$ be the sojourn time of the $i$-th customer leaving the phantom system parameterized by $t$, and $\bar{\eta}_i(t) = 1 - \eta_{\tilde{n}(i)}(t)$. The estimator is then constructed as:

$$Y_n^2(c,N) = \frac{2\lambda}{aTN}\sum_{t=1}^{2M}\sin(2\pi\omega t)\sum_{i=nN+1}^{nN+N} X_i(t)\bar{\eta}_i(t)I_c[\tilde{n}(i)] \tag{9}$$

Since the phantom systems appear in (9) in an additive way, then for each $t$,

$$\frac{1}{n}\sum_{m=0}^{n-1}\frac{\lambda}{N}\sum_{i=nN+1}^{nN+N} X_i(t)\bar{\eta}_i(t)I_c[\tilde{n}(i)] =$$

$$\left(\frac{\lambda N_c(nN,t)}{nN}\right)\frac{1}{N_c(nN,t)}\sum_{i=1}^{nN} X_i(t)\bar{\eta}_i(t)I_c[\tilde{n}(i)],$$

where $N_c(nN,t) = \sum_{i=1}^{nN} I_c[\tilde{n}(i)]\bar{\eta}_i(t)$ is the number of class $c$ customers among the first $nN$ to complete service in the phantom system labelled by $t$. For any $N$, we have $\lim_{n\to\infty} N_c(nN,t)/N = (\lambda_c(t)/\lambda)$ almost surely. Also, the sample average of the sojourn times $X_i(t)\bar{\eta}_i(t)I_c[\tilde{n}(i)]$ converges almost surely to the stationary average $X_c(\lambda(t))$, where $\lambda(t)$ has $\lambda_1(t) = \lambda_1(0) + a\sin(2\pi\omega t)$ and $\lambda_c(t), c = 2,\ldots,C$ are constant.

This estimator is consistent in the average sense of Kushner and Vázquez-Abad (1994) for the weighted finite difference (3), that is,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1} E(Y_m^2(c,N)) = \frac{\partial}{\partial\lambda_1}[\lambda_c X_c(\lambda)] + \mathcal{O}(a^2)$$

Since the perturbations are now performed in parallel, most of the common problems of harmonic estimation that are related to the undesired effects of perturbing the system are no longer present. In particualr, the oscillation frequencies $\omega_c$ do not have to approach zero, since we have consistency of the estimation of the stationary average sojourn times for each phantom system. Also, to estimate the gradients, the same frequencies can be used for all the control variables $\lambda_c, c = 1,\ldots,C$. In this case, $2MC$ are constructed. While the effort here does increase explicitly with $C$, the number of parallel phantom systems needed simultaneously is always fixed. In most applications, the utilization factor may be close to one, while the number of different classes of customers is typically not very large. Therefore, the phantom HG may yield a more practical estimator, albeit biased.

## 6 SIMULATION RESULTS

We present the results from the simulation of a single server queue with two customer classes and a non-preemptive service discipline. Class 1 customers have higher priority than class 2 customers. The arrival processes are Poisson with parameter $\lambda_i$ for class $i$, and the services are exponential with parameter $\mu = 1.0 > \lambda_0 = \lambda_1 + \lambda_2$. For this model, the theoretical values of the stationary averages and their gradients can be evaluated analytically. All the simulations were performed on a PC (486, 33 MHz, 8M).

Three systems were simulated. In our first system, $\rho = 0.60$ with $\lambda_1 = 0.45$ and $\lambda_2 = 0.15$. In the second, $\rho = 0.75$ with $\lambda_1 = 0.60$ and $\lambda_2 = 0.15$. In the third, $\rho = 0.85$ with $\lambda_1 = 0.65$ and $\lambda_2 = 0.20$.

We used a total of $N$ number of customers and performed a Monte Carlo simulation accumulating the values (non-reset estimation) to obtain the estimates $\{Y_n^k(i,c,N), n = 1,\ldots,S\}$, $S = 50$, for method $k = 1,2$. The sample mean was calculated, as well as the sample mean square error

$$\text{MSE} = (1/S)\sum_{n=1}^{S}[Y_n^k(i,c,N) - \frac{\partial(\lambda_c X_c(\lambda))}{\partial\lambda_i}]^2$$

**The Phantom RPA Method:** The results for the Phantom RPA method are reported in table 1. We give the theoretical value $D_i(c) = \partial(\lambda_c X_c(\lambda))/\partial\lambda_i$. We show in this table the mean number of customers per busy period (NBP) as well as the maximum number of customers in a single busy period (Max). We used $N = 3000$.

A comparison between tables 1 and 2 shows the quality of the phantom RPA method. However, this method requires the calculation of as many phantom systems in parallel as customers in each busy period. Even if long busy periods are rare, the code uses all of the memory required. Actually, we could not get the results for $\rho = 0.85$ because the simulation consumed all of the available memory in the computer.

**The Phantom HG Method:** We used $a_c = 0.01\lambda_c$ for the estimation of the derivatives with respect to $\lambda_c$. Recall that when $T = 3$, the phantom HG method reduces to the phantom finite difference (FD). In table 2 we give the results of the phantom HG. We used a fixed value of $N$ for each system (see table 2) for $T = 3, 5, 9$. In our experiments, the variance showed a dramatic reduction in passing from $T = 3$ to $T = 5$

| | $\rho = 0.60, N = 3,000$ Max = 106 | | | $\rho = 0.75, N = 10,000$ Max = 275 | | |
|---|---|---|---|---|---|---|
| | Theoretical | Mean | MSE | Theoretical | Mean | MSE |
| $D_1(1)$ | 3.80 | 3.75 | 0.13 | 7.19 | 7.01 | 0.40 |
| $D_1(2)$ | 2.45 | 2.40 | 0.49 | 8.81 | 8.59 | 3.31 |
| $D_2(1)$ | 0.82 | 0.85 | 0.03 | 1.50 | 1.51 | 0.10 |
| $D_2(2)$ | 5.43 | 5.34 | 1.05 | 14.50 | 14.29 | 4.92 |
| NBP | - | 2.50 | 0.02 | - | 4.00 | 0.02 |

Table 1: Results for the Phantom RPA Method

to $T = 9$. We performed other simulations at higher values of $T$ (up to $T = 51$) that showed marginal variance reduction at the expense of computation time.

We compare the results for $T = 5$ and 9 with the phantom FD ($T = 3$) using higher values of $N$ (shown in table 2). The computational effort of these longer simulation runs using the phantom finite difference is comparable to the computational effort of the phantom HG with $T = 9$ using the smaller $N$. The theoretical values $D_i(c)$ correspond to the theoretical weighted average of the finite differences (3) that approximate $\partial(\lambda_c X_c(\lambda))/\partial\lambda_i$.

It is clear from table 2 that the phantom HG method with $T = 9$ performs no worse than the phantom FD method ($T = 3$) with a larger horizon $N$. In many problems, the code for calculating the parallel phantom systems can be optimized to reduce the time of computation even further. This is the subject of on-going research. The advantages of using a shorter finite horizon $N$ with, say, $T = 9$ are clear: first, it allows for more frequent updating when the estimation is used in the optimization context. Second, if we can implement $U$ parallel processors that can compute the four corresponding phantom systems, the actual time of computation could be greatly reduced. Thus we can improve accuracy in the estimation with shorter runs.

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| $\rho = 0.60$ | | | | | |
|---|---|---|---|---|---|
| | | Phantom FD ($T = 3$) | | Phantom HG ($N = 3,000$) | |
| | | $N = 3,000$ | $N = 6,000$ | $T = 5$ | $T = 9$ |
| | Theoretical | Mean (MSE) | Mean (MSE) | Mean (MSE) | Mean (MSE) |
| $D_1(1)$ | 3.76 | 3.71 (1.47) | 3.50 (0.95) | 3.71 (0.62) | 3.49 (0.36) |
| $D_1(2)$ | 2.42 | 2.29 (1.83) | 2.07 (1.07) | 2.28 (1.06) | 2.15 (0.67) |
| $D_2(1)$ | 0.81 | 1.02 (1.20) | 0.87 (0.54) | 0.86 (0.53) | 0.79 (0.29) |
| $D_2(2)$ | 5.38 | 4.86 (9.80) | 4.88 (3.88) | 4.83 (4.95) | 4.95 (2.93) |
| Time | | 37 sec. | 76 sec. | 45 sec. | 78 sec. |
| $\rho = 0.75$ | | | | | |
| | | Phantom FD ($T = 3$) | | Phantom HG ($N = 10,000$) | |
| | | $N = 10,000$ | $N = 20,000$ | $T = 5$ | $T = 9$ |
| | Theoretical | Mean (MSE) | Mean (MSE) | Mean (MSE) | Mean (MSE) |
| $D_1(1)$ | 7.11 | 6.96 (1.65) | 6.87 (1.07) | 6.94 (1.07) | 6.91 (0.98) |
| $D_1(2)$ | 8.73 | 7.95 (7.06) | 7.39 (5.09) | 7.80 (5.08) | 7.66 (4.84) |
| $D_2(1)$ | 1.49 | 1.36 (1.80) | 1.16 (0.98) | 1.36 (0.86) | 1.34 (0.76) |
| $D_2(2)$ | 14.36 | 13.71 (38.29) | 13.01 (16.22) | 13.78 (17.95) | 13.48 (14.88) |
| Time | | 140 sec. | 276 sec. | 184 sec. | 287 sec. |
| $\rho = 0.85$ | | | | | |
| | | Phantom FD ($T = 3$) | | Phantom HG ($N = 20,000$) | |
| | | $N = 20,000$ | $N = 45,000$ | $T = 5$ | $T = 9$ |
| | Theoretical | Mean (MSE) | Mean (MSE) | Mean (MSE) | Mean (MSE) |
| $D_1(1)$ | 9.70 | 9.46 (2.52) | 9.19 (1.13) | 9.42 (1.69) | 9.51 (0.93) |
| $D_1(2)$ | 34.37 | 30.55 (88.68) | 29.51 (43.59) | 30.07 (72.78) | 31.26 (68.60) |
| $D_2(1)$ | 1.84 | 1.71 (2.74) | 1.90 (1.30) | 1.93 (1.15) | 1.74 (0.87) |
| $D_2(2)$ | 42.17 | 43.00 (342.52) | 39.33 (49.21) | 42.27 (174.85) | 36.30 (92.27) |
| Time | | 322 sec. | 724 sec. | 413 sec. | 708 sec. |

Table 2: Results for the Phantom HG Method

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