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Smoothed perturbation analysis algorithm for A G/G/1 routing problem

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Abstract

The smoothed perturbation analysis (SPA) algorithm is proposed for estimating the derivative of the mean delay with respect to the routing probability for a routing problem in data-communication networks. The algorithm requires minmum knowledge about the system and is very suitable for on-line optimization of data-communication networks. It is shown that the SPA algorithm is unbiased.

1 Introduction

In this paper we consider the derivative estimation problem for a single server queueing system. The system is a single server with a switch that controls the arrival stream to the server. The descipline is first-come-first-serve. Whenever a customer arrives, with probability p it will get into the queue and with probability $(1-p)$ it will be rejected. The switching mechanism is independent of everything else. Both the interarrival time distribution and the transmission time distribution are general(G/G/1). The mean arrival rate is denoted by λ. We assume that the system is stable, i.e., the probability that the queue content goes to infinity is zero. We will mainly consider the the mean delay per transmitted customer as our performance measure in this paper. The decision parameter is the switching probability p. We want to estimate $\frac{\partial EL}{\partial p}$ where EL denotes the average performance measure. This derivative, sometimes called the marginal delay, is important in distributed optimization of data-communication networks [6,20,1,2]. The purpose of this paper is to present an estimation algorithm which has advantages over other algorithms. The algorithm is developed based on the Smoothed Perturbation Analysis (SPA) approach. In Poisson arrival case the algorithm turned out to be similar to the algorithm proposed in [1], which has been discussed in [6,20] for possible applications in distributed optimization of data-communication networks. The algorithm requires minimum knowledge about the system and therefore is particularly suitable for on-line optimization purpose. However in [1] it is concluded that the algorithm is biased even for M/M/1 case. Through our derivation it is shown that the algorithm is unbiased for general G/G/1 system. Simulation results are consistent with our analysis. We will point out the error in [1] that yields the wrong conclusion.

In the next section we briefly describe two recently developed derivative estimators to motivate the SPA approach.

2 Derivative estimators

Derivative estimation of discrete event systems such as queueing systems has recently become a very active research area [12,8,18,16,17]. Infinitesimal Perturbation Analysis (IPA) [12,3,21] and Likelihood Ratio method [8,18,17] are valuable accomplishments in this area. For systems where IPA gives strongly consistent estimate it is often the most efficient sensitivity estimator [22,4]. IPA tries to estimate the derivative directly along a single realization instead of approximating it by the ratio of finite differences. For many practically important systems, however, the IPA method does not give consistent estimates [3]. The Likelihood ratio method, othe other hand, is quite general. However the variance of the estimate goes up with the length of the sample path, makes it more suitable for systems with short regenerative cycles.

Smoothed Perturbation Analysis (SPA) [9,10] is an effort to overcome the difficulty of the IPA method while retaining its advantages. SPA is also an attempt to combine queueing theory analysis and the knowledge gained from a realization of the system state process to get new theoretical and practical results.

The fundamental idea of SPA, known as "conditioning", is to do conditional expectation before differentiation, thereby allowing the smoothing property of the conditional expectation to "smooth out" the discontinuity of the sample performance function. The motivation behind this approach stems from the difficulty of IPA that occurs in dealing with discontinuous sample performance functions [13,3,11].

To be precise, consider a stochastic system defined on a underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$. A sample path of such a system is denoted as $\xi(\theta,\omega)$, where θ is regarded as a decision parameter and $\omega \in \Omega$. The space Ω is taken to be $[0,1]^\infty$ and each ω interpreted as an output sequence from a "random number generator" that generates uniformly distributed random numbers; then the function ξ can be thought of as a simulation algorithm. Sometimes we simply denote the sample path by ξ. Usually θ is a parameter of a cumulative distribution function $G(\theta,\cdot)$ involved in the evolution of the system state process. A typical example is that $G(\theta,\cdot)$ is the c.d.f. of the transmission time of a server in a communication network and θ is the mean transmission time of the server. Another example of decision parameter is the routing probability in our routing problem. Note that p is a parameter of the distribution of the switch status which is a random variable taking only two possible values (on or off). Sometimes $\xi(\theta,\omega) = (x_1(\theta,\omega), x_2(\theta,\omega), ..., x_{n(\theta,\omega)}(\theta,\omega))$ is used to denote a sample path, where the $x_i(\theta,\omega)$ are the sample random variables involved in the simulation. Note that for different ω the number of $x_i(\theta,\omega)$ is generally different.

The sample performance function of the system is defined as a real-valued function $L(\xi(\theta,\omega))$. For any θ, $L(\xi(\theta,\omega))$ is a random variable on $(\Omega, \mathcal{F}, \mathcal{P})$. The performance $J(\theta)$ is the expected value of the sample performance function:

$$J(\theta) = \mathsf{E}L(\xi(\theta,\omega)). \tag{1}$$

In this paper we concerned about the estimation of

$$\frac{dJ}{d\theta} = \frac{d}{d\theta}\mathsf{E}L \tag{2}$$

in a routing problem. Following is a brief description of the Infinitesimal Perturbation Analysis estimator and the Likelihood Ratio estimator.

1. **Infinitesimal Perturbation Analysis (IPA) Estimator** [12]:

   The IPA estimator is defined by

$$\begin{aligned}\left[\frac{dJ}{d\theta}\right]_{IPA} &\triangleq \frac{\partial L(\xi(\theta,\omega))}{\partial\theta} \\ &= \lim_{\Delta\theta\to 0}\frac{L(\xi(\theta+\Delta\theta,\omega)) - L(\xi(\theta,\omega))}{\Delta\theta} \\ &= \sum_{i=1}^{n}\frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial\theta}\end{aligned} \tag{3}$$

   for appropriately defined $\frac{\partial L}{\partial x_i}$ and $\frac{\partial x_i}{\partial\theta}$ [21,7]. Assuming ergodicity, this estimate converges to $\mathsf{E}\frac{dJ}{d\theta}$ with probability one as the simulation time goes to ∞. The IPA estimator therefore only applies to those systems where the sample performance $L(\xi(\theta,\omega))$ satisfies

$$\frac{d}{d\theta}\mathsf{E}L(\xi(\theta,\omega)) = \mathsf{E}\frac{\partial}{\partial\theta}L(\xi(\theta,\omega)). \quad (4)$$

Note that in contrast to the traditional finite difference estimator which estimate $\Delta J/\Delta\theta$ the IPA estimator requires only 1 simulation run [12]. On top of that, the IPA method avoids taking the ratio of two small numbers, which is an improtant source of inaccuracy [22].

2. **Score Function (SF) Estimator** [18]

   Score Function estimator [18] is a special case of the *Likelihood Ratio (LR)* estimator [8]. It can demonstrate the features of a general Likelihood Ratio estimator. The SF estimator is defined by

$$\left[\frac{dJ}{d\theta}\right]_{SF} \triangleq L(x_1,x_2,...,x_n)\frac{d}{d\theta}\ln f_\theta(x_1,x_2,...,x_n) \quad (5)$$

   where $L(x_1,x_2,...,x_n)$ denotes the value of $L((x_1(\theta,\omega),x_2(\theta,\omega)$ evaluated at $x_1,x_2,...,x_n$ and f_θ denotes the joint density of the n random variables involved. Here the value of n is fixed at the nominal value. This estimate is unbiased due to

$$\begin{aligned}\frac{dJ}{d\theta} &= \frac{d}{d\theta}\mathsf{E}L(\xi(\theta,\omega))\\ &= \frac{d}{d\theta}\int L(x_1,x_2,...,x_n)f_\theta(x_1,x_2,...,x_n)dx\\ &= \int L\frac{d\ln f_\theta}{d\theta}f_\theta dx\\ &= \mathsf{E}\left[L\frac{d}{d\theta}\ln f_\theta\right] \quad (6)\end{aligned}$$

   The variance of the LR estimate goes up with the length of the simulation, makes it more suitable for systems with short regenarative cycles [4].

Smoothed Pertubation Analysis [9,10] is proposed to overcome the difficulty of IPA. The variance of the SPA estimate goes to zero when the simulation time goes to infinity. In our routing problem the SPA estimate does not require any knowledge about the distribution of the interarrival times and transmission times. Neither it needs the independence assumptions about the interarrival times and transmission times. It is well known that for a network with loops the independence assuptions are not practical. Therefore the SPA estimate is particularly suitable for on-line optimization of data-communication networks.

3 Smoothed Perturbation Analysis (SPA) method

As mentioned before, the basic idea of SPA is to make use of the smoothing property of the conditional expectation to smooth out the discontinuity of the sample performance function, thereby allowing the interchange of expectation and differentiation for most practically important cases.

We first introduce an important concept in SPA: the ***characterization*** of a simulation run. Let $\mathcal{F}_t$ be an increasing family of σ-algebras on $(\Omega,\mathcal{F},P)$, which is generated by the simulated time t, and let T be the duration of the simulation run. The characterization z is an $\mathcal{F}_T$-measurable random vector. Intuitively, $z(\xi(\theta,\omega))$ can be understood as follows. In simulation, ξ is a sequence of data from which we can calculate the queue contents, status of routing switches, transmission times, interarrival times, certain kinds of residual lifetimes, etc. A characterization $z(\xi(\theta,\omega))$ is simply a set of data obtainable from ξ. Choice of an appropriate characterization depends on the particular concrete problem being addressed.

We assume that for most practically important cases the following equality holds, due to the smoothing property of the conditional expectation:

$$\begin{aligned}\frac{\partial}{\partial\theta}EL(\xi(\theta,\omega)) &= \lim_{\Delta\theta\to 0}E\frac{E[\Delta L(\xi(\theta,\omega))|z]}{\Delta\theta}\\ &= E\lim_{\Delta\theta\to 0}\frac{E[\Delta L(\xi(\theta,\omega))|z]}{\Delta\theta} \quad (7)\end{aligned}$$

where $\Delta L(\xi(\theta,\omega)) \triangleq L(\xi(\theta+\Delta\theta,\omega)) - L(\xi(\theta,\omega)))$. It can be easily seen later that this assumption is valid for our routing problem.

Thus we have the SPA estimator

$$\left[\frac{\partial}{\partial\theta}EL(\xi(\theta,\omega))\right]_{SPA} = \lim_{\Delta\theta\to 0}\frac{E[\Delta L(\xi(\theta,\omega))|z]}{\Delta\theta} \quad (8)$$

Comparing (7) and (4) it is easy to see that in general (7) is easier to satisfy than (4), since the inner integral on the right hand side of (7) "smooths" the function that is to be differentiated. More precisely, assuming that $\lim_{\Delta\theta\to 0}(E[\Delta L|z]/\Delta\theta])$ exists w.p.1, we then have

$$E[\Delta L|z] = \left(\lim_{\Delta\theta\to 0}\frac{E[\Delta L|z]}{\Delta\theta}\right)\Delta\theta + r(\Delta\theta,z). \quad (9)$$

If $\lim_{\Delta\theta\to 0}(E[r(\Delta\theta,z)]/\Delta\theta) = 0$, then we have

$$\begin{aligned}\lim_{\Delta\theta\to 0}\frac{E\Delta L}{\Delta\theta} &= \lim_{\Delta\theta\to 0}\frac{EE[\Delta L|z]}{\Delta\theta}\\ &= \lim_{\Delta\theta\to 0}E\left[\lim_{\Delta\theta\to 0}\frac{E[\Delta L|z]}{\Delta\theta}+\frac{r(\Delta\theta,z)}{\Delta\theta}\right]\\ &= E\lim_{\Delta\theta\to 0}\frac{E[\Delta L|z]}{\Delta\theta}+\lim_{\Delta\theta\to 0}\frac{Er(\Delta\theta,z)}{\Delta\theta}\\ &= E\lim_{\Delta\theta\to 0}\frac{E[\Delta L|z]}{\Delta\theta} \quad (10)\end{aligned}$$

i.e., (8) holds. We will see later that in our routing problem we do have

$$\lim_{\Delta\theta\to 0}\frac{E[r(\Delta\theta,z)]}{\Delta\theta} = 0. \quad (11)$$

It is worth noting that the IPA method is a special case of the SPA method in the sense that, if we take z to be the sample path ξ itself, the SPA estimator reduces to the usual IPA estimator. On the other hand, if we take z as a constant, the SPA estimator then degenerates to the theoretical calculation of the derivative of the average performance. Between the two extremes a whole spectrum of possibilities exists.

A common feature of the LR and the SPA(including IPA) estimators is that they both seek to estimate the derivative directly rather than approximating it by $\Delta J/\Delta\theta$ via repeated simulation. This feature enables the SPA to avoid the twin evils of nonlinear effects (when $\Delta\theta$ is large) and "noise" magnification when $\Delta\theta$ is small).

4 SPA algorithm for general performance measure

To develop the SPA estimation algorithm we need to understand the structure of the sample path of the system. The sample path of our system can be generated as follows. At the instant of the ith arrival an interarrival time is generated to schedule the $(i+1)$th arrival. An uniformly distributed random number u_i is then independtly generated and compared with the routing probability p. If $u_i \le p$ then this arrival will be accepted, otherwise it is rejected. At the instant of a departure either a transmission time is generated to schedule the next departure time or the server will become idle until the next arrival begin to get its transmission.

Suppose the simulation run of our simple routing system teminates at the nth departure. Denote the length of the run by $T(\xi(\theta,\omega))$. $T(\xi(\theta,\omega))$ is finite with probability 1 for finite n. Now suppose that the routing probability p is decreased by Δp. The Infinitesimal Perturbation Analysis (IPA) estimator does not work here: any negative perturbation of p may cause the rejection of an originally accepted customer, hence causes a discontinuity of the sample performance function such as average interdeparture time and average delay. IPA estimates are 0 here. This can be seen more clearly from the following.

According to the perturbation analysis convention we call the path generated at the nominal value of the decision parameter p the nominal path and the path generated at the perturbed value of the decision parameter $p - \Delta p$ the perturbed path. For a sample path ξ with finite length, with probability 1 there exists a $\delta(\xi) > 0$ such that when $\Delta p < \delta(\xi)$ the perturbed path $\xi(p - \Delta p, \omega)$ and the nominal path $\xi(p, \omega)$ are exactly the same. Consequently the difference of the sample performances $\Delta L(\xi)$ is 0 w.p.1. Thus the IPA estimate

$$\lim_{\Delta p \to 0} \frac{\Delta L(\xi(p,\omega))}{\Delta p}$$

is 0 w.p.1. Since for most performances the derivatives are apparently not zero, the IPA estimates are biased in our routing problem.

Now introduce a negative pertubation $-\Delta p$ to the switching probability. On the perturbed path there will be fewer customers accepted by the switch. We call the path where there is one fewer accepted customer than the nomninal path the one-removal path. Notation PP_i is used to denote the ith one-removal path, i.e., the path where the ith customer is removed from the nominal path. The characterization of the nominal path is chosen as the switch status (which can be either on or off) sequence, i.e., we have

$$z(\xi) \triangleq (s_j(\xi), j = 1, \cdots, N(\xi)) \tag{12}$$

where $s_j(\xi)$ is the switch status at the jth arrival and $N(\xi)$ is the total number of arrivals in the simulation run ξ. More precisely,

$$s_j(\xi(p,\omega)) \triangleq 1\{u_j(\xi(p,\omega) \leq p\}. \tag{13}$$

In other words, $s_j(\xi(p,\omega)) = 1$ means that the jth arrival is accepted on the nominal path and $s_j(\xi(p,\omega)) = 0$ means that the jth arrival is rejected on the nominal path. Later on we can see that we actually don't need to be able to observe the underlying u_j for implementing the estimation algorithm. In the following we use Δs_j to denote $s_j(\xi(p - \Delta p, \omega)) - s_j(\xi(p,\omega))$ and $\Delta s_i = -1$ to mean $\Delta s_1 = 0, \cdots, \Delta s_{i-1} = 0, \Delta s_i = -1, \Delta s_{i+1} = 0, \cdots, \Delta s_n = 0$ for simplicity.

Now the effect of $-\Delta p$ to the conditional expected performance measure $E[L(\xi)|z]$, denoted as $E[\Delta L(\xi)|z]$, can be expressed as follows.

$$\begin{aligned}
E[\Delta L|z] &= \sum_{i=1}^{n} P(\Delta s_i = -1|z) E[\Delta L|\Delta s_i = -1, z] \\
&+ \sum_{i\neq j}\sum_{j} P(\Delta s_i = -1, \Delta s_j = -1|z) E[\Delta L|\Delta s_i = -1, \\
&\qquad \Delta s_j = -1,\ z] \\
&+ \cdots + P(\Delta s_i = -1, \cdots, \Delta s_n = -1) E[\Delta L|\Delta s_1 = -1, \\
&\qquad \cdots, \Delta s_n = -1, z] \\
&= \sum_{i=1}^{n} P_i E[\Delta L|\Delta s_i = -1, z] \\
&+ \sum_{i\neq j}\sum_{j} P_{ij} E[\Delta L|\Delta s_i = -1, \Delta s_j = -1, z] \\
&+ \cdots + P_{1\cdots n} E[L|\Delta s_i = 1, \cdots, \Delta s_n = 1, z]
\end{aligned}$$

where $P_i \triangleq P(\Delta s_i = -1|z)$ is the probability that the ith originally accepted customer is removed due to the perturbation $-\Delta p$ given that $s_i = 1$; $E[\Delta L|\Delta s_i = -1, z]$ is the change of the sample performance caused by this removal; $P_{ij} \triangleq P(\Delta s_i = -1, \Delta s_j = -1|z)$ is the probability that both the ith and the jth originally accepted customers are removed simultaneously due to the perturbation $-\Delta p$, $E[\Delta L|\Delta s_i = -1, \Delta s_j = -1, z]$ is the effect of the two simultaneous removals; and so on.

Since the random variables $s_i, i = 1, \cdots, N$ are independent, we have

$$\begin{aligned}
P_i &= P(\Delta s_i = -1|z) \\
&= P(\Delta s_i = -1|s_i = 1) \\
&= P(s_i(\xi(p - \Delta p, \omega)) = 0|s_i = 1) \\
&= \frac{P(p - \Delta p \leq u_i < p)}{P(u_i < p)} \\
&= \frac{\Delta p}{p}
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
P_{ij} &= P(\Delta s_i = -1, \Delta s_j = -1|z) \\
&= P(\Delta s_i = -1, \Delta s_j = -1|s_i = 1, s_j = 1) \\
&= P(s_i(\xi(p - \Delta p, \omega)) = 0, s_j(\xi(p - \Delta p, \omega)) = 0|s_i = 1, s_j = 1) \\
&= \frac{P(p - \Delta p \leq u_i < p) P(p - \Delta p \leq u_j < p)}{P(u_i < p) P(u_j < p)} \\
&= \frac{(\Delta p)^2}{p^2}.
\end{aligned} \tag{16}$$

In general we have

$$P_{\underbrace{i\cdots j}_{k}} = (\frac{\Delta p}{p})^k. \tag{17}$$

Since P_i is linear in Δp and $E[\Delta L|\Delta s_i = -1, z]$ is independent of Δp, $\lim_{\Delta p \to 0} \frac{E[\Delta L|z]}{\Delta p}$ does exist w.p.1 and can be written as

$$\lim_{\Delta p \to 0} \frac{E[\Delta L|z]}{\Delta p} = \sum_{i=1}^{n} \lim_{\Delta p \to 0} \frac{P_i}{\Delta p} E[\Delta L|\Delta s_i = -1, z]. \tag{18}$$

Thus we have

$$E[\Delta L|z] = \sum_{i=1}^{n} P_i E[\Delta L|\Delta s_i = -1, z] + r(\Delta p, z). \tag{19}$$

Now assume the perfromance measure $EL(\xi)$ has the property that

$$|E[\Delta L| \underbrace{\Delta s_i = -1, \cdots, \Delta s_j = -1}_{k}, z]| \leq B(z) \leq \infty \quad for \quad k \geq 2 \tag{20}$$

where $B(z)$ satisfies $EB(z) < \infty$.

We then have

$$\begin{aligned}
\left|\frac{E[r(\Delta p, z)]}{\Delta p}\right| &\leq \frac{1}{\Delta p} E[\sum_{i\neq j}\sum_{j} P_{ij} |E[\,\Delta L|\Delta s_i = -1, \Delta s_j = -1, \\
&\qquad z]| + \cdots + P_{1\cdots n}|E[L|z]|\,] \\
&\leq \frac{1}{\Delta p} E\left[\sum_{i\neq j}\sum_{j} P_{ij} B(z) + \cdots + P_{1\cdots n} B(z)\right] \\
&= EB(z) \sum_{k=2}^{n} \binom{n}{k} \frac{(\Delta p)^{k-1}}{p^k} \\
&= EB(z) \sum_{k=2}^{n} \binom{n}{k} \frac{(\Delta p)^{k-1}}{p^k} \\
&\to 0.
\end{aligned} \tag{21}$$

The requirement for unbiasedness

$$\lim_{\Delta p \to 0} \frac{E[r(\Delta p, z)]}{\Delta p} = 0 \tag{22}$$

is thus satisfied. Note that the above assumptions are not restrictive at all. For example, we will show that the mean delay in our routing problem have the above property (20) later in section 5.

To emphasize the "trick" of SPA, which in our case is to differentiate probabilities instead of conditional expected performance measures, we introduce the notation $\frac{\partial}{\partial p} p_i$ as follows.

$$\begin{aligned}
\frac{\partial}{\partial p} p_i &\triangleq \lim_{\Delta p \to 0} \frac{P_i}{\Delta p} \\
&= \frac{1}{p}.
\end{aligned} \tag{23}$$

Since

$$\lim_{\Delta p \to 0} E[\Delta L|\Delta s_i = -1, z] = E[\Delta L|\Delta s_i = -1, z], \tag{24}$$

the SPA estimator can be written as

$$\left[\frac{\partial}{\partial p}EL\right]_{SPA} = \sum_{i=1}^{n}\frac{\partial}{\partial p}p_i \lim_{\Delta p\to 0} E[\Delta L|\Delta s_i=-1,z]$$
$$= \sum_{i=1}^{n}\frac{1}{p}E[\Delta L|\Delta s_i=-1,z] \qquad (25)$$

Although $E[\Delta L|\Delta s_i=-1,z]$ is still difficult to calculate, it is easy to get an unbiased estimate of this quantity. In the following we will see that a sample of ΔL caused by removing the ith accepted customer ($\Delta s_i=-1$) can be used to replace the conditional expected value $E[\Delta L|\Delta s_i=-1,z]$.

5 SPA algorithm for the mean interdeparture time

Let us consider the SPA derivative estimator for the mean interdeparture time first. The derivative of the mean departure time $EL(\xi)$ with respect to the routing probability p can be calculated analytically. The mean interdeparture time is apparently $\frac{1}{\lambda p}$. The derivative $\frac{\partial}{\partial p}EL$ therefore is $-\frac{1}{\lambda p^2}$. The reason that we derive the SPA estimator for $\partial EL(\xi)/\partial p$ here is because this derivation can help us to see clearly why SPA algorithms should work for this system. The calculation of $E[\Delta L|\Delta s_i=-1,z]$ is very simple for the average interdeparture time here. The sample average interdeparture time for the ith one-removal path $PP_i, i=1,\cdots,n-1$ is

$$\frac{T(\xi)}{n-1} \qquad (26)$$

and we have

$$E[\Delta L|\Delta s_i=-1,z] = -E\left[\frac{T(\xi)}{n-1}-\frac{T(\xi)}{n}\mid \Delta s_i=-1,z\right]$$
$$= -\frac{1}{n(n-1)}E\left[T(\xi)\mid \Delta s_i=-1,z\right]$$
$$= -\frac{1}{n(n-1)}E\left[T(\xi)\mid z\right]$$

where the negative sign comes from the fact that the perturbation of the decision parameter is negative. The last equality is due to the fact that $T(\xi)$ is independent of Δs_i for $i=1,\cdots,n-1$. For the case of $i=n$ the length of the perturbed path is shorter than $T(\xi)$, since the removal of the last transmitted customer makes the sample path terminated at the departure of the $n-1$th transmitted customer. However we are going to ignore this "tail effect", since this effect is small when n is large.

Now we have

$$\left[\frac{\partial}{\partial p}EL\right]_{SPA} = -\sum_{i=1}^{n}\frac{\partial}{\partial p}p_i\frac{1}{n(n-1)}E\left[T(\xi)\mid z\right]$$
$$= -\sum_{i=1}^{n}\frac{1}{pn(n-1)}E\left[T(\xi)\mid z\right] \qquad (27)$$

The above estimator can be further simplified as follows.

Since

$$E\left[\left[\frac{\partial}{\partial p}EL\right]_{SPA}\right] = -E\left[\sum_{i=1}^{n}\frac{1}{pn(n-1)}E\left[T(\xi)\mid z\right]\right]$$
$$= -\sum_{i=1}^{n}\frac{1}{pn(n-1)}E\left[E\left[T(\xi)\mid z\right]\right]$$
$$= -\sum_{i=1}^{n}\frac{1}{pn(n-1)}E\left[T(\xi)\right]$$
$$= E\left[-\sum_{i=1}^{n}\frac{1}{pn(n-1)}T(\xi)\right] \qquad (28)$$

we can use

$$-\sum_{i=1}^{n}\frac{1}{pn(n-1)}T(\xi) \qquad (29)$$

as our estimator.

The unbiasedness of the above estimate is easy to check. We have

$$E\left[\left[\frac{\partial}{\partial p}EL\right]_{SPA}\right] = -\sum_{i=1}^{n}\frac{1}{pn(n-1)}E\left[T(\xi)\right]$$
$$= -\sum_{i=1}^{n}\frac{1}{pn(n-1)}\frac{n}{\lambda p}$$
$$= -\sum_{i=1}^{n}\frac{1}{\lambda p^2(n-1)}$$
$$\to -\frac{1}{\lambda p^2} \qquad (30)$$

which shows that estimate (29) is asymptotically unbiased. Note that for any finite run estimate (29) is biased due to the tail effect which we ignored for the convenience of the calculation. As mentioned before, the effect of the removal of the nth customer should be different from others, since the simulation time is also shortened. The "theoretical" SPA estimate (25) is unbiased as being proved.

This result is trivial. Neverthless it demonstrates that in estimating the derivative we need only consider one jump (in our case this is just one removal of originally accepted customers) at one time. This is true for quite general discrete event systems [10] and is particularly clear in this example, since the probability that more than one jump happen is clearly of higher order than Δp. A more interesting performance measure in this system, the mean delay can also be treated the same way. The only thing that needs more consideration is the calculation of $E[\Delta L|\Delta s_i=-1,z]$, which we will discuss in the next section.

6 SPA algorithm for the mean delay

It is clear from the above that for estimating the first derivative we only need to consider the effect of removing only one customer at one time. The sample average delay $L(\xi)$ for sample path ξ is defined as

$$L(\xi)=\frac{A(\xi)}{n} \qquad (31)$$

where $A(\xi)$ is the total time that all the n customers spent in the system on the nominal path ξ. The total time that the $(n-1)$ customers spent in the system for the ith one-removal path PP_i is denoted by $A_i(\xi)$. We then have

$$E[\Delta L(\xi)|\Delta s_i=-1,z] = E\left[\frac{A(\xi)}{n}-\frac{A_i(\xi)}{(n-1)}\mid \Delta s_i=-1,z\right]$$
$$= E\left[\frac{A(\xi)-A_i(\xi)}{n-1}-\frac{A(\xi)}{n(n-1)}\mid \Delta s_i=-1,z\right]$$
$$= E\left[\frac{A(\xi)-A_i(\xi)}{n-1}-\frac{L(\xi)}{n-1}|\Delta s_i=-1,z\right] \qquad (32)$$

The SPA estimator then takes the form

$$\left[\frac{\partial}{\partial p}EL\right]_{SPA} = \sum_{i=1}^{n}\frac{\partial}{\partial p}p_i \lim_{\Delta p\to 0}E[\Delta L|\Delta s_i=-1,z]$$
$$= \sum_{i=1}^{n}\frac{1}{p}E\left[\frac{A(\xi)-A_i(\xi)}{n-1}-\frac{L(\xi)}{n-1}\mid \Delta s_i=-1,z\right]$$
$$= \frac{1}{(n-1)p}\sum_{i=1}^{n}E\left[A(\xi)-A_i(\xi)-L(\xi)\mid \Delta s_i=-1,z\right]$$
$$= \frac{1}{(n-1)p}\sum_{i=1}^{n}E\left[A(\xi)-A_i(\xi)-L(\xi)\mid z\right]. \qquad (33)$$

The last equality is due to the independence of $A(\xi), A_i(\xi)$ and $L(\xi)$ with respect to Δs_i.

Since

$$E\left[\frac{1}{(n-1)p}\sum_{i=1}^{n} E\left[A(\xi)-A_i(\xi)-L(\xi)\mid \Delta s_i=-1,z\right]\right] = E\left[\frac{1}{(n-1)p}\sum_{i=1}^{n}\left[A(\xi)-A_i(\xi)-L(\xi)\right]\right] \quad (34)$$

we can use

$$\frac{1}{(n-1)p}\sum_{i=1}^{n}\left[A(\xi)-A_i(\xi)-L(\xi)\right] \quad (35)$$

as our estimator.

To prove that the SPA estimate is unbiased we have to show that

$$\lim_{\Delta\theta\to 0}\frac{E[r(\Delta\theta,z)]}{\Delta\theta}=0. \quad (36)$$

As mentioned before, to do this we only need to show that the delay EL has property (20). This can be done as follows.

We have

$$\begin{aligned}
&\left|E[\Delta L(\xi)|\underbrace{\Delta s_i=-1,\cdots,\Delta s_j=-1}_{k},z]\right| \\
&= \left|E\left[\frac{A(\xi)}{n}-\frac{A_{i\cdots j}(\xi)}{n-k}\mid z\right]\right| \\
&< |E[\frac{A(\xi)}{n}|z]|+|E[\frac{A_{i\cdots j}(\xi)}{n-k}\mid z]| \\
&< \left|E\left[\frac{A(\xi)}{n-k}|z\right]\right|+\left|E\left[\frac{A(\xi)}{n-k}\mid z\right]\right| \\
&\triangleq \frac{2n}{n-k}L(z) \quad for \quad k<n
\end{aligned} \quad (37)$$

where $A_{i\cdots j}$ denotes the total time the $n-k$ remainning customers spent in the system and $L(z)\triangleq E[\frac{A}{n}|z]$. It can be easily seen that (20) is satisfied for $k<n$. In the case of $k=n$ we define $E[L(\xi(p-\Delta p,\omega))|\Delta s_1=-1,\cdots,\Delta s_n=-1,z]=0$ for natural reasons. Thus we have

$$|E[\Delta L|\Delta s_1=-1,\cdots,\Delta s_n=-1,z]|=|E[L|z]| \quad (38)$$

and (20) is also satisfied. As a consequence the SPA estimate is unbiased.

For implementing the SPA estimator we need to calculate $\sum_{i=1}^{n}[A(\xi)-A_i(\xi)]$. This can be done as follows [20].

It is easy to see that removing a customer from one given busy period has no effect on customers transmitted in other busy periods, so that we only need to consider the effect of the removal on customers from the same busy period. For the mth busy period, denoted by BP_m, let $c_k^l(m)$ be the amount of system time that the kth customer in BP_m would save if the lth customer were to be removed. If we denote by $d_l(m)$ and $a_l(m)$, respectively, the departure and arrival time of the lth customer in BP_m relative to the beginning of the busy period, say, then one can easily obtain the following recursive formulas:

$$c_k^l(m)=0, \qquad for \quad k<l, \quad (39)$$

$$c_k^l(m)=d_l(m)-a_l(m) \quad for \quad k=l, \quad (40)$$

$$c_{l+1}^l(m)=d_l-max(a_{l+1}(m),d_{l-1}(m)) \quad where\ we\ take\ d_0(m)=0 \quad (41)$$

and for $k>l+1$

$$c_k^l(m)=min(c_{k-1}^l(m),d_{l-1}(m)-a_l(m)). \quad (42)$$

Over M busy periods, containing each $n_m, m=1,...,M$ customers, the total effect is therefore

$$\sum_{i=1}^{n}[A(\xi)-A_i(\xi)]=\sum_{m=1}^{M}\sum_{l=1}^{n_m}\sum_{k=l}^{n_m}c_k^l(m) \quad (43)$$

and the estimate is

$$\begin{aligned}
\left[\frac{\partial}{\partial p}EL\right]_{SPA} &= \frac{1}{p(n-1)}\sum_{i=1}^{n}[(A(\xi)-A_i(\xi))-L(\xi)] \\
&= \frac{1}{p(n-1)}\sum_{m=1}^{M}\sum_{l=1}^{n_m}\sum_{k=l}^{n_m}c_k^l(m)-\frac{1}{p}L(\xi).
\end{aligned} \quad (44)$$

When the external arrival stream (before the switch) is a Poisson process with rate λ, perturbing p is equivalent to perturbing the "net" arrival rate λp. In [1] an algorithm for estimating the derivative of the mean delay per unit time (instead of per customer as in our case), denoted by D, with respect to the net arrival rate is proposed for the M/G/1 case. Let the derivative be denoted by $D'\triangleq\partial D/\partial(\lambda p)$. The algorithm can be described as

$$\hat{D}'=\frac{1}{n}\sum_{m=1}^{M}\sum_{l=1}^{n_m}\sum_{k=l}^{n_m}c_k^l(m). \quad (45)$$

Since

$$\begin{aligned}
\frac{\partial EL}{\partial p} &= \lambda\frac{\partial EL}{\partial(\lambda p)} \\
&= \frac{1}{p}\left[\frac{\partial D}{\partial(\lambda p)}-EL\right],
\end{aligned} \quad (46)$$

it can be seen that the SPA algorithm (44) and the algorithm (45) are essentially the same for Poisson arrivals. However it is concluded in [1] that algorithm (45) is asymptotically biased except for M/D/1 case. In [1] even the asymptotical bias of algorithm (45) is calculated for the M/M/1 case. From our analysis it is clear that the algorithms are unbiased. The simulation results shown in Fig. 1 demonstrate this for the M/M/1 case. The error in the analytical calculation in [1] can be briefly explained as follows. In [1] an effort was made to calculate the theoretical value of

$$E\hat{D}'=E\frac{1}{n}\sum_{m=1}^{M}\sum_{l=1}^{n_m}\sum_{k=l}^{n_m}c_k^l(m). \quad (47)$$

Conditional expectation method is used to carry out the calculation. Let N_m be the number of customers transmitted in the mth busy period, it is clear that

$$E\hat{D}'=EE\left[\frac{1}{n}\sum_{m=1}^{M}\sum_{l=1}^{n_m}\sum_{k=l}^{n_m}c_k^l(m)\mid N_m\right]. \quad (48)$$

In calculating the conditional expectation, however, the original given distributions were used as the conditional distribution. These two distributions are different. For example, busy periods with fewer customers tend to have shorter transmission times than the typical transmission time. To be more precise consider the M/M/1 case. The conditional expected transmission time conditioned on $N_m=1$ is apparently shorter than $1/\mu$, where μ is the transmission rate.

Another remark about the SPA algorithm presented in this paper is one can see from the above derivation of the algorithm that it is not difficult to get an estimation algorithm for the second derivatives by taking into consideration the two simultaneous removal phenomena. The estimate of the second derivatives can be very useful in optimization of communication networks [2].

As a final remark, we want to point out that the SPA algorithm presented in this paper does not need any knowledge about the distribution of the interarrival times and the transmission times, which are necessary for other derivative estimators such as the modified IPA estimator [14] or LR estimator. This is especially important for optimization of real systems. For example, usually the interarrival times and the transmission times are not independent. To apply the LR or IPA method one has to know the joint distribution of all the random variables involved, which is not practical. The SPA algorithm presented here sqeezes the information about the distributions while estimating the derivative by imaginarily removing each of the accepted customers. In other words the SPA method combines the distrbution identification and the derivative estimation together. While in LR or IPA approach one has to first estimate the relavent distributions and then estimate the derivative using the estimated distributions. It could be difficult to decide how much time one should spend on the identification of the distributions. Needless to say, the SPA algorithm presented here would converge slower than the modified IPA method [14] with known distributions. This is the price one has to pay for lacking of the knowledge about the system.

7 Conclusion

In this paper we present a smoothed perturbation analysis algorithm for estimating derivative in a routing problem. The algorithm is shown to be unbiased. The result can be used in the distributed optimization of data-communication networks. The algorithm is similar to the algorithm proposed in [1]. The error in [1] which yield the wrong conclusion about the biasedness of the algorithms is pointed out. The SPA algorithm requires minimum knowledge about the system and therefore can be used for on-line optimization purpose.

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References

[1] M.Bello, S. M. thesis, Dep. Elec. Eng. and Comput. Sci., Massachusetts Inst. Technol., Cambridge, Sept. 1976.

[2] D.P.Bertsekas, E.M.Gafni and R.G.Gallager, 'Second Derivative Algorithms for Minimum Delay Routing in Networks', *IEEE trans. on Communication*, Vol.Com-32,911-919 Aug.1984.

[3] Cao, X.R., 'Convergence of parameter sensitivity estimates in a stochastic experiment,' *IEEE Transactions on Automatic Control, AC-30*, pp.845-853, 1985.

[4] Cao, X.R., 'Sensitivity estimates based on one realization of a stochastic system', submitted to *Journal of Statistical Computation and Simulation.*

[5] Cassandras, C.G., Abidi, M.V., and Towsley D., 'Distributed Routing with On-line Marginal Delay Estimation', submmited to *IEEE Transactions on Communication, 1987.*

[6] Gallager, R.G., 'A minimum delay routing algorithm using distributed computation,' *IEEE Transactions on Communication, COM-25*, pp.73-85, 1977.

[7] Glynn, P.W., 'Process differentiable representation of a random variable', *Technical Report, University of Wisconsin*, 1986.

[8] Glynn, P.W. and Sanders, J.L., 'Monte Carlo optimization of stochastic systems: two new approaches,' *Proceedings of the 1986 ASME Computers in Engineering Conference*, 1986.

[9] Gong, W.B. and Ho, Y.C., 'Smoothed perturbation analysis of discrete-event dynamic systems', *IEEE Transactions on Automatic Control*, Vol.10, pp.858-866, 1987.

[10] Gong, W.B., 'Smoothed perturbation analysis of discrete-event dynamic systems', Ph.D. Thesis, Division of Applied Sciences, Harvard University, 1987.

[11] Heidelberger, P., Cao, X.R., Zazanis, M. and Suri,R., 'Convergence properties of infinitesimal perturbation analysis', *Management Science*, to appear, 1988.

[12] Ho, Y.C. and Cao, X.R., 'Perturbation analysis of queueing networks,' *Journal of Optimization Theory and Applications, 40*, pp.559-582, 1983.

[13] Ho, Y.C. and Cao, X.R., 'Performance Sensitivity to Routing Changes in Queueing Networks and Flexible Manufacturing systems Using Perturbation Analysis', *IEEE Journal of Robotics and Automation,* RA-1, pp. 615-172, 1985

[14] Ho, Y. C. and Vakili, P. 'Infinitesimal perturbation analysis algorithm for a routing problem', Allerton Control Conference, 1987.

[15] Kleinrock, 'Queueing Syustems' John & Wiely, 1975.

[16] Reiman, M.I. and Simon, B., 'Open queueing systems in light traffic', *Mathematics of Operations Research*, to appear, 1988.

[17] Reiman, M.I. and Weiss, A., 'Sensitivity analysis for simulations via likelihood ratios', Manuscript, AT& T Bell Laboratories, 1986.

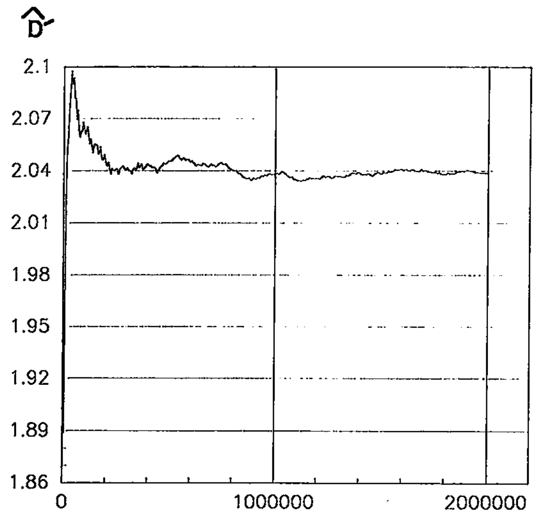
[18] Rubinstein, R., 'Sensitivity analysis and performance extrapolation for simulation models', *Math. of Simulation and Computers*, 1987.

[19] Sasaki, G. and Hajek, B., 'Optimal Dynamic Routing in Single Commodity Networks by Iterative Methods', *IEEE Trans. Communications*, Vol.Com-35, Nov.1987

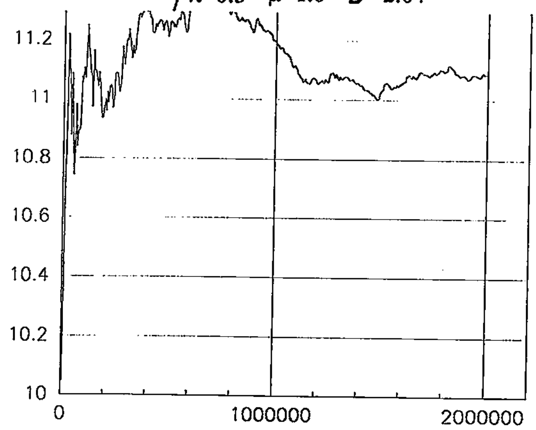
[20] A.Segall, 'The Modeling of Adaptive Routing in Data Communication Networks', *IEEE Trans. Communications*, Vol.Com-25,85-95,Jan.1977.

[21] Suri, R., 'Infinitesimal perturbation analysis for general discrete event systems',*Journal of the ACM*, July, 1987.

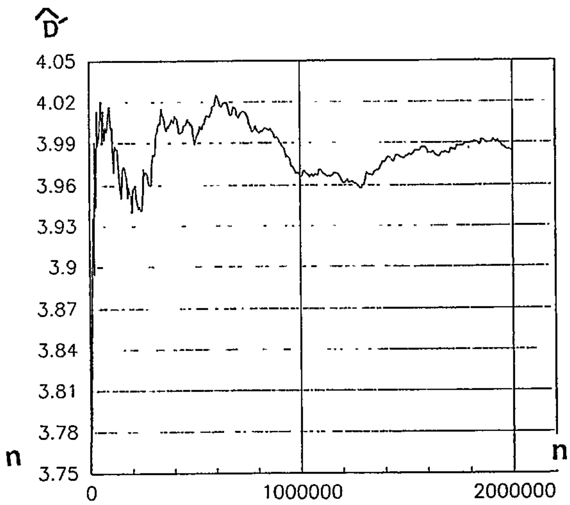
[22] Zazanis, M. and Suri, R., 'Comparison of perturbation analysis with conventional estimates for regenerative stochastic systems', submitted to *Operations Research*, 1984.



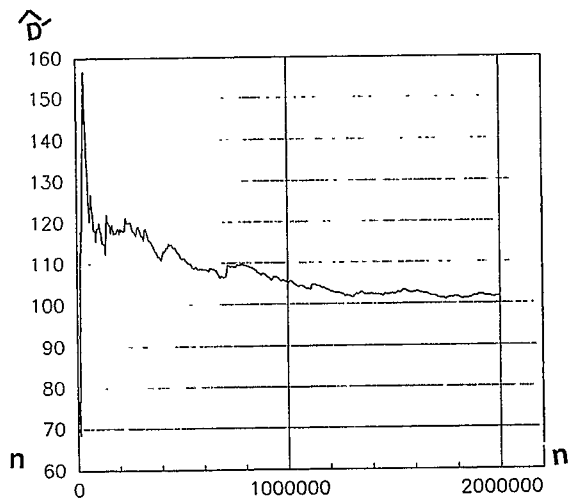
λ=0.3 μ=1.0 $\hat{D}$=2.04



λ=0.7 μ=1.0 $\hat{D}$=11.11



$\rho\lambda=0.5$ $\mu=1.0$ $D'=4.00$



$\rho\lambda=0.9$ $\mu=1.0$ $D'=100$

Fig. 1 Algorithm (45) for M/M/1 system