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MULTINOMIAL SELECTION PROCEDURES FOR USE IN SIMULATIONS

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ABSTRACT

We describe single-stage and closed sequential procedures for selecting the most probable cell of a multinomial distribution. These procedures are then reformulated as nonparametric techniques for selecting the best one of a number of competing simulated systems or alternatives. We discuss performance characteristics of the procedures and make recommendations concerning their use.

1 INTRODUCTION

This article is concerned with the problem of selecting the most probable (or "best") cell from a multinomial distribution. We will show how procedures for solving this multinomial problem can be used to answer the question "Which one of a number of competing systems is the best?" The term "best" can take on a variety of interpretations depending on the problem at hand; e.g., we might be interested in finding:

- The most popular candidate in a political race,
- The inventory policy that maximizes profit,
- The manufacturing line layout that minimizes makespan, or
- The most precise scale.

Similar problems often arise in the context of discrete-event simulation when the experimenter is interested in determining the best one of a number of alternative simulated systems.

In the remainder of this section, we briefly review the multinomial distribution and provide motivation for the *indifference-zone* procedures to be discussed in the subsequent sections. In §2, we present single-stage and closed sequential procedures for selecting the most probable cell of a multinomial distribution. §3 is concerned with simulation applications of the multinomial selection problem, and §4 provides conclusions.

1.1 The Multinomial Distribution

Suppose that n independent trials of the same experiment are conducted, each having t mutually exclusive and exhaustive possible outcomes (or cells). Let p_i $(0 < p_i < 1, \sum_{i=1}^{t} p_i = 1)$ denote the single-trial probability of the event associated with the ith cell $(1 \leq i \leq t)$, and let Y_{in} be the number of outcomes falling in cell i $(1 \leq i \leq t)$ after n observations have been taken. Then $0 \leq Y_{in} \leq n$ $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} Y_{in} = n$. The t-variate discrete random variable $\boldsymbol{Y}_n = (Y_{1n}, \ldots, Y_{tn})$ has the probability mass function

$$P\{\boldsymbol{Y}_n = (y_1, \ldots, y_t)\} = \frac{n!}{\prod_{i=1}^{t} y_i!} \prod_{i=1}^{t} p_i^{y_i},$$

and we say that $\boldsymbol{Y}_n$ has the *multinomial* distribution with parameters n and $\boldsymbol{p} = (p_1, \ldots, p_t)$. The binomial distribution is the special case of the multinomial for which there are $t = 2$ cells.

The next two examples are from Bechhofer, Santner, and Goldsman (1993).

Example 1 Suppose that a fair die has three red faces, two blue faces, and one green face, i.e., the probability vector associated with red, blue, and green is $\boldsymbol{p} = (3/6, 2/6, 1/6)$. If the die is tossed $n = 5$ times, then the probability of observing ex-

actly 3 reds, 0 blues, and 2 greens is

$$P\{\boldsymbol{Y}_5 = (3,0,2)\} = \frac{5!}{3!0!2!}(\frac{3}{6})^3(\frac{2}{6})^0(\frac{1}{6})^2 = 0.03472.$$

Example 2 Continuing with Example 1, suppose that we did not know $\boldsymbol{p}$ and that we wanted to select the color having the largest probability of occurring on a single trial. (Of course, that color is red.) Our selection rule is to take whatever color occurs the most often during the $n = 5$ trials, using randomization to break ties. The probability that we *correctly* select red is

$$\begin{aligned} &P\{\text{red wins in 5 trials}\} \\ &= P\{Y_{1,5} > Y_{2,5} \ \&\ Y_{3,5}\} + \frac{1}{2}P\{Y_{1,5} = Y_{2,5} > Y_{3,5}\} \\ &\quad + \frac{1}{2}P\{Y_{1,5} = Y_{3,5} > Y_{2,5}\}. \end{aligned}$$

The following table lists the $\boldsymbol{Y}_5$'s leading to a *correct selection* (CS) of red, along with their associated probabilities, incorporating randomization when ties occur.

$\boldsymbol{Y}_5$	$P\{\boldsymbol{Y}_5$ & CS$\}$
(5,0,0)	0.03125
(4,1,0)	0.10417
(4,0,1)	0.05208
(3,2,0)	0.13889
(3,1,1)	0.13889
(3,0,2)	0.03472
(2,2,1)	(0.5)(0.13889)
(2,1,2)	(0.5)(0.06944)
	0.60416

Thus, the probability of correctly selecting red based on $n = 5$ trials is 0.6042. This probability can be improved by increasing the sample size n.

1.2 The Indifference-Zone Approach

This subsection discusses the *indifference-zone* (IZ) approach for selecting the most probable cell of a multinomial distribution. This approach was popularized by Bechhofer (1954) and Bechhofer, Elmaghraby, and Morse (1959). First, we establish some notation and ground rules.

Statistical Assumptions: We take independent observations $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{tj})$ $(j \geq 1)$ from a multinomial distribution having $t \geq 2$ cells with unknown probabilities $\boldsymbol{p} = (p_1, \ldots, p_t)$, where $0 < p_i < 1$ $(1 \leq i \leq t)$ and $\sum_{i=1}^{t} p_i = 1$. Then $X_{ij} = 1$ [0] if the jth observation is [is not] from cell i $(1 \leq i \leq t, j \geq 1)$.

We denote the ordered values of the p_i's by $p_{[1]} \leq \cdots \leq p_{[t]}$. Neither the values of the $p_{[s]}$'s nor the pairings of the p_i's with the $p_{[s]}$'s $(1 \leq i, s \leq t)$ are assumed to be known. The observed value of $\boldsymbol{X}_j$ is given by $\boldsymbol{x}_j = (x_{1j}, \ldots, x_{tj})$ $(j \geq 1)$. The cumulative sum for cell i after $m \geq 1$ multinomial observations have been taken is given by $y_{im} = \sum_{j=1}^{m} x_{ij}$ $(1 \leq i \leq t)$, and the ordered values of the y_{im}'s are given by $y_{[1]m} \leq \cdots \leq y_{[t]m}$.

Our experimentation will attempt to achieve the following goal.

Goal 1 To select the cell associated with $p_{[t]}$, i.e., the most probable or "best" category.

We say that a *correct selection* is made if Goal 1 is achieved. If $p_{[t]}$ and $p_{[t-1]}$ are very close, say $p_{[t]}/p_{[t-1]} < \delta^*$ for some $\delta^* > 1$ specified by the experimenter, then for all practical purposes, it does not matter which of the two associated cells we select. On the other hand, if $p_{[t]}$ and $p_{[t-1]}$ are quite different, say $p_{[t]}/p_{[t-1]} \geq \delta^*$, we would prefer to make a correct selection. The constant δ^* can be interpreted as the "smallest $p_{[t]}/p_{[t-1]}$ ratio worth detecting." For these reasons, the set

$$\Omega_{\delta^*} \equiv \{\boldsymbol{p} \,|\, p_{[t]}/p_{[t-1]} \geq \delta^*\}$$

is called the *preference zone*, and $\Omega^c_{\delta^*}$ is called the *indifference zone.*

We are interested in statistical procedures that satisfy the following indifference-zone probability requirement.

Probability Requirement: For constants (δ^*, P^*) with $1 < \delta^* < \infty$ and $1/t < P^* < 1$, specified prior to the start of experimentation, we require

$$P\{\text{CS}|\boldsymbol{p}\} \geq P^* \text{ whenever } \boldsymbol{p} \in \Omega_{\delta^*}. \quad (1)$$

The probability in (1) depends on the entire vector $\boldsymbol{p} = (p_1, \ldots, p_t)$ and on the number n of independent multinomial observations taken. In the next section, we discuss two procedures for selecting the most probable multinomial cell.

2 TWO MULTINOMIAL PROCEDURES

This section discusses a single-stage procedure and a closed sequential procedure for achieving Goal 1 while guaranteeing the probability requirement (1).

2.1 A Single-Stage Procedure

The following *single-stage* procedure was proposed by Bechhofer, Elmaghraby and Morse (1959) to guarantee (1).

Procedure $\mathcal{M}_{BEM}$

For the given t, and (δ^*, P^*) specified prior to the start of sampling, find n from Table 1.

Sampling rule: Take a random sample of n multinomial observations $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{tj})$ $(1 \leq j \leq n)$ in a *single* stage.

Terminal decision rule: Calculate the ordered sample sums $y_{[1]n} \leq \cdots \leq y_{[t]n}$. Select the cell that yielded the largest sample sum, $y_{[t]n}$, as the one associated with $p_{[t]}$. Use randomization in case of ties.

The n-values found in Table 1 and more complete tables from Gibbons, Olkin, and Sobel (1977) and Bechhofer, Santner, and Goldsman (1993) are given for a variety of $(t; \delta^*, P^*)$. They are computed so that procedure $\mathcal{M}_{BEM}$ achieves the nominal probability of correct selection P^* when the event probabilities $\boldsymbol{p}$ are in the *least-favorable* (LF) configuration, i.e., the configuration that minimizes $P\{\text{CS}|\boldsymbol{p}\}$ for $\boldsymbol{p} \in \Omega_{\delta^*}$. For procedure $\mathcal{M}_{BEM}$, Kesten and Morse (1959) proved that the LF configuration is $p_{[1]} = p_{[t-1]} = 1/(\delta^* + t - 1)$, $p_{[t]} = \delta^*/(\delta^* + t - 1)$.

Example 3 Suppose that a soft drink producer wishes to determine which of three colas is the most popular. The company will ask each of n individuals which of the three brands they most prefer. The company will declare as best that cola corresponding to the largest sample proportion of positive responses. The sample size n will be chosen is such a way that the company is guaranteed that $P\{\text{CS}|\boldsymbol{p}\} \geq 0.95$ whenever the ratio of the largest to second largest true (but unknown) proportions is at least 1.4. From Table 1 with $t = 3$, $P^* = 0.95$, and $\delta^* = 1.4$, we find that $n = 186$ individuals must be interviewed. If it turns out that 50 of these people prefer Brand A, 120 prefer Brand B, and 16 prefer Brand C, then the company can state that Brand B is the most popular and assert that if $\boldsymbol{p} \in \Omega_{\delta^*}$, then the $P\{\text{CS}|\boldsymbol{p}\}$ is at least 95%.

2.2 A Closed Sequential Procedure

Sometimes it is not necessary to take all of the n observations dictated by procedure $\mathcal{M}_{BEM}$. For instance, it is clear from the data in Example 3 that, since Brand B had garnered a very large margin of victory, it would have been possible to curtail sampling *before* $n = 186$ observations had been taken. For this and other reasons, the use of *sequential* procedures can often lead to significant savings in the total number of observations to termination relative to single-stage procedures that guarantee the same probability requirement (1).

Bechhofer, Kiefer and Sobel (1968) proposed an *open* sequential sampling procedure ($\mathcal{M}_{BKS}$) for selecting the most probable cell from a multinomial distribution. By "open," we mean that, before sampling begins, the experimenter cannot place an upper bound on the number of observations that the procedure will ultimately require before termination—certainly, this is not a desirable property! Bechhofer and Goldsman (1985b, 1986) studied the performance characteristics of procedure $\mathcal{M}_{BKS}$ and found that the procedure always "overprotects," i.e., the *achieved* $P\{\text{CS}|\text{LF}\}$ of the procedure typically exceeds its specified lower bound P^* by a substantial amount. Further, the distribution of the random number of observations N taken by procedure $\mathcal{M}_{BKS}$ is highly skewed to the right; sometimes the procedure requires prohibitively large N to terminate sampling (with resulting large $\mathsf{E}\{N|\boldsymbol{p}\}$ and $\mathsf{Var}\{N|\boldsymbol{p}\}$). These remarks led the authors to study the effects of *truncation*—stopping (or *closing*) a procedure once the number of observations hits a prespecified limit. The truncation number n_0 must be chosen in such a way as to maintain $P\{\text{CS}|\text{LF}\} \geq P^*$, while *reducing* $\mathsf{E}\{N|\boldsymbol{p}\}$ and $\mathsf{Var}\{N|\boldsymbol{p}\}$ *uniformly* in $\boldsymbol{p}$.

A second augmentation of procedure $\mathcal{M}_{BKS}$ features the use of *curtailment*—stopping early when the termination decision becomes apparent. Suppose that, at some point in sampling, the current leading cell achieves an insurmountable lead given the limited number of potential remaining observations before truncation were to terminate sampling. Then it can be shown that curtailment permits early termination of the procedure at no loss in $P\{\text{CS}|\boldsymbol{p}\}$.

The following closed sequential procedure for selecting the most probable multinomial cell incorporates truncation and curtailment.

Procedure $\mathcal{M}_{BG}$

For the given t, and (δ^*, P^*) specified prior to the start of sampling, find the truncation number n_0 from Table 1.

Sampling rule: At the jth stage of experimentation $(j \geq 1)$, take the observation $\boldsymbol{X}_j = (X_{1j}, \ldots, X_{tj})$.

Table 1: Sample Size n for Single-Stage Procedure $\mathcal{M}_{BEM}$, and Truncation Numbers n_0 and $\mathsf{E}\{N|\mathrm{LF}\}$ and $\mathsf{E}\{N|\mathrm{EP}\}$ for Closed Sequential Procedure $\mathcal{M}_{BG}$ (* denotes results obtained via Monte Carlo simulation; see Bechhofer and Goldsman 1986 for details)

			$\mathcal{M}_{BEM}$	$\mathcal{M}_{BG}$						$\mathcal{M}_{BEM}$	$\mathcal{M}_{BG}$		
t	P^*	δ^*	n	n_0	$\mathsf{E}\{N\|\mathrm{LF}\}$	$\mathsf{E}\{N\|\mathrm{EP}\}$	t	P^*	δ^*	n	n_0	$\mathsf{E}\{N\|\mathrm{LF}\}$	$\mathsf{E}\{N\|\mathrm{EP}\}$
		3.0	1	1	1.00	1.00			3.0	8	9	4.91	5.75
		2.8	3	3	2.39	2.50			2.8	9	9	6.00	6.79
		2.6	3	3	2.40	2.50			2.6	10	11	7.05	8.16
		2.4	3	3	2.42	2.50			2.4	12	15	8.29	9.91
	0.75	2.2	3	3	2.43	2.50		0.75	2.2	15	17	10.44	12.26
		2.0	5	5	3.09	3.25			2.0	20	24	13.78	16.45
		1.8	5	7	3.44	3.63			1.8	29	35	19.42	23.34
		1.6	9	9	5.96	6.26			1.6	46	57	31.11	37.65
		1.4	17	19	11.35	12.06			1.4	92	*124	62.31	76.01
		1.2	55	67	36.75	39.28			1.2	326	*495	219.69	270.89
		3.0	7	∞	3.20	4.00			3.0	16	19	9.84	13.85
		2.8	7	7	4.63	5.34			2.8	19	22	11.29	15.94
		2.6	7	9	5.23	6.26			2.6	22	26	13.20	18.87
		2.4	9	11	5.72	6.94			2.4	26	31	15.93	22.77
2	0.90	2.2	11	15	6.33	7.84	4	0.90	2.2	33	39	19.79	28.39
		2.0	15	15	8.90	10.59			2.0	43	53	25.71	37.31
		1.8	19	27	11.04	13.91			1.8	61	*75	36.94	53.77
		1.6	31	41	17.00	21.48			1.6	98	*126	58.69	86.83
		1.4	59	79	32.92	41.84			1.4	196	*274	116.89	176.56
		1.2	199	267	112.28	143.50			1.2	692	*1050	413.68	627.68
		3.0	9	11	5.25	6.94			3.0	23	26	12.97	20.34
		2.8	11	15	5.65	7.84			2.8	26	30	14.74	23.29
		2.6	13	13	7.54	9.66			2.6	31	36	17.19	27.36
		2.4	15	17	8.47	11.38			2.4	37	44	20.68	33.38
	0.95	2.2	19	23	9.43	13.13		0.95	2.2	46	56	25.75	42.16
		2.0	23	27	13.09	17.90			2.0	61	*74	33.86	55.47
		1.8	33	35	18.03	24.30			1.8	87	*106	47.80	79.05
		1.6	49	59	26.56	37.09			1.6	139	*180	76.06	127.76
		1.4	97	151	48.31	72.36			1.4	278	*380	152.72	264.25
		1.2	327	455	166.54	245.31			1.2	979	*1500	537.10	962.45
		3.0	5	5	3.24	3.48			3.0	11	*12	7.44	8.95
		2.8	6	6	3.70	4.15			2.8	12	*13	8.39	9.96
		2.6	6	7	3.94	4.38			2.6	14	*17	9.80	11.93
		2.4	7	8	5.40	5.94			2.4	17	*20	11.91	14.55
	0.75	2.2	9	10	6.00	6.68		0.75	2.2	22	*25	14.99	18.20
		2.0	12	13	7.97	8.93			2.0	29	*34	19.81	24.35
		1.8	17	18	11.34	12.74			1.8	41	*50	28.44	34.88
		1.6	26	32	17.60	20.25			1.6	68	*86	45.68	57.14
		1.4	52	71	34.02	39.84			1.4	137	*184	92.68	117.62
		1.2	181	*285	117.89	140.85			1.2	486	*730	329.36	421.47
		3.0	11	12	6.97	8.93			3.0	21	*24	13.13	18.76
		2.8	13	15	7.77	10.36			2.8	24	*28	15.01	21.87
		2.6	15	16	9.17	11.83			2.6	29	*34	17.43	25.69
		2.4	18	22	10.43	14.25			2.4	35	*42	21.17	31.66
3	0.90	2.2	22	25	13.30	17.77	5	0.90	2.2	44	*52	26.63	39.37
		2.0	29	34	17.17	23.30			2.0	58	*71	35.16	52.75
		1.8	40	50	23.71	32.89			1.8	83	*104	50.34	75.69
		1.6	64	83	37.26	52.61			1.6	134	*172	80.93	123.96
		1.4	126	*170	73.42	105.39			1.4	271	*374	163.55	252.55
		1.2	437	*670	254.85	369.78			1.2	964	*1460	585.00	923.61
		3.0	17	20	8.90	13.57			3.0	29	*34	16.42	27.19
		2.8	19	22	10.48	15.79			2.8	34	*39	19.19	31.50
		2.6	22	25	12.27	18.27			2.6	40	*46	22.55	36.87
		2.4	26	31	14.48	22.09			2.4	48	*58	27.04	45.37
	0.95	2.2	32	41	17.56	27.76		0.95	2.2	61	*74	34.05	57.80
		2.0	42	52	23.03	35.97			2.0	81	*98	45.01	76.28
		1.8	59	71	32.63	50.41			1.8	115	*142	64.37	109.09
		1.6	94	125	50.32	81.43			1.6	185	*240	103.53	180.02
		1.4	186	*266	98.88	165.90			1.4	374	*510	209.64	365.49
		1.2	645	*960	346.42	577.82			1.2	1331	*2000	741.70	1350.73

Stopping rule: At stage m ($m \geq 1$), calculate the ordered sample sums $y_{[1]m} \leq \cdots \leq y_{[t]m}$. Stop sampling when, for the first time, *either*

$$z_m \equiv \sum_{i=1}^{t-1} (1/\delta^\star)^{(y_{[t]m} - y_{[i]m})} \leq (1 - P^\star)/P^\star \quad (2)$$

or

$$m = n_0 \quad (3)$$

or

$$y_{[t]m} - y_{[t-1]m} \geq n_0 - m. \quad (4)$$

Terminal decision rule: Let N denote the value of m at the termination of sampling. Select the cell that yielded the largest sample sum, $y_{[t]N}$, as the one associated with $p_{[t]}$. Use randomization in case of ties.

The truncation numbers n_0 given in Table 1 for selected $(t; \delta^\star, P^\star)$ are taken from more complete tables given in Bechhofer and Goldsman (1986).

Remark 1 Stopping criterion (2) is simply that of the open procedure $\mathcal{M}_{BKS}$. Stopping criterion (3) truncates sampling if n_0 multinomial observations have been taken. Stopping criterion (4) curtails sampling if the cell currently in second place can do no better than *tie* the cell currently in first place.

We give several examples to illustrate how procedure $\mathcal{M}_{BG}$ works.

Example 4 For $t = 3$, $P^\star = 0.75$, and $\delta^\star = 3.0$, Table 1 tells us to truncate sampling at $n_0 = 5$ observations. Consider the following observations.

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	0	1	0	0	2	0

We stop sampling by criterion (2) since $z_2 = (1/3)^2 + (1/3)^2 = 2/9 \leq (1 - P^\star)/P^\star = 1/3$. We select cell 2.

Example 5 Again suppose that $t = 3$, $P^\star = 0.75$, and $\delta^\star = 3.0$; so $n_0 = 5$. Consider the following multinomial observations:

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	1	0	0	1	1	0
3	0	1	0	1	2	0
4	1	0	0	2	2	0
5	1	0	0	3	2	0

Since $m = n_0 = 5$, criterion (3) stops sampling, and we select cell 1.

Example 6 Again suppose that $t = 3$, $P^\star = 0.75$, and $\delta^\star = 3.0$; so $n_0 = 5$. Consider the following observations:

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	1	0	0	1	1	0
3	0	1	0	1	2	0
4	1	0	0	2	2	0
5	0	0	1	2	2	1

Since $m = n_0 = 5$, criterion (3) tells us to stop sampling. Since we have a tie, we randomly select between cells 1 and 2.

Example 7 Again suppose that $t = 3$, $P^\star = 0.75$, and $\delta^\star = 3.0$; so $n_0 = 5$. Consider the following observations:

m	x_{1m}	x_{2m}	x_{3m}	y_{1m}	y_{2m}	y_{3m}
1	0	1	0	0	1	0
2	1	0	0	1	1	0
3	0	1	0	1	2	0
4	0	0	1	1	2	1

Since cells 1 and 3 can do no better than tie cell 2 (if we were to take the potential remaining $n_0 - m = 5 - 4 = 1$ observation), criterion (4) tells us to stop, and we select cell 2.

Remark 2 It can be shown that procedure $\mathcal{M}_{BG}$ has the same LF configuration of the p_i's as does procedure $\mathcal{M}_{BEM}$.

Table 1 illustrates performance characteristics of procedure $\mathcal{M}_{BG}$. In particular, we tabulate the expected number of stages in the LF configuration, $\mathsf{E}\{N|\mathrm{LF}\}$, as well as the expected number of stages required in the equal-probability (EP) configuration, $\mathsf{E}\{N|\mathrm{EP}\}$, for which $p_1 = \cdots = p_t = 1/t$. (The LF configuration can be regarded as a "worst case" configuration for all $\boldsymbol{p}$ in the preference zone, while the EP configuration is a worst case configuration for all $\boldsymbol{p}$ in the t-dimensional unit simplex.) We see that the expected number of stages for procedure $\mathcal{M}_{BG}$ is almost always less than the fixed sample size n of procedure $\mathcal{M}_{BEM}$, usually by a substantial margin.

Remark 3 Many other sequential procedures have been proposed for the goal of selecting the most probable multinomial cell using the indifference-zone approach. For instance, Cacoullos and Sobel (1966) proposed an inverse sampling procedure for which sampling stops when the frequency of any cell reaches a

preassigned number; Alam (1971) gave a procedure that stops when the difference between the largest and second-largest cell frequencies reaches a preassigned number; Ramey and Alam (1979) (and Goldsman and Bechhofer 1985a) investigated a stopping rule based on *both* of the above-mentioned criteria; Chen (1992) studied a stopping rule that combines Ramey and Alam's stopping rule, the curtailment rule of the Bechhofer and Kulkarni (1984), and truncation.

In terms of the expected number of stages to termination, procedure $\mathcal{M}_{BG}$ compares quite favorably (over a broad range of practical $(t; \delta^\star, P^\star)$-values) to the other procedures discussed in Remark 3. Thus, if sequential sampling is an option of the experimenter and appropriate truncation numbers are available, we recommend the use of procedure $\mathcal{M}_{BG}$.

3 APPLICATIONS TO SIMULATION

This section discusses a nonparametric application of the multinomial selection problem. In particular, we give an interpretation that enables us to select that one of t competing simulations having the highest probability of producing the "most desirable" output statistic from a given vector-observation of the competing simulations' output statistics. This nonparametric interpretation follows from remarks due to Bechhofer and Sobel (1958) and further studied by Dudewicz (1971).

Let $W_{1j}, \ldots, W_{tj}$ $(j \geq 1)$ be independent output statistics from $t \geq 2$ simulations; the W_{ij}'s can be discrete or continuous random variables with unknown probability density or mass functions. For example, W_{ij} could represent the cost incurred in the jth independent simulation replication of the ith inventory policy under consideration. Suppose we take independent and identically distributed vector-observations $\boldsymbol{W}_j = (W_{1j}, \ldots, W_{tj})$ $(j \geq 1)$. For a particular vector-observation $\boldsymbol{W}_j$, suppose that the experimenter can determine which one of the t observations W_{ij} $(1 \leq i \leq t)$ is the "most desirable." The term "most desirable" is based on some criterion of goodness designated by the experimenter, and can be quite general. For instance, as described in Bechhofer, Santner, and Goldsman (1993), the "most desirable" observation might correspond to:

- The largest crop yield based on a vector-observation of t agricultural plots using competing fertilizers.
- The smallest sample average customer waiting time based on a simulation run of each of t competing queueing strategies.
- The smallest estimated variance of customer waiting times (from the above simulations).
- The smallest sample proportion of customer waiting times (from the above simulations) that are greater than some designated bound w.

Suppose that $X_{ij} = 1$ [0] if W_{ij} is [is not] the "most desirable" of the components of $\boldsymbol{W}_j$ $(1 \leq i \leq t,\ j \geq 1)$ (ties are not allowed). Then $\boldsymbol{X}_j$ $(j \geq 1)$ has a multinomial distribution with probability $\boldsymbol{p}$, where

$$p_i = P\{W_{i1} \text{ is the "most desirable" component of } \boldsymbol{W}_1\} \quad (1 \leq i \leq t).$$

The problem of finding the cell having the largest p_i can be interpreted as that of finding the component of $\boldsymbol{W}_j$ having the highest probability of yielding the "most desirable" observation from a particular vector-observation; this reformulated problem can be approached using the multinomial selection methods described in this article.

Example 8 We will use procedure $\mathcal{M}_{BG}$ to determine which of $t = 3$ job shop set-ups is most likely to yield reasonable times-in-system for a certain manufactured product. Due to the complicated configurations of the candidate job shops, it is necessary to simulate the three competitors. Suppose that the jth simulation run of configuration i $(1 \leq i \leq 3,\ j \geq 1)$ yields W_{ij}, the proportion of 1000 times-in-system greater than 20 minutes. Management has decided that the "most desirable" component of $\boldsymbol{W}_j$ will be that component corresponding to the smallest W_{ij} $(1 \leq i \leq 3)$. If p_i denotes the probability that configuration i yields the smallest component of $\boldsymbol{W}_j$, then we wish to select the job shop configuration that corresponds to $p_{[3]}$. By the above remarks, this problem is the same as that of selecting the multinomial cell associated with $p_{[3]}$. Suppose that we specify $P^\star = 0.75$ and $\delta^\star = 3.0$. The truncation number from Table 1 for procedure $\mathcal{M}_{BG}$ is $n_0 = 5$. We apply the procedure to the data found in the table below, and select cell 2 (i.e., shop configuration 2).

m	w_{1m}	w_{2m}	w_{3m}	$\boldsymbol{x}_m$	$\boldsymbol{y}_m$
1	0.13	0.09	0.14	(0,1,0)	(0,1,0)
2	0.24	0.10	0.07	(0,0,1)	(0,1,1)
3	0.17	0.11	0.12	(0,1,0)	(0,2,1)
4	0.13	0.08	0.02	(0,0,1)	(0,2,2)
5	0.14	0.13	0.15	(0,1,0)	(0,3,2)

4 CONCLUSIONS

Multinomial selection procedures are quite useful in the simulation arena. In particular, sequential procedures such as procedure $\mathcal{M}_{BG}$ are readily adapted as

nonparametric methods for selecting the "best" of a number of competing simulations.

There are a number of extensions of and complements to the above work. For instance,

- To facilitate the use of the procedures discussed in this article, Adelman, Goldsman, and Hartmann (1993) provide an algorithm that computes on-line truncation numbers as well as the resulting performance characteristics of the procedures.
- Bechhofer, Goldsman, and Jennison (1989) consider a *multivariate* generalization of the single-stage procedure discussed in §2.1.
- Dudewicz (1971) and Auclair (1993) investigate robustness aspects of multinomial nonparametric procedures.
- Another important multinomial problem is that of finding the *least* probable cell. Chen (1992), among others, gives a procedure for this problem.
- Gupta (1956) devised the *subset selection* approach, resulting in a set of statistical techniques that are complementary to indifference-zone procedures. Chen (1988) provides a tutorial on subset selection procedures for the multinomial problem.

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