Proceedings of the 1993 Winter Simulation Conference
G.W. Evans, M. Mollaghasemi, E.C. Russell, W.E. Biles (eds.)

COMBINED CORRELATION INDUCTION STRATEGIES FOR DESIGNED SIMULATION EXPERIMENTS

Chimyung Kwon

Department of Applied Statistics
Dong-A University
Pusan, Korea 640-714

Jeffrey D. Tew

Department of Industrial and Systems Engineering
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

ABSTRACT

In this paper we examine three methods for combining the variance reduction techniques of antithetic variates and control variates to estimate the mean response in a designed simulation experiment. In Combined Method I, we perform h independent pairs of simulation runs as follows—on the second run of each such pair, we use random number streams that are antithetic (complementary) to the streams used on the first run of the pair to drive the non-control-variate components of the simulation model; and we use independent random number streams to drive the control-variate components of the simulation model. In Combined Method II, we also perform h independent pairs of runs; but on each pair of runs we use independent random number streams to drive the non-control-variate model components, and we use antithetic random number streams to drive the control-variate components. In Combined Method III, all of the random number streams driving the second run of each pair of runs are antithetic to the streams driving the first run of the pair. For each of these three methods we derive the variance of the resulting estimator of the mean response to make a theoretical comparison of the efficiency of each method. We implemented these three methods, along with the classical method of control variates, in a simulation model of a resource-constrained activity network to show how each combined method is implemented in practice and to evaluate the performance of each combined method experimentally. The results indicate that: (a) Combined Method III outperformed all other methods, and (b) the effectiveness of Combined Method III as well as the choice of whether to use Combined Method I or Combined Method II depends on the degree of correlation between the control variates and the response.

1 INTRODUCTION

In this paper we propose three ways of combining the two standard variance reduction techniques of antithetic variates and control variates to yield more precise estimators of the mean response.

For a single-model simulation experiment (i.e., one system configuration or design point), the methods of antithetic variates and control variates are probably among the most commonly applied variance reduction techniques (Law and Kelton 1991, Chapter 11). The method of antithetic variates assigns complementary streams of random numbers to pairs of simulation runs taken at a single design point to induce a negative correlation between the corresponding responses. Let y_1 and y_2 denote two responses obtained by antithetic replicates of a single design point. Suppose that we estimate $\mu_y = \mathrm{E}(y_i)$ $(i = 1, 2)$ by the sample mean response $\frac{1}{2}(y_1 + y_2)$. Then we observe that in general,

$$\mathrm{var}[\frac{1}{2}(y_1 + y_2)] = \frac{1}{2}\mathrm{var}(y_1) + \frac{1}{2}\mathrm{cov}(y_1, y_2).$$

In this equation, if the covariance between y_1 and y_2 obtained by antithetic replicates is negative, then the variance of the sample mean is less than that obtained by two independent replicates (for a more detailed discussion of antithetic variates see Section III.6 of Kleijnen 1974, Section 2.2 of Bratley, Fox, and Schrage 1987, and Section 11.3 of Law and Kelton 1991).

In contrast to the approach of antithetic variates, the method of control variates attempts to exploit *intrinsic* correlations between the target response and selected auxiliary outputs (control variates) when all of these quantities are generated within a single run. Let y_i and $\mathbf{c}_i$ respectively denote the response of interest and the $s \times 1$ vector of control variates obtained from the ith simulation run. We assume that

$(i = 1, 2, \ldots, 2h)$:

$$\begin{bmatrix} y_i \\ \mathbf{c}_i \end{bmatrix} \sim \mathrm{N}_{s+1}\left(\begin{bmatrix} \mu_y \\ \mathbf{0}_{(s\times1)} \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \boldsymbol{\sigma}'_{y\mathbf{C}} \\ \boldsymbol{\sigma}_{y\mathbf{C}} & \boldsymbol{\Sigma}_{\mathbf{C}} \end{bmatrix} \right), \quad (1)$$

where σ_y^2 is the unconditional variance of y_i, $\boldsymbol{\sigma}_{y\mathbf{C}}$ is the $s \times 1$ covariance vector between y_i and $\mathbf{c}_i$, and $\boldsymbol{\Sigma}_{\mathbf{C}}$ is the $s \times s$ covariance matrix of $\mathbf{c}_i$. Also, without loss of generality, we assume $\mathrm{E}(\mathbf{c}_i) = \mathbf{0}_{(s\times1)}$. In the context of performing $2h$ independent replications of the simulation, joint normality of the response and the controls ensures that the response can be represented by the following linear model:

$$\mathbf{y} = \mu_y \mathbf{1}_{(2h\times1)} + \mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\epsilon}, \quad (2)$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_{2h})'$ is the vector of replicated responses, μ_y is the mean response, $\mathbf{1}_{(2h\times1)}$ is a $2h \times 1$ vector of ones, $\mathbf{C}$ is a $2h \times s$ control variate matrix whose ith row consists of $\mathbf{c}_i'$, $\boldsymbol{\alpha} = \boldsymbol{\Sigma}_{\mathbf{C}}^{-1}\boldsymbol{\sigma}_{y\mathbf{C}}$ is the $s \times 1$ vector of control coefficients, and $\boldsymbol{\epsilon}$ is the $2h \times 1$ vector of error terms (see Lavenberg, Moeller, and Welch 1982). The least squares estimators of $\boldsymbol{\alpha}$ and μ_y in the linear model (2) are given by, respectively:

$$\hat{\boldsymbol{\alpha}} = (\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y} \quad \text{and} \quad \hat{\mu}_y = \bar{y} - \bar{\mathbf{c}}'\hat{\boldsymbol{\alpha}}, \quad (3)$$

where $\bar{y} = (2h)^{-1}\mathbf{1}'_{(1\times2h)}\mathbf{y} = (2h)^{-1}\sum_{i=1}^{2h} y_i$ and $\bar{\mathbf{c}}' = (2h)^{-1}\mathbf{1}'_{(1\times2h)}\mathbf{C} = (2h)^{-1}\sum_{i=1}^{2h} \mathbf{c}_i'$ respectively denote the sample mean of the responses and the control vectors computed across $2h$ replications, and $\mathbf{P} = \mathbf{I}_{(2h\times2h)} - \frac{1}{2h}\mathbf{1}_{(2h\times1)}\mathbf{1}'_{(1\times2h)}$ (see Searle 1971, p. 341). Under the normality assumption in (1), the components of the error vector $\boldsymbol{\epsilon}$ are independent identically distributed (IID) variates with distribution $N(0, \sigma_{y|\mathbf{C}}^2)$ where

$$\sigma_{y|\mathbf{C}}^2 = \mathrm{var}(y_i|\mathbf{c}_i) = \sigma_y^2 - \boldsymbol{\sigma}'_{y\mathbf{C}}\boldsymbol{\Sigma}_{\mathbf{C}}^{-1}\boldsymbol{\sigma}_{y\mathbf{C}} \quad (4)$$

(see Theorem 2.5.1 of Anderson 1984); and the least squares estimator $\hat{\mu}_y$ is an unbiased estimator for μ_y. Lavenberg, Moeller, and Welch (1982) showed that the unconditional variance of $\hat{\mu}_y$ is given by

$$\mathrm{var}(\hat{\mu}_y) = \left(\frac{2h-2}{2h-s-2}\right)(1 - R_{y\mathbf{C}}^2)\frac{\sigma_y^2}{2h}, \quad (5)$$

where $R_{y\mathbf{C}}^2 = \sigma_y^{-2}\boldsymbol{\sigma}'_{y\mathbf{C}}\boldsymbol{\Sigma}_{\mathbf{C}}^{-1}\boldsymbol{\sigma}_{y\mathbf{C}}$ is the square of the multiple correlation coefficient between y_i and $\mathbf{c}_i$. Lavenberg, Moeller, and Welch also defined the quantity $\frac{2h-2}{2h-s-2}$ as the loss factor due to the estimation of the unknown control coefficient vector $\boldsymbol{\alpha}$ in (2); and they identified $(1 - R_{y\mathbf{C}}^2)$ as the minimum variance ratio which represents the potential for reducing the variance of the estimator of μ_y by the control variates. Thus, the efficiency of control variates is measured by the product of the loss factor and the minimum variance ratio.

There are three obvious ways to use control variates in conjunction with antithetic variates: (a) apply the antithetic-variates method to the response but not the control variates; (b) apply the antithetic-variates method to the control variates but not the response; and (c) apply the antithetic-variates method to both the response *and* the control variates. These constitute the three combined methods discussed in Section 2.

This paper is organized as follows. Section 2 develops the three combined methods for jointly applying control variates and antithetic variates. Section 3 describes the simulation model that was used in an experimental comparison of the three combined methods as well as the classical methods of control variates and direct simulation (i.e., no variance reduction technique used). Section 4 presents a summary of the experimental performance evaluation of these five techniques. Section 5 presents the conclusions of this research and gives recommendations for future work.

2 SIMULATION EFFICIENCY OF COMBINED METHODS

In computer simulation, random number streams that drive a simulation model are under the control of the experimenter and *completely* determine the simulation output. Let the random number stream $\mathbf{r}_{ij} = (r_{ij1}, r_{ij2}, \ldots)'$ denote the potentially infinite sequence of random numbers used to drive the jth random component in the simulation model at the ith replicate, where $\{r_{ijk} : k = 1, 2, \ldots\}$ are IID $U(0,1)$ variates. The random components in a queueing simulation model may include, among other things, the sequence of service times sampled at a particular service center, the sequence of interarrival times sampled from a given arrival process at a service center, etc. Here, we are saying that, to each random component in the simulation model, we assign a separate random number stream to generate realizations of the corresponding stochastic simulation input process. We assume that g random number streams are required to drive the entire simulation model, and we let $\mathbf{R}_i$ denote the complete set of streams used for the ith replication of the model (for $i = 1, 2, \ldots, 2h$):

$$\mathbf{R}_i = (\mathbf{r}_{i1}, \mathbf{r}_{i2}, \ldots, \mathbf{r}_{ig}).$$

We now consider the random number assignment strategy of jointly utilizing antithetic variates and control variates to estimate the mean response μ_y. To this end, we separate $\mathbf{R}_i$ into two mutually exclusive and exhaustive subsets of random number streams, such that (for $i = 1, 2, \ldots, 2h$)

$$\mathbf{R}_i \equiv (\mathbf{R}_{i1}, \mathbf{R}_{i2}).$$

The first subset, $\mathbf{R}_{i1}$, consisting of $(g - s^\star)$ random number streams, is used to drive the non-control-variate random components in the model so that $\mathbf{c}_i$ is independent of $\mathbf{R}_{i1}$. The second subset, $\mathbf{R}_{i2}$, consisting of $s^\star$ random number streams, is used to drive the control-variate random components in the model so that $\mathbf{c}_i$ is a function of $\mathbf{R}_{i2}$. These properties are summarized as follows (for $i = 1, 2, \ldots, 2h$):

$$\mathbf{c}_i, \mathbf{R}_{i1} \text{ are independent and } \mathbf{c}_i = \mathbf{c}_i(\mathbf{R}_{i2}).$$

Applying the method of antithetic variates to the appropriate random components in the simulation model may induce correlations between: (a) responses, (b) control variates, and (c) responses *and* control variates, across replicates.

Specifically, we consider the following methods: (a) use antithetic variates for all random components except the control variates, (b) use antithetic variates on only the control variates, and (c) use antithetic variates for all random components. Through statistical analysis and simulation experimentation, we will explore how these methods may improve the simulation efficiency in reducing the variance of the estimator, and what conditions are necessary for each method to ensure an improvement in variance reduction.

2.1 Combined Method I

In this subsection, we present a method for combining antithetic variates and control variates based on correlated replicates in which only the non-control-variate random components in the model are used for correlation induction. Recall from the discussion given in the Introduction, the basic idea of this method (as well as the methods presented in the next two subsections) is to group the replicates into h antithetic pairs. Within the jth pair of replicates, Combined Method I uses $(\mathbf{R}_{2j-1,1}, \mathbf{R}_{2j-1,2})$ as the input to run $2j - 1$ and $(\bar{\mathbf{R}}_{2j-1,1}, \mathbf{R}_{2j,2})$ as the input to run $2j$, where $\mathbf{R}_{2j-1,1}$, $\mathbf{R}_{2j-1,2}$, and $\mathbf{R}_{2j,2}$ are mutually independent sets of random number streams; and $\bar{\mathbf{R}}_{2j-1,1}$ is the set of random number streams that are antithetic (complimentary) to those comprising $\mathbf{R}_{2j-1,1}$

$$\bar{\mathbf{R}}_{2j-1,1} \equiv (\bar{\mathbf{r}}_{2j-1,k}),$$

where ($k = 1, 2, \ldots, g - s^\star$ for $j = 1, \ldots, h$) and $\bar{\mathbf{r}}_{2j-1,k}$ denotes the random number stream complimentary to $\mathbf{r}_{2j-1,k}$ so that (for $k = 1, 2, \ldots, g - s^\star$).

$$\bar{\mathbf{r}}_{2j-1,k} \equiv \mathbf{1} - \mathbf{r}_{2j-1,k} = \begin{pmatrix} 1 - r_{2j-1,k,1} \\ 1 - r_{2j-1,k,2} \\ \vdots \end{pmatrix}.$$

Across pairs of replicates, this method uses independent random number streams. Thus, the jth pair of responses, y_{2j-1} and y_{2j} ($j = 1, 2, \ldots, h$), are negatively correlated by the use of antithetic streams on the non-control-variate random components. However, across *all* $2h$ replicates, the control variates $\mathbf{c}_i$ ($i = 1, 2, \ldots, 2h$) are independently generated by the assignment of randomly selected random number streams, $\{\mathbf{R}_{i,2} : i = 1, \ldots, 2h\}$, used to drive the control-variate random components for each replicate. Because we randomly select the random number streams to drive the control variates, the response y_i ($i = 1, 2, \ldots, 2h$) is independent of the control variate vector $\mathbf{c}_k$ when $i \neq k$ ($k = 1, 2, \ldots, 2h$). Based on the above discussion, we have the following properties for Combined Method I:

Property I-1: Homogeneity of response variances across replicates,

$$\text{var}(y_i) = \sigma_y^2 \text{ for } i = 1, 2, \ldots, 2h. \tag{6}$$

Property I-2: Homogeneity of response correlations across replicate pairs and independence of responses observed on different pairs of replications,

$$\text{cov}(y_i, y_k) = \left\{ \begin{array}{l} -\rho_1 \sigma_y^2 \\ \quad i = k + 1 \\ \quad k = 1, 3, \ldots, 2h - 1 \\ 0 \\ \quad \text{otherwise.} \end{array} \right\}. \tag{7}$$

Property I-3: Homogeneity of response–control-variates covariances across replicates and independence of the response and control variates observed on different replications,

$$\text{Cov}(y_i, \mathbf{c}_k) = \left\{ \begin{array}{l} \boldsymbol{\sigma}_{y\mathbf{c}} \\ \quad i = k = 1, 2, \ldots, 2h \\ \mathbf{0}_{(s \times 1)} \\ \quad i \neq k \end{array} \right\}. \tag{8}$$

Property I-4: Homogeneity of control-variates covariances across replicates and independence of control variates observed on different replications,

$$\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_k) = \left\{ \begin{array}{l} \mathbf{\Sigma_C} \\ \quad i = k = 1, 2, \ldots, 2h \\ \mathbf{0}_{(s\times s)} \\ \quad i \neq k \end{array} \right\}. \tag{9}$$

We make the following assumption about Combined Method I:

Assumption I-1: $0 < \rho_1 < 1$.

In view of (7), Assumption I-1 postulates a negative induced correlation, $-\rho_1$, between the responses of antithetic pairs of runs. Techniques for structuring the simulation experiment to ensure the validity of Assumption I-1 are described in Subsection 2.4 and Section 3.

Under the four properties listed above, the variance of the mean response, $\bar{y}_j = \frac{1}{2}(y_{2j-1} + y_{2j})$, and mean control variate vector, $\bar{\mathbf{c}}_j = \frac{1}{2}(\mathbf{c}_{2j-1} + \mathbf{c}_{2j})$, within the jth replicate pair ($j = 1, 2, \ldots, h$) are respectively given by:

$$\mathrm{var}(\bar{y}_j) = \frac{1}{2}(1 - \rho_1)\sigma_y^2, \tag{10}$$

and

$$\mathrm{cov}(\bar{\mathbf{c}}_j) = \frac{1}{2}\mathbf{\Sigma_C}. \tag{11}$$

Also, the covariance matrix between $\bar{y}_j$ and $\bar{\mathbf{c}}_j$ is given by:

$$\mathrm{cov}(\bar{y}_j, \bar{\mathbf{c}}_j) = \frac{1}{2}\boldsymbol{\sigma}_{y\mathbf{C}}. \tag{12}$$

The joint normality assumption of the response and control variates gives the joint distribution of $\bar{y}_j$ and $\bar{\mathbf{c}}_j$ ($j = 1, 2, \ldots, h$) as the following multivariate normal distribution:

$$\mathrm{N}_{s+1}\left(\begin{bmatrix} \mu_y \\ \mathbf{0}_{(s\times 1)} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_1)\sigma_y^2 & \boldsymbol{\sigma}_{y\mathbf{C}}' \\ \boldsymbol{\sigma}_{y\mathbf{C}} & \mathbf{\Sigma_C} \end{bmatrix} \right). \tag{13}$$

Consequently given $\bar{\mathbf{c}}_j$, the conditional distribution of $\bar{y}_j$, is normal with expectation $\mathrm{E}(\bar{y}_j|\bar{\mathbf{c}}_j) = \mu_y + \bar{\mathbf{c}}_j'\boldsymbol{\alpha}$ and variance

$$\mathrm{var}(\bar{y}_j|\bar{\mathbf{c}}_j) = \frac{1}{2}((1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}_{y\mathbf{C}}'\mathbf{\Sigma_C}^{-1}\boldsymbol{\sigma}_{y\mathbf{C}}) = \frac{1}{2}\tau_1^2, \tag{14}$$

where

$$\tau_1^2 \equiv ((1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}_{y\mathbf{C}}'\mathbf{\Sigma_C}^{-1}\boldsymbol{\sigma}_{y\mathbf{C}})$$

(see Theorem 2.5.1 of Anderson 1984). As with the case of the linear relationship in (2), the $h \times 1$ vector of mean paired responses, $\bar{\mathbf{y}}$, can be represented as:

$$\bar{\mathbf{y}} \equiv \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_h \end{bmatrix} = \mu_y \mathbf{1}_{(h\times 1)} + \bar{\mathbf{C}}\boldsymbol{\alpha} + \boldsymbol{\epsilon}^\star, \tag{15}$$

where $\bar{\mathbf{C}}$ is a $h \times s$ matrix of control variates whose jth row is $\bar{\mathbf{c}}_j'$. Regression analysis on this linear model yields the following controlled estimator of the mean response:

$$\begin{aligned} \hat{\mu}_y &= \frac{1}{h}\mathbf{1}'_{(1\times h)}(\bar{\mathbf{y}} - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{Q}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{Q}\bar{\mathbf{y}}) \\ &= \frac{1}{h}\mathbf{1}'_{(1\times h)}(\mathbf{I}_{(h\times h)} - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{Q}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{Q})\bar{\mathbf{y}}, \end{aligned} \tag{16}$$

where $\mathbf{Q} = \mathbf{I}_{(h\times h)} - \frac{1}{h}\mathbf{1}_{(h\times 1)}\mathbf{1}'_{(1\times h)}$. From this expression for $\hat{\mu}_y$, it can easily be shown (see Appendix in Kwon and Tew 1993) that its unconditional variance is

$$\begin{aligned} \mathrm{var}(\hat{\mu}_y) &= \frac{\tau_1^2}{2h^2}\left[h + \frac{hs}{(h-s-2)}\right] \\ &= \frac{\sigma_y^2}{2h}(1 - \rho_1 - (R_{y\mathbf{C}}^{(1)})^2)\left(\frac{h-2}{h-s-2}\right), \end{aligned} \tag{17}$$

where $R_{y\mathbf{C}}^{(1)}$ is the multiple correlation coefficient between y_i and $\mathbf{c}_i$ ($i = 1, 2, \ldots, 2h$). Thus, provided Assumption I-1 holds, Combined Method I will result in a reduction of the variance of $\hat{\mu}_y$, if the effects due to antithetic variates (ρ_1) and the control variates $(R_{y\mathbf{C}}^{(1)})^2$, together, compensate for the loss factor $\frac{h-2}{h-s-2}$.

2.2 Combined Method II

In this subsection, we consider the second method for combining control variates and antithetic variates based on correlated replications in which only the control-variates components of the simulation model are used for correlation induction. Contrary to the random number assignment in the previous section, this method uses antithetic random number streams for the control-variate random components and independent random number streams for all other random components in the model. With this replication strategy, we induce negative correlations between the responses, between the control variates, and between the response and the control variates within

h pairs of the responses and the control variates, respectively, obtained from $(\mathbf{R}_{2j-1,1}, \mathbf{R}_{2j-1,2})$ and $(\mathbf{R}_{2j,1}, \bar{\mathbf{R}}_{2j-1,2})$ $(j = 1, 2, \ldots, h)$. However, across pairs of replications, we get independent outputs. Based on the above discussion and the development given for Combined Method I, we note the following properties for Combined Method II:

Property II-1: Homogeneity of response variances across replicates,

$$\mathrm{var}(y_i) = \sigma_y^2 \quad \text{for} \quad i = 1, 2, \ldots, 2h. \tag{18}$$

Property II-2: Homogeneity of response correlations across replicate pairs and independence of responses observed on different pairs of replications,

$$\mathrm{cov}(y_i, y_k) = \left\{ \begin{array}{l} -\rho_2 \sigma_y^2 \\ \quad i = k+1 \\ \quad k = 1, 3, \ldots, 2h-1 \\ 0 \\ \quad \text{otherwise} \end{array} \right\}. \tag{19}$$

Property II-3: Homogeneity of response-control-variates covariances across replicates, homogeneity of response-control-variates covariances across replicate pairs, and independence of the response and control variates observed on different replicate pairs,

$$\mathrm{Cov}(y_i, \mathbf{c}_k) = \left\{ \begin{array}{l} \boldsymbol{\sigma}_{y\mathbf{c}} \\ \quad i = k = 1, 2, \ldots, 2h \\ \boldsymbol{\sigma}_{y\mathbf{c}}^{(2)} \\ \quad i = k+1 \text{ or } k = i+1 \\ \quad \min\{i, k\} = \\ \quad\quad 1, 3, \ldots, 2h-1 \\ \mathbf{0}_{(s \times 1)} \\ \quad \text{otherwise} \end{array} \right\}. \tag{20}$$

Property II-4: Homogeneity of control-variates covariances across replicates, homogeneity of control-variates covariances across replicate pairs, and independence of control variates observed on different pairs of replications,

$$\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_k) = \left\{ \begin{array}{l} \boldsymbol{\Sigma}_{\mathbf{C}} \\ \quad i = k = 1, 2, \ldots, 2h \\ \boldsymbol{\Sigma}_{\mathbf{C}}^{(2)} \\ \quad i = k+1 \\ \quad k = 1, 3, \ldots, 2h-1 \\ \mathbf{0}_{(s \times s)} \\ \quad \text{otherwise} \end{array} \right\}. \tag{21}$$

We make the following assumption about Combined Method II:

Assumption II-1: $0 < \rho_2 < 1$.

In view of (19), Assumption II-1 postulates a negative induced correlation, $-\rho_2$, between the responses of antithetic pairs of runs. Techniques for structuring the simulation experiment to ensure the validity of Assumption II-1 are described in Subsection 2.4 and Section 3.

Under the four properties given above, the variance of $\bar{y}_j$, the covariance of $\bar{\mathbf{c}}_j$, and the covariances between $\bar{y}_j$ and $\bar{\mathbf{c}}_j$ are, respectively $(j = 1, 2, \ldots, h)$:

$$\mathrm{var}(\bar{y}_j) = \frac{1}{2}(1 - \rho_2)\sigma_y^2, \tag{22}$$

$$\mathrm{cov}(\bar{\mathbf{c}}_j) = \frac{1}{2}(\boldsymbol{\Sigma}_{\mathbf{C}} + \boldsymbol{\Sigma}_{\mathbf{C}}^{(2)}), \tag{23}$$

and

$$\begin{aligned} \mathrm{cov}(\bar{y}_j, \bar{\mathbf{c}}_j) &= \frac{1}{4}\mathrm{cov}(y_{2j-1} + y_{2j}, \mathbf{c}_{2j-1} + \mathbf{c}_{2j}) \\ &= \frac{1}{2}(\boldsymbol{\sigma}_{y\mathbf{C}} + \boldsymbol{\sigma}_{y\mathbf{C}}^{(2)}). \end{aligned} \tag{24}$$

Based on the joint normality assumption of the response and control variates and the properties given in (18), (19), (20) and (21), we have the joint distribution of $\bar{y}_j$ and $\bar{\mathbf{c}}_j$ ($j = 1, \ldots, h$) is a $s+1$-dimensional multivariate normal distribution with mean

$$\mathrm{E}\begin{bmatrix} \bar{y}_j \\ \bar{\mathbf{c}}_j \end{bmatrix} = \begin{bmatrix} \mu_y \\ \mathbf{0}_{(s \times 1)} \end{bmatrix}, \tag{25}$$

and variance

$$\mathrm{Var}\begin{bmatrix} \bar{y}_j \\ \bar{\mathbf{c}}_j \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1-\rho_2)\sigma_y^2 & (\boldsymbol{\sigma}_{y\mathbf{C}} + \boldsymbol{\sigma}_{y\mathbf{C}}^{(2)})' \\ \boldsymbol{\sigma}_{y\mathbf{C}} + \boldsymbol{\sigma}_{y\mathbf{C}}^{(2)} & \boldsymbol{\Sigma}_{\mathbf{C}} + \boldsymbol{\Sigma}_{\mathbf{C}}^{(2)} \end{bmatrix}. \tag{26}$$

Since $\bar{\mathbf{c}}_j$ $(j = 1, 2, \ldots, h)$ are independent, following the development of (17) we get by an analogous approach:

$$\mathrm{var}(\hat{\mu}_y) = \frac{\tau_2^2}{2h}\left(\frac{h-2}{h-s-2}\right)$$

$$= \frac{(1-\rho_2)\sigma_y^2}{2h}(1-(R_{y\mathbf{c}}^{(2)})^2)\left(\frac{h-2}{h-s-2}\right), \quad (27)$$

where

$$\tau_2^2 \equiv (1-\rho_2)\sigma_y^2 - (\boldsymbol{\sigma}_{y\mathbf{c}}+\boldsymbol{\sigma}_{y\mathbf{c}}^{(2)})'(\boldsymbol{\Sigma}_\mathbf{c}+\boldsymbol{\Sigma}_\mathbf{c}^{(2)})^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}}+\boldsymbol{\sigma}_{y\mathbf{c}}^{(2)}) \quad (28)$$

and

$$(R_{y\mathbf{c}}^{(2)})^2 = ((1-\rho_2)\sigma_y^2)^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}}+\boldsymbol{\sigma}_{y\mathbf{c}}^{(2)})' (\boldsymbol{\Sigma}_\mathbf{c}+\boldsymbol{\Sigma}_\mathbf{c}^{(2)})^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}}+\boldsymbol{\sigma}_{y\mathbf{c}}^{(2)}), \quad (29)$$

which is the square of the multiple correlation coefficient between $\bar{y}_j$ and $\bar{\mathbf{c}}_j$. Thus, provided that Assumption II-1 holds, Combined Method II will result in a reduction of the variance of $\hat{\mu}_y$, if the effects due to antithetic variates (ρ_2) and the control variates $(R_{y\mathbf{c}}^{(2)})^2$, together, compensate for the loss factor $\frac{h-2}{h-s-2}$.

2.3 Combined Method III

In this subsection we present Combined Method III, which jointly implements control variates and antithetic variates based on correlated replicates induced by utilizing *all* random components in the simulation model for correlation induction. Unlike the random number assignment strategies discussed in the previous two sections, we apply antithetic variates to *all* random components in the model. This assignment strategy induces correlations across h pairs of both the response and the control variates. That is, negative correlations are induced between the responses, between the control variates, *and* between the response and the control variates within h pairs of replicates. However, the induced correlation between the response and the control variates is different from that of Combined Methods I and II. Across the h pairs of replicates, the mean response and the mean of the control variates (within a pair of replicates) are independently observed by the assignment of different sets of randomly chosen random number streams. As before, we identify a set of properties for this strategy with regard to the covariance structure of the responses and control variates in the experiment. Based on the above discussion and the developments for the other two combined methods, we have the following properties for Combined Method III:

Property III-1: Homogeneity of response variances across replicates,

$$\mathrm{var}(y_i) = \sigma_y^2 \quad \text{for } i = 1,2,\ldots,2h. \quad (30)$$

Property III-2: Homogeneity of response correlations across replicate pairs and independence of responses observed on different pairs of replicates,

$$\mathrm{cov}(y_i, y_k) = \left\{\begin{array}{l} -\rho_3\sigma_y^2 \\ \quad i = k+1 \\ \quad k = 1,3,\ldots,2h-1 \\ 0 \\ \quad \text{otherwise} \end{array}\right\}. \quad (31)$$

Property III-3: Homogeneity of response–control-variates covariances across replicates, homogeneity of response–control-variates covariances across replicate pairs, and independence of the response and control variates observed on different replicate pairs,

$$\mathrm{Cov}(y_i, \mathbf{c}_k) = \left\{\begin{array}{l} \boldsymbol{\sigma}_{y\mathbf{c}} \\ \quad i = k = 1,2,\ldots,2h \\ \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)} \\ \quad i = k+1 \text{ or } k = i+1 \\ \quad \min\{i,k\} = \\ \quad\quad 1,3,\ldots,2h-1 \\ \mathbf{0}_{(s\times 1)} \\ \quad \text{otherwise} \end{array}\right\}. \quad (32)$$

Property III-4: Homogeneity of control-variates covariances across replicates, homogeneity of control-variates covariances across replicate pairs, and independence of control variates observed on different pairs of replicates,

$$\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_k) = \left\{\begin{array}{l} \boldsymbol{\Sigma}_\mathbf{c} \\ \quad i = k = 1,2,\ldots,2 \\ \boldsymbol{\Sigma}_\mathbf{c}^{(3)} \\ \quad i = k+1 \\ \quad k = 1,3,\ldots,2h-1 \\ \mathbf{0}_{(s\times s)} \\ \quad \text{otherwise} \end{array}\right\}. \quad (33)$$

We make the following assumption about Combined Method III:

Assumption III-1: $0 < \rho_3 < 1$.

In view of (30), Assumption III-1 postulates a negative induced correlation, $-\rho_3$, between the responses of antithetic pairs of runs. Techniques for structuring the simulation experiment to ensure the validity of Assumption III-1 are described in Subsection 2.4 and Section 3.

Under the four properties given above we obtain analogous results to those given in Section 2.2 ($j = 1, 2, \ldots, h$):

$$\mathrm{var}(\bar{y}_j) = \frac{1}{2}(1-\rho_3)\sigma_y^2 \quad (34)$$

$$\mathrm{cov}(\bar{\mathbf{c}}_j) = \frac{1}{2}(\mathbf{\Sigma_c} + \mathbf{\Sigma_c^{(3)}}), \quad (35)$$

and

$$\mathrm{cov}(\bar{y}_j, \bar{\mathbf{c}}_j) = \frac{1}{2}(\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)}). \quad (36)$$

Under the joint normality assumption of the response and control variates and the properties given in (29), (30), (31) and (32), the joint distribution of $\bar{y}_j$ and $\bar{\mathbf{c}}_j$ is given as:

$$\mathrm{E}\begin{bmatrix} \bar{y}_j \\ \bar{\mathbf{c}}_j \end{bmatrix} = \begin{bmatrix} \mu_y \\ \mathbf{0}_{(s\times 1)} \end{bmatrix}, \quad (37)$$

and variance

$$\mathrm{Var}\begin{bmatrix} \bar{y}_j \\ \bar{\mathbf{c}}_j \end{bmatrix} = \frac{1}{2}\begin{bmatrix} (1-\rho_3)\sigma_y^2 & (\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)})' \\ \boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)} & \mathbf{\Sigma_c} + \mathbf{\Sigma_c^{(3)}} \end{bmatrix}. \quad (38)$$

Since $\bar{\mathbf{c}}_j$ ($j = 1, 2, \ldots, h$) are independent, following the development of (26) we get by an analogous approach:

$$\begin{aligned} \mathrm{var}(\hat{\mu}_y) &= \frac{\tau_3^2}{2h}\left(\frac{h-2}{h-s-2}\right) \\ &= \frac{(1-\rho_3)\sigma_y^2}{2h}(1-(R_{y\mathbf{c}}^{(3)})^2)\left(\frac{h-2}{h-s-2}\right), \end{aligned} \quad (39)$$

where

$$\begin{aligned} \tau_3^2 \equiv\ & (1-\rho_3)\sigma_y^2 \\ & -(\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)})'(\mathbf{\Sigma_c} + \mathbf{\Sigma_c^{(3)}})^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)}) \end{aligned} \quad (40)$$

and

$$\begin{aligned} (R_{y\mathbf{c}}^{(3)})^2 =\ & ((1-\rho_3)\sigma_y^2)^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)})' \\ & (\mathbf{\Sigma_c} + \mathbf{\Sigma_c^{(3)}})^{-1}(\boldsymbol{\sigma}_{y\mathbf{c}} + \boldsymbol{\sigma}_{y\mathbf{c}}^{(3)}), \end{aligned} \quad (41)$$

which is the square of the multiple correlation coefficient between $\bar{y}_j$ and $\bar{\mathbf{c}}_j$. Provided Assumption III-1 holds, Combined Method III will result in a reduction of the variance of $\hat{\mu}_y$, if the effects due to antithetic variates (ρ_3) and the control variates $(R_{y\mathbf{c}}^{(3)})^2$, together, compensate for the loss factor $\frac{h-2}{h-s-2}$.

2.4 Comparison of the Combined Methods and Control Variates Method

In this subsection we give a brief and formal comparison of the three combined methods presented in the three previous subsections and the method of control variates. This comparison assumes the validity of the assumptions given for each of these four methods and is done with respect to the unconditional variances of the estimators for the mean response given in equations (5), (17), (27), and (39), respectively. We can say something in general about the assumptions. If the simulation response y and the simulation model are structured so that y is *monotonic* in each random-number input (either nonincreasing or nondecreasing), then it is guaranteed that ρ_1, ρ_2, and ρ_3 are all nonnegative (see Bratley, Fox, and Schrage 1987).

First, we consider the three combined methods presented earlier. Comparing Combined Methods I and II via equations (17) and (27) yields that Combined Method I is preferred to Combined Method II if

$$(1-\rho_1-(R_{y\mathbf{c}}^{(1)})^2) < (1-\rho_2)(1-(R_{y\mathbf{c}}^{(2)})^2). \quad (42)$$

Similarly, working with (17) and (39) yields that Combined Method III is better than Combined Method I, provided that

$$(1-\rho_1-(R_{y\mathbf{c}}^{(1)})^2) > (1-\rho_3)(1-(R_{y\mathbf{c}}^{(3)})^2). \quad (43)$$

Also, working with (27) and (39) yields that Combined Method III is better than Combined Method II if

$$(1-\rho_3)(1-(R_{y\mathbf{c}}^{(3)})^2) < (1-\rho_2)(1-(R_{y\mathbf{c}}^{(2)})^2). \quad (44)$$

As we discussed earlier, the loss factors for the three combined methods are the same; hence, they cancel when constructing the comparisons. Thus, the preference of the three methods is determined according to their minimum variance ratios given in (17), (27), and (39), respectively. Of course, other ordering schemes for the terms in equations (17), (27), and (39) could be conjectured. Clearly, it is not easy to identify an ordered relationship among $(R_{y\mathbf{c}}^{(1)})^2$, $(R_{y\mathbf{c}}^{(2)})^2$, and $(R_{y\mathbf{c}}^{(3)})^2$ since these terms involve the unknown elements $\mathbf{\Sigma_c}$, $\mathbf{\Sigma_c^{(2)}}$, $\mathbf{\Sigma_c^{(3)}}$, $\boldsymbol{\sigma}_{y\mathbf{c}}$, $\boldsymbol{\sigma}_{y\mathbf{c}}^{(2)}$, $\boldsymbol{\sigma}_{y\mathbf{c}}^{(3)}$. Nevertheless, we hope to give some clarity to this problem with the experimental results given in Section 4.

Next, we compare these three combined methods to the method of control variates. A comparison of equations (5) and (17) yields that Combined Method I is better than the control variates method if

$$(1-\rho_1-(R_{y\mathbf{c}}^{(1)})^2)\left(\frac{h-2}{h-s-2}\right) < (1-R_{y\mathbf{c}}^2)\left(\frac{2h-2}{2h-s-2}\right). \quad (45)$$

Also, comparing equations (5), (27), and (39) shows that Combined Method II is better than the method of control variates if

$$(1-\rho_2)(1-(R_{y\mathbf{c}}^{(2)})^2)\left(\frac{h-2}{h-s-2}\right) < (1-R_{y\mathbf{c}}^2)\left(\frac{2h-2}{2h-s-2}\right); \quad (46)$$

and Combined Method III yields a better result than the method of control variates, provided

$$(1-\rho_3)(1-(R_{y\mathbf{c}}^{(3)})^2)\left(\frac{h-2}{h-s-2}\right) < (1-R_{y\mathbf{c}}^2)\left(\frac{2h-2}{2h-s-2}\right). \quad (47)$$

Note that the loss factor of each combined method is greater than that of the method of control variates. Hence, for preference of each combined method to the method of control variates, the associated minimum variance ratio of the combined method should, at least, compensate for the increase in the associated loss factor. The effects of antithetic variates and control variates on the minimum variance ratio for Combined Method I are represented by an additive form in reducing the variance of the estimator of the mean response. Next, we present our computational results based on the application of these three combined methods to a classic simulation model.

3 EXAMPLE

We conducted a set of simulation experiments on a resource-constrained stochastic activity network in order to evaluate the performance of the variance reduction methodologies presented in Section 2. This section contains a brief description of this system. Section 4 contains a summary of the numerical results obtained from these simulation experiments.

3.1 A Resource-Constrained Stochastic Activity Network

We consider the resource-constrained stochastic activity network depicted in Figure 1 of Kwon and Tew (1993) which is similar to that described in Chapter 5 of Pritsker (1974). (Our network differs from Pritsker's in that we have substituted exponential distributions for triangular distributions in the activity durations.) This network is a model of a repair and retrofit project, and it consists of 11 nodes and 16 activities.

Table I
Activity Resource Requirements

Activity	Mechanics	Technicians
1	3	0
2	0	2
3	2	0
4	1	0
5	2	0
6	4	1
7	0	2
8	2	0
9	0	0
10	1	0
11	1	0
12	3	0
13	1	1
14	1	2
15	1	1
16	2	0

Two types of resources are used in this network—mechanics and technicians. There are 5 mechanics and 3 technicians. As depicted in Table I, each activity is assigned a 2-tuple indicating the required number of mechanics and the required number of technicians. In addition, each activity is assigned an activity duration distribution corresponding to a specific exponential distribution (with the exception of activity 9, which has duration 0). This latter assignment scheme is given in Table II. An activity cannot start until all of its predecessors have been completed and the required units of each resource can be assigned to that activity. Among the activities whose predecessors have all been completed and which are waiting for the allocation of a required resource, mechanics are assigned to activities in the following order: 4, 8, 10, 13, 15, 3, 5, 11, 14, 1, 16, 6, and 12. Technicians are dispatched to waiting activities in the following order: 15, 13, 14, 6, 7, and 2. Thus available resources are assigned to waiting activities according to the shortest expected processing time (SPT) of the waiting activities.

The response of interest y is the observed network completion time. We also consider two control variates which are used individually in performing both the control variate and combined procedures. In particular, control variate c_1 is the sum of observed activity durations for the path consisting of activities 2, 7, and 14; and control variate c_2 is the sum of observed activity durations for the path consisting of activities 1, 6, 12, and 13. These control variates were selected

Table II
Activity Duration Distributions

Activity	Description	Distribution
1	disassemble power units	exponential(3.00)
2	test and repair instrumentation	exponential(9.00)
3	clear main frame	exponential(2.00)
4	procure new subassembly	exponential(1.00)
5	pull old assembly	exponential(2.00)
6	clean, inspect, and repair power units	exponential(4.00)
7	calibrate instrumentation	exponential(8.00)
8	inspect and repair main frame	exponential(1.00)
9	dummy	0.0
10	change tags	exponential(1.00)
11	install new assembly	exponential(2.00)
12	assemble and test power units	exponential(5.00)
13	safety inspection	exponential(1.00)
14	systems check	exponential(2.00)
15	retrofit check	exponential(1.00)
16	check all fittings	exponential(3.00)

through a series of preliminary simulation runs conducted by Tew and Wilson (1993); c_1 was found to be highly correlated with y and c_2 was found to not be so highly correlated with y. Thus, we selected one highly effective control variate (c_1) and one less effective control variate (c_2) to consider in our example. This was done in order to illustrate the possible pitfalls that can be encountered when using antithetic variates in conjunction with control variates. Note that both c_1 and c_2 include only the corresponding sampled activity durations; they do *not* include the time spent waiting for the allocation of resources to these activities.

We used the SLAM II simulation language (Pritsker 1986) to implement a model of the stochastic activity network described above. (The SLAM II code used by the authors as well as tables of the random number seeds used and the observed responses are available from the second author upon request.) For each variance reduction method, an experiment consisted of 20 (= $2h$) replicates and 50 independent *macroreplicates* (see Section 4). That is, for each variance reduction method, 1000 overall replications were made. The *macroreplicates* were included in order to ensure unbiased estimates of all variances concerned.

In modeling this system, we used two separate random number streams to drive the random components of the model (the non-control-variate random components ($\mathbf{r}_1$) and the control-variate random components ($\mathbf{r}_2$)). That is, for the case of using control variate c_1, $\mathbf{r}_1$ was used to drive activities 1, 3, 4, 5, 6, 8, 10, 11, 12, and 13 and $\mathbf{r}_2$ was used to drive activities 2, 7, and 14 (an analogous arrangement was used for the situation when we used control variate c_2). Thus, in the notation of Section 2, $\mathbf{R}_{i1} = \{\mathbf{r}_{i1}\}$ and $\mathbf{R}_{i2} = \{\mathbf{r}_{i2}\}$. For each of the two control variates, the estimation of μ_y under each of the following methodologies was considered: (a) direct simulation, (b) control variates, (c) Combined Method I, (d) Combined Method II, and (e) Combined Method III. In each case, 50 independent estimates of μ_y were obtained (one from each macroreplicate). These independent estimates of μ_y were used to estimate $\text{var}(\hat{\mu}_y)$ and represented what we thought to be a sufficiently large sample size for meaningful comparisons of the five methodologies.

4 NUMERICAL RESULTS

This section provides a summary of the simulation results obtained from the example discussed in Section 3. For each of the two control variates considered, c_1 and c_2, these results are organized into two parts: (a) we present performance statistics (Table III) on the observed variance reduction for the response variable considered, where the results under control variates, Combined Method I, Combined Method II, and Combined Method III, are compared to those of direct simulation; and (b) we present sample estimates (Table IV) of ρ_1, ρ_2, ρ_3, $R^2_{y\mathbf{c}}$, $(\bar{R}^{(1)}_{y\mathbf{c}})^2$, $(\bar{R}^{(2)}_{y\mathbf{c}})^2$, and $(\bar{R}^{(3)}_{y\mathbf{c}})^2$, in order to evaluate the quality of the comparative statements given in the inequalities of Section 2.4. If we let: (a) N (= 50) be the number of macroreplicates in the experiment, (b)

$$m = \begin{cases} 1 & \text{for direct simulation} \\ 2 & \text{for control variates} \\ 3 & \text{for Combined Method I} \\ 4 & \text{for Combined Method II} \\ 5 & \text{for Combined Method III} \end{cases}$$

be the variance reduction method used, and (c) $\hat{\mu}_y(m,n)$ be the point estimator of μ_y on the nth (= $1, 2, \ldots, N$) macroreplicate using the mth (= $1, 2, \ldots, 5$) variance reduction technique, then an *external*, unbiased estimator of the variance of $\hat{\mu}_y(m,n)$ is given by

$$\hat{V}_y(m) = \frac{1}{N-1}\sum_{n=1}^{N}[\hat{\mu}_y(m,n) - \bar{\hat{\mu}}_y(m)]^2, \qquad (48)$$

where

$$\bar{\hat{\mu}}_y(m) = \frac{1}{N}\sum_{n=1}^{N}\hat{\mu}_y(m,n).$$

(Note that the macroreplicates are not necessary for the direct simulation method. However, we used independent macroreplicates on all 5 methods in order

Table III
Variance Reduction in var($\hat{\mu}_{y_i}$)

Method	Control Variate	
	c_1	c_2
2	70.72	15.51
3	74.68	37.83
4	76.74	22.97
5	77.57	41.23

Table IV
Correlation Estimates

Estimator	Control Variate	
	c_1	c_2
$\hat{R}^2_{yc}$	.6753	.1542
$\hat{\rho}_1$	.0892	.3327
$(\hat{R}^{(1)}_{yc})^2$	.7586	.0661
$\hat{\rho}_2$	.2875	.1458
$(\hat{R}^{(2)}_{yc})^2$	.5761	.2032
$\hat{\rho}_3$	.2864	.2864
$(\hat{R}^{(3)}_{yc})^2$	.5733	.1582

to maintain consistency in the presentation of the results. We hope that this will help avoid any confusion on the part of the reader in the interpretation of our results.)

For the resource-constrained stochastic activity network described in Section 3.1 we considered the response of observed network completion time (y). The first row of Table III contains the observed variance reductions, relative to direct simulation, for each of the other four methodologies under consideration when the control variate c_1 was used. The second row of Table III contains the analogous observations when control variate c_2 was used.

In order to better understand these results, we computed maximum-likelihood estimates (MLEs) of the squared-multiple correlation coefficient and the induced negative correlation (where applicable) for each of the four methodologies (see Section 3.6 of Morrison 1976). These estimates are given in Table IV. Again, the first row contains the results obtained when c_1 was used and the second row contains the results obtained when c_2 was used.

The results presented in Table III clearly indicate that the combined methods *can* result in significant improvements over the method of control variates. Specifically, we see that, for c_1, all three Combined Methods each resulted in a larger variance reduction than control variates. Also, for c_2 we see that again all three Combined Methods each resulted in a larger variance reduction than control variates. Clearly, in both cases, Combined Method III outperformed *all* of the other methodologies considered and the degree to which it improved upon the method of control variates seems to depend on the degree of correlation between the control variate and the response. That is, when a good control variate (c_1) was used, Combined Method III resulted in a modest improvement (9.68%) over the method of control variates. However, when a poor control variate (c_2) was used, Combined Method III resulted in a much more substantial improvement (165.82%) over the method of control variates. It should be noted that Cheng (1982) suggests ways for increasing the magnitude of the induced correlation between two random variables and that through their application the performances of all three of the combined methods presented in this paper may be significantly improved. A comparison of the results for c_1 and c_2 further suggests that when the control variate under consideration is strongly correlated with the response variable, Combined Method II may preferable to Combined Method I and the reverse may be recommended when the control variate is not strongly correlated with the response variable.

In our example clearly the response is linearly dependent on those input variables that are to be treated antithetically or as control variates. A good control (c_1) will pick up a lot of this dependence; antithetic variates, whether applied to controls or non-controls, will therefore only give marginal improvement. This we see in the c_1 row of Table III. If however, the control variates are not well chosen (c_2), but the response does depend linearly on input variables, then this will be picked up by applying antithetics to the non-control inputs. This we see in the c_2 row of Table III.

Inspection of the correlation estimates when control variate c_2 is used and given in the second row of Table IV indicates that the ranking schemes given in equations (40) through (42) and equations (43) through (45) are consistent with the observed variance reductions. However, the ranking schemes are not consistent with the observed variance reductions when c_1 is used. We observe that for our data the expressions in (40), (41), and (42) are probably not significantly different; and precise conclusions are difficult to make. These results point out the importance of the development of an accurate testing procedure for selecting the *best* methodology in a given situation.

5 SUMMARY AND CONCLUSIONS

In the past, both control variates and antithetic variates have been shown to often be effective variance reduction techniques for estimating the mean of a response of interest for simulation experiments. In this paper we have presented three methods that can easily be implemented in a simulation experiment without significant additional programming effort; each combines both control variates and antithetic variates. Under general assumptions, these combined methods are shown to yield superior performance, based on estimator variance, compared to control variates alone. These claims are supported by computational results. These results also show that, for the combined methods, if the requisite assumptions for that method are violated then an improvement upon the control variates method may not be achieved.

We hope that this work will stimulate greater application of these combined correlation methods in simulation experiments and further investigative research.

ACKNOWLEDGEMENTS

The authors would like to James R. Wilson for his many helpful comments and suggestions.

REFERENCES

Anderson, T. W. 1984. *An introduction to multivariate statistical analysis.* New York: John Wiley & Sons.

Arnold, S. F. 1981. *The theory of linear models and multivariate analysis.* New York: John Wiley & Sons.

Bratley, P., B. L. Fox, and L. E. Schrage. 1987. *A guide to simulation*, 2d. ed. New York: Springer-Verlag.

Cheng, R. C. H. 1982. The use of antithetic variates in computer simulations. *Journal of the Operational Research Society*, 33:229–237.

Kleijnen, J. P. C. 1974. *Statistical techniques in simulation, part I.* New York: Marcel Dekker.

Kwon, C. and J. D. Tew. 1993. Strategies for combining antithetic variates and control variates in designed simulation experiments. *Management Science*, (to appear).

Law, A. M., and W. D. Kelton. 1991. *Simulation modeling & analysis*, 2d. ed. New York: McGraw Hill.

Lavenberg, S. S., T. L. Moeller, and P. D. Welch. 1982. Statistical results on control variates with application to queueing network simulation. *Operations Research*, 30:182–202.

Morrison, D. F. 1976. *Multivariate statistical methods.* 2nd ed. New York: McGraw Hill.

Pritsker, A. A. B. 1974. *The GASP IV simulation language.* New York: John Wiley & Sons.

Pritsker, A. A. B. 1986. *Introduction to simulation and SLAM II.* 3d ed. New York: Halsted Press.

Searle, S. R. 1971. *Linear models.* New York: John Wiley & Sons.

Tew, J. D., and J. R. Wilson. 1992. Validation of simulation analysis methods for the Schruben-Margolin correlation-induction strategy. *Operations Research* 40:87-103.

Tew, J. D., and J. R. Wilson. 1993. Estimating simulation metamodels using combined correlation-based variance reduction techniques. *IIE Transactions*, (to appear).

AUTHOR BIOGRAPHIES

CHIMYUNG KWON is an Assistant Professor in the Department of Applied Statistics at Dong-A University in Pusan, Korea. He received a B.S. in industrial engineering from Seoul National University in 1978, a M.S. in industrial engineering from Seoul National University in 1983, and a Ph.D. in industrial engineering from Virginia Polytechnic Institute and State University in 1991. His current research interests include variance reduction techniques and the design of simulation experiments.

JEFFREY D. TEW is an Assistant Professor in the Industrial and Systems Engineering Department at Virginia Polytechnic Institute and State University. He received a B.S. in mathematics from Purdue University in 1979, a M.S. in statistics from Purdue University in 1981, and a Ph.D. in industrial engineering from Purdue University in 1986. His current research interests include variance reduction techniques, simulation optimization, and the design of simulation experiments. He is a member of ACM, ASA, IIE, IMS, ORSA,