RANKING AND SELECTION TUTORIAL: 2-FACTOR NORMAL MEANS PROCEDURES

David Goldsman

School of ISyE

Georgia Institute of Technology

Atlanta, GA 30332

ABSTRACT

We discuss sampling procedures for selecting that one of a number of normal populations (with common known variance) which has the largest mean. We present procedures appropriate for single-factor experiments, and then give procedures devised for 2-factor experiments without interaction between the factor-levels. In all cases, the procedures guarantee a prespecified probability of selecting the correct population.

1. INTRODUCTION

This tutorial introduces the reader to *ranking and selection*, a field that has enjoyed a great deal of popularity in the statistical literature over the last thirty years. We concentrate specifically on the problem of selecting that one of a number of normal populations (with common known variance) which has the largest mean.

The paper is organized as follows. In Section 2, we present some basic notation and formally state the normal means problem. Some procedures for use in single-factor experiments for finding the normal population with the largest mean are outlined in Section 3. Section 4 gives relevant notation and formally states the normal means problem for the 2-factor case. Section 5 generalizes the procedures of Section 3 for use in 2-factor experiments without interaction.

2. THE SINGLE-FACTOR NORMAL MEANS PROBLEM

Suppose $\Pi_1, \ldots, \Pi_k$ are k independent normal populations with unknown means μ_i and *common known variance* σ^2, $i = 1, \ldots, k$. Without loss of generality, *we henceforth assume that* $\sigma^2 = 1$. Denote the ordered but unknown μ_i's by $\mu_{[1]} \leq \mu_{[2]} \leq \cdots \leq \mu_{[k]}$. Of course, we do not know the values of the $\mu_{[i]}$'s nor how they are paired with the Π_i's. Our *goal* is to determine which of the Π_i's has the largest mean $\mu_{[k]}$; we call that Π_i the "best" population.

A typical procedure for choosing the best population usually requires the experimenter to take a certain number of observations in a prescribed manner from each Π_i; the selection is then made using statistics calculated from these observations. Since the observations are realizations of random variables, it is possible that the experimenter will not choose the best Π_i. However, if the best population is indeed chosen, we say that a *correct selection* (CS) has been made.

We shall limit consideration to procedures that guarantee the following *probability requirement* on the P{CS} [cf. Bechhofer (1954)].

$$P\{CS\} \geq P^* \text{ whenever } \mu_{[k]} - \mu_{[k-1]} \geq \delta^* \quad (PR)$$

δ^* and P^* are two constants which must be specified prior to the start of experimentation subject to the restrictions that $\delta^* > 0$ and $1/k < P^* < 1$. δ^* is chosen as the smallest value of $\mu_{[k]} - \mu_{[k-1]}$ that the experimenter regards as "worth detecting." That is, if $\mu_{[k]} - \mu_{[k-1]}$ were to be $\geq \delta^*$, the experimenter would prefer to

choose the π_i associated with $\mu_{[k]}$; if $\mu_{[k]} - \mu_{[k-1]}$ were to be $< \delta^*$, the experimenter would, for all practical purposes, be indifferent about choosing either the population corresponding to $\mu_{[k]}$ or that corresponding to $\mu_{[k-1]}$. Hence, $\Omega_{\delta^*} \equiv \{\mu \mid \mu_{[k]} - \mu_{[k-1]} \geq \delta^*\}$ is called the *preference-zone*, and its complement is the *indifference-zone*. Procedures which guarantee (PR) are often referred to as indifference-zone procedures.

3. SINGLE-FACTOR INDIFFERENCE-ZONE PROCEDURES

In this section, which closely follows Goldsman (1986), we give a number of indifference-zone procedures for the normal means problem discussed in Section 2.

3.1 A Single-Stage Procedure

A *single-stage* procedure is one for which the size of the sample to be taken from each π_i is a constant determined prior to the start of experimentation. We first examine a single-stage procedure due to Bechhofer (1954).

Procedure P_{SS}:

1. For given k and known $\sigma^2 = 1$, specify $\{\delta^*, P^*\}$, with $\delta^* > 0$ and $1/k < P^* < 1$.

2. Determine n_{SS}, the smallest integer constant satisfying

$$\int_{-\infty}^{\infty} [\Phi(x + \delta^* n_{SS}^{1/2})]^{k-1} d\Phi(x) \geq P^*,$$

where $\Phi(\cdot)$ is the standard normal c.d.f. [n_{SS} can also be calculated easily from the tables in, e.g., Bechhofer (1954).]

3. Take n_{SS} independent observations from each of the k populations.

4. Let x_{ij} denote the jth observation from π_i, i = 1,...,k and j = 1,...,n_{SS}. Calculate

$$A_i \leftarrow \sum_{j=1}^{n_{SS}} x_{ij}, \quad i = 1,...,k.$$

5. Select as best that population corresponding to the largest A_i.

Example 1: Suppose that k = 3, $\sigma^2 = 1$, $P^* = 0.75$, and $\delta^* = 0.2$. From the tables in Bechhofer (1954), we can calculate that $n_{SS} = 52$. So we must take 52 independent observations from each of the three populations.

Although a single-stage procedure such as P_{SS} is very easy to implement, it has the drawback of being *conservative*; that is, it cannot react to "favorable" configurations of μ by reducing the number of observations to be taken. Sequential procedures, to be discussed next, *can* react to such conditions.

3.2 An Open Sequential Procedure

A *sequential* procedure is one which can involve more than a single stage of sampling. Such a procedure is *open* if, prior to the start of sampling, we can not place an upper bound on the number of observations to be taken from each π_i. Otherwise, the procedure is *closed*. The next procedure is an open sequential procedure from Bechhofer, Kiefer, and Sobel (1968).

Procedure P_{BKS}:

1. For given k and known $\sigma^2 = 1$, specify $\{\delta^*, P^*\}$, with $\delta^* > 0$ and $1/k < P^* < 1$.

2. Set the *stage counter* n $\leftarrow$ 0.

3. Set n $\leftarrow$ n + 1. Take one observation x_{in} from each π_i, i = 1,...,k. [Thus, we actually take observations a *vector-at-a-time*.]

4. For all i, calculate

$$A_{in} \leftarrow \sum_{j=1}^{n} x_{ij}.$$

Further, let $A_{max} \leftarrow \max_{1 \leq i \leq k} A_{in}$.

5. Calculate

$$Z_n \leftarrow \sum_{i=1}^{k} \exp\{-\delta^*(A_{max} - A_{in})\} - 1.$$

6. If $Z_n \leq (1-P^*)/P^*$, stop sampling, and select as best that π_i corresponding to A_{max}. Otherwise, go to Step 3.

Let N_{BKS} denote the random number of stages required for P_{BKS} to terminate (i.e., the value of the counter n at termination). The distribution

of N_{BKS} depends on the underlying configuration of the population means μ; two configurations are of special interest:

Least favorable configuration (LFC):

$$\mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta^*.$$

Equal means configuration (EMC): $\mu_{[1]} = \mu_{[k]}.$

It can be shown that the LFC maximizes $E[N_{BKS}|\mu]$ over all $\mu \in \Omega_{\delta^*}$. Thus, the LFC is the "worst case" configuration for $\mu \in \Omega_{\delta^*}$. Similarly, the EMC is the "worst case" configuration over the *entire* space of μ's since $E[N_{BKS}|\mu] \le E[N_{BKS}|\mu=EMC]\ \forall \mu$. If (unknown to the experimenter) μ is in the EMC, P_{BKS} might require an inordinate number of stages.

Example 1 (continued): Again suppose that $k = 3$, $\sigma^2 = 1$, $P^* = 0.75$, and $\delta^* = 0.2$. Bechhofer and Goldsman (B-G) (1987b) show that P_{BKS} yields $P\{CS|\mu=LFC\} \approx 0.777$, $E[N_{BKS}|\mu=LFC] \approx 38.84$, and $E[N_{BKS}|\mu=EMC] \approx 47.98$. Therefore, *for this example*, these expected values are *less than* the 52 observations required by each population for the single-stage P_{SS}.

Two unappealing features of P_{BKS} are the facts that it is open and that it generally gives $P\{CS|\mu=LFC\}$ which is somewhat larger than the desired P^*. (In the previous example, $P\{CS|\mu=LFC\}$ $\approx 0.777 > 0.75 = P^*$.) B-G (1987b) study a *closed* version (P_{BKS-T}) of P_{BKS} which employs automatic *truncation* of sampling after a certain predetermined number of stages have been conducted. The truncation point (n_0) is the smallest upper bound on the number of stages which still guarantees (PR); tedious Monte Carlo experimentation is required in order to determine n_0 for given $\{k,\sigma^2,\delta^*,P^*\}$. For the above example, P_{BKS-T} (with $n_0 = 68$ stages) yields $P\{CS|\mu=LFC\} \approx 0.751$, $E[N_{BKS-T}|\mu=LFC] \approx 35.94$, and $E[N_{BKS-T}|\mu=EMC] \approx 41.89$. P_{BKS-T} always results in less probability "overprotection" than P_{BKS}, eliminates the possibility of extreme values of N_{BKS}, and reduces $E[N_{BKS}|\mu]$ (sometimes substantially).

3.3 Other Sequential Procedures

Other single-factor indifference-zone normal means procedures are due to Paulson (1964), Alam (1970), Fabian (1974), and Tamhane and Bechhofer (1977,1979). The reader should also consult the texts of Gibbons, Olkin, and Sobel (1977) [in addition, see Bechhofer (1980)] and Gupta and Panchapakesan (1979), both of which give more comprehensive treatments on ranking and selection techniques (primarily from a single-factor point of view).

4. THE 2-FACTOR NORMAL MEANS PROBLEM

This section parallels the development of Section 2. We are now concerned with 2-factor experiments without interaction. References are B-G (1987a,1987c).

Suppose that we have a·b independent normal populations π_{ij} $(1 \le i \le a,\ 1 \le j \le b)$ with unknown means μ_{ij} and a *common known variance* σ^2; we henceforth assume without loss of generality that $\sigma^2 = 1$. Further, we assume the usual *additive* fixed-effects ANOVA model,

$$\mu_{ij} = \mu + \alpha_i + \beta_j ,$$

where $\Sigma_i \alpha_i = \Sigma_j \beta_j = 0$; that is, there is *no interaction* between the factor-levels. α_i (β_j) is the "effect" of the ith (jth) level of factor A (B) on μ.

Analogous to the single-factor case, denote the ordered but unknown α_i's and β_j's by $\alpha_{[1]} \le \cdots \le \alpha_{[a]}$ and $\beta_{[1]} \le \cdots \le \beta_{[b]}$. We do not know the values of the $\alpha_{[i]}$'s and $\beta_{[j]}$'s nor how they are paired with the π_{ij}'s. Our *goal* is to determine which of the a·b π_{ij}'s has the largest mean (i.e., the π_{ij} associated with $\alpha_{[a]}$ and $\beta_{[b]}$); we call that π_{ij} the "best" population. If the best population is indeed chosen by a procedure, a CS has been made. We limit consideration to procedures that guarantee the following 2-factor indifference-zone probability requirement on the $P\{CS\}$.

$$P\{CS\} \ge P^* \text{ whenever } \begin{cases} \alpha_{[a]} - \alpha_{[a-1]} \ge \delta^*_\alpha \\ \beta_{[b]} - \beta_{[b-1]} \ge \delta^*_\beta \end{cases}$$

δ^*_α, δ^*_β, and P^* are three constants which must be specified prior to the start of experimentation

subject to the restrictions that $\delta_\alpha^* > 0$, $\delta_\beta^* > 0$, and $1/ab < P^* < 1$. For ease of exposition, we only consider the case $\delta_\alpha^* = \delta_\beta^* = \delta^*$, say.

5. 2-FACTOR INDIFFERENCE-ZONE PROCEDURES

Here, we parallel the development of Section 3. We give a number of indifference-zone procedures for the 2-factor normal means problem discussed in Section 4.

5.1 A Single-Stage Procedure

The following single-stage procedure is due to Bechhofer (1954).

Procedure P_{SS2}:

1. For given a and b and known $\sigma^2 = 1$, specify $\{\delta^*, P^*\}$, with $\delta^* > 0$ and $1/ab < P^* < 1$.

2. Determine n_{SS2}, a constant satisfying

$$\int_{-\infty}^{\infty} [\Phi(x + \delta^*(bn_{SS2})^{1/2})]^{a-1} d\Phi(x)$$

$$\cdot \int_{-\infty}^{\infty} [\Phi(x + \delta^*(an_{SS2})^{1/2})]^{b-1} d\Phi(x) \geq P^*$$

[A table of n_{SS2}-values is given in B-G (1987c).]

3. Take n_{SS2} independent observations from each of the a·b populations.

4. Let x_{ijs} denote the sth observation from Π_{ij}, $i = 1,\ldots,a$, $j = 1,\ldots,b$, and $s = 1,\ldots,n_{SS2}$. Calculate the quantities

$$A_i \leftarrow \sum_{j=1}^{b} \sum_{s=1}^{n_{SS2}} x_{ijs}, \quad i = 1,\ldots,a$$

and

$$B_j \leftarrow \sum_{i=1}^{a} \sum_{s=1}^{n_{SS2}} x_{ijs}, \quad j = 1,\ldots,b.$$

5. Select as best that Π_{ij} corresponding to the largest A_i and the largest B_j.

Example 2: Suppose that a = 3, b = 6, $\sigma^2 = 1$, $P^* = 0.9025 = (.95)^2$, and $\delta^* = 0.2$. From a table in B-G (1987c), we find that $n_{SS2} = 64$. So we must take 64 independent observations from each of the 18 Π_{ij}'s.

5.2 An Open Sequential Procedure

The next procedure is the 2-factor generalization of P_{BKS}, and is itself a special case of a more general treatment of Koopman-Darmois populations without interaction [cf. B-G (1987a)].

Procedure P_{BKS2}:

1. For given a and b and known $\sigma^2 = 1$, specify $\{\delta^*, P^*\}$, with $\delta^* > 0$ and $1/ab < P^* < 1$.

2. Set the *stage counter* n ← 0.

3. Set n ← n + 1. Take one observation x_{ijn} from each Π_{ij}, $i = 1,\ldots,a$, $j = 1,\ldots,b$. [Thus, we actually take observations a *matrix*-at-a-time.]

4. Calculate the quantities

$$A_{in} \leftarrow \sum_{j=1}^{b} \sum_{s=1}^{n} x_{ijs}, \quad i = 1,\ldots,a ,$$

$$B_{jn} \leftarrow \sum_{i=1}^{a} \sum_{s=1}^{n} x_{ijs}, \quad j = 1,\ldots,b ,$$

$$A_{max} \leftarrow \max_{1 \leq i \leq a} A_{in}, \quad \text{and}$$

$$B_{max} \leftarrow \max_{1 \leq j \leq b} B_{jn}.$$

5. Calculate

$$U_n \leftarrow \sum_{i=1}^{a} \exp\{-\delta^*(A_{max} - A_{in})\} - 1 ,$$

$$V_n \leftarrow \sum_{j=1}^{b} \exp\{-\delta^*(B_{max} - B_{jn})\} - 1 ,$$

$$\text{and} \quad Z_n \leftarrow U_n + V_n + U_n \cdot V_n.$$

6. If $Z_n \leq (1-P^*)/P^*$, stop sampling, and select as best that Π_{ij} corresponding to A_{max} and B_{max}. Otherwise, go to Step 3.

Let N_{BKS2} denote the random number of stages required for P_{BKS2} to terminate. The 2-factor LFC is given by

$$\alpha_{[1]} = \alpha_{[a-1]} = \alpha_{[a]} - \delta^*,$$

$$\beta_{[1]} = \beta_{[b-1]} = \beta_{[b]} - \delta^*,$$

and the EMC is given by

$$\alpha_{[a]} = \beta_{[b]} = 0.$$

Example 2 (continued): Again suppose that $a = 3$, $b = 6$, $\sigma^2 = 1$, $P^* = 0.9025$, and $\delta^* = 0.2$. B-G (1987c) show that P_{BKS2} yields $P\{CS|\mu=LFC\} \gtrsim 0.922$, $E[N_{BKS2}|\mu=LFC] \gtrsim 44.80$, and $E[N_{BKS2}|\mu=EMC] \gtrsim 117.76$. Recall that P_{SS2} required 64 observations from each population; so *for this example*, P_{BKS2} is more parsimonious than P_{SS2} in the LFC.

B-G (1987c) also study a version (P_{BKS2-T}) of P_{BKS2} which employs truncation and still guarantees the probability requirement. For the above example, P_{BKS2-T} (with a truncation point of $n_0 = 80$ stages) yielded $P\{CS|\mu=LFC\} \gtrsim 0.904$, $E[N_{BKS2-T}|\mu=LFC] \gtrsim 43.53$, and $E[N_{BKS2-T}|\mu=EMC] \gtrsim 70.88$. By design, P_{BKS2-T} always outperforms P_{BKS2} in terms of these and other criteria. Further, the Monte Carlo work in B-G (1987c) suggests that P_{BKS2-T} substantially outperforms P_{SS2} in the LFC, and that P_{BKS2-T} never fares too much worse than P_{SS2} in the EMC.

5.3 Why Conduct 2-Factor Experiments?

One might think that conducting a sequential 2-factor procedure, which samples observations a matrix-at-a-time, is less efficient than carrying out two independent single-factor experiments, which take observations a vector-at-a-time. A major finding of B-G (1987a) shows that this is not the case. Although the 2-factor procedures take more observations per stage (a·b) than do the corresponding single-factor procedures (a and b, respectively), the 2-factor procedures usually terminate sampling in far *fewer* stages. Indeed, B-G (1987a) find that, when the situation at hand is appropriate, an experimenter can usually save *substantially* on the total number of scalar observations by using a 2-factor procedure instead of two single-factor experiments – about 40% when the population means are in the LFC and much more so when the means are in the EMC!

6. CONCLUSIONS

This tutorial served to give the simulation practitioner a flavor of some of the ranking and selection procedures that are available in the literature. In this paper, we concentrated on the single- and 2-factor normal means problem (with common known variance), but ranking and selection procedures exist to solve a wide variety of other interesting problems. In fact, a number of papers have recently appeared which formulate and/or utilize ranking and selection techniques in the simulation environment, e.g., Sullivan and Wilson (1985).

REFERENCES

Alam, K. (1970). "A Two-Sample Procedure for Selecting the Population with the Largest Mean from k Normal Populations." *Ann. Inst. Statist. Math.*, 22, 127–136.

Bechhofer, R.E. (1954). "A Single-Sample Multiple Decision Procedure for Ranking Means of Normal Populations with Known Variances." *Ann. Math. Stat.*, 25, 16–39.

Bechhofer, R.E. (1980). Book Review of Gibbons, Olkin, and Sobel (1977). *J. Amer. Statist.*, 75, No. 371, 751–756.

Bechhofer, R.E. and D. Goldsman (1987a). "Sequential Selection Procedures for Multi-Factor Experiments Involving Koopman-Darmois Populations with Additivity." *Statistical Decision Theory and Related Topics, IV* (Eds. S.S. Gupta and J.O. Berger), Academic Press, New York.

Bechhofer, R.E. and D. Goldsman (1987b). "Truncation of the Bechhofer-Kiefer-Sobel Procedure for Selecting the Normal Population which has the Largest Mean." To appear in *Comm. Stat. – B*.

Bechhofer, R.E. and D. Goldsman (1987c). "Truncation of the Bechhofer-Kiefer-Sobel Procedure for Selecting the Normal Population which has the Largest Mean (II): 2-Factor Experiments without Interaction." Tech. Report, School of OR&IE, Cornell Univ., Ithaca, NY.

Bechhofer, R.E., J. Kiefer, and M. Sobel (1968). *Sequential Identification and Ranking Procedures.* The Univ. of Chicago Press, Chicago.

Fabian, V. (1974). "Note on Anderson's Sequential Procedures with Triangular Boundary," *Annals of Statistics*, 2, 170–175.

Gibbons, J.D., I. Olkin, and M. Sobel (1977). *Selecting and Ordering Populations*. John Wiley, New York.

Goldsman, D. (1986). "Tutorial on Indifference-Zone Ranking and Selection Procedures." *Proceedings of the 1986 Winter Simulation Conference*, 370–375.

Gupta, S.S. and S. Panchapakesan (1979). *Multiple Decision Procedures*. John Wiley, New York.

Paulson, E. (1964). "A Sequential Procedure for Selecting the Population with the Largest Mean from k Normal Populations." *Ann. Math. Stat.*, 35, 174–180.

Sullivan, D.W. and J.R. Wilson (1985). "Restricted Subset Selection Procedures for Simulation." Rsearch Mem. #85–15, School of IE, Purdue Univ., W. Lafayette, Ind.

Tamhane, A.C. and R.E. Bechhofer (1977). "A Two-Stage Minimax Procedure with Screening for Selecting the Largest Normal Mean." *Comm. Stat. – Theor. Meth.*, A6, 1003–1033.

Tamhane, A.C. and R.E. Bechhofer (1979). "A Two-Stage Minimax Procedure with Screening for Selecting the Largest Normal Mean (II): An Improved PCS Lower Bound and Associated Tables." *Comm. Stat. – Theor. Meth.*, A8, 337–358.

AUTHOR'S BIOGRAPHY

DAVE GOLDSMAN is an Assistant Professor of Industrial and Systems Engineering at Georgia Tech. He holds degrees from Syracuse and Cornell, where he spent last summer as a Visiting Scientist. His research interests include simulation output analysis, and ranking and selection.

Dave Goldsman
School of ISyE
Georgia Tech
Atlanta, GA 30332
(404) 894-2365