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AN EMPIRICAL COMPARISON OF FOURIER AND WALSH SPECTRA FOR SYSTEM IDENTIFICATION

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Identification of significant factors has traditionally been a costly process. Initial results indicate that the use of spectral methods can reduce the number of simulation runs required to obtain the desired information. Two spectral methods will be compared to the traditional method of testing for factor significance. The paper introduces the concepts needed for the spectral methods, and the empirical results will be presented orally at the conference.

System identification is an area of simulation analysis which should be of particular interest to practitioners. It refers to the task of trying to identify which factors in a systems model have a significant impact upon the response of interest.

Statisticians distinguish between parameters, which are fixed attributes of the system, and factors, which are attributes that can be altered by the experimenter. The simulation environment is somewhat unique, in that the experimenter has total control over the model. He can alter the level settings of parameters which cannot be changed in the real system in order to test the model's sensitivity to errors which arise in parameter estimation. Hence we will use the term factor throughout this paper to denote either a factor or a parameter, where a statistician would distinguish between the two.

Most experimenters face resource constraints in terms of budgets, manpower, time, and computing availability, to give some examples. It is desirable to identify factors which do not significantly alter the results of the simulation, and to do so as efficiently as possible. The experimenter can then concentrate on accurate estimation of those factors to which the simulation is highly sensitive. This may also lead to a simplification of the system model, possibly yielding a more efficient simulation.[13]

The system response can be visualized as a multi-dimensional surface in a space defined by the factors. If a given factor has no impact on the response, then the slope of the response surface in that dimension will be zero no matter what the factor levels are for any other factor. Conversely, if the slope of the response surface is non-zero for any configuration of settings of the other factors, then the designated factor should be considered significant.

The branch of statistics which deals with how to choose factor settings so as to gain the most information from each experimental unit is known as experimental design. We will be considering experimental designs for linear models. It is assumed that the true response can be adequately approximated by a linear combination of polynomial terms. We are effectively performing a Taylor's series expansion for the response surface. The order of the polynomial terms determines which factor configurations should be used to most efficiently identify significant terms in the polynomial approximation. For example, the slope of a straight line is best determined by taking half of the observations at the extreme upper setting of the factor, and the remainder at the extreme lower setting. Identifying a quadratic function requires observations from the extreme values and from the midrange value as well.

The traditional approach has been to set all factors to the levels specified for one of the design points of an optimally designed experiment, and to make a run of the simulation at that configuration of factors. The basic experimental unit is then regarded to be a single run of the simulation. If runs are independently seeded, each run is an independent observation. The drawback to this approach is that the number of runs required increases as a geometric function of the number of factors.

Schruben and Cogliano developed a different approach to the problem.[5,14] Their work makes use of the fact that spectral estimators at frequencies more than a bandwidth apart are approximately independent. It is therefore possible to obtain many virtually independent estimates from a single simulation run if analysis is performed in the frequency domain.

The basic design of a spectral experiment consists of selecting frequencies which can be uniquely distinguished in the model output. Each factor is varied at a different one of these uniquely identifiable frequencies. This is called a signal run. Another run is made in which the factors are held at fixed values. This is called the noise run because with no variation in the factors, any variation in the output is attributed to noise. The spectra are then estimated for the two runs, and the signal to noise ratio is computed by taking the ratio of the spectral estimators at each frequency. This ratio can then be plotted and examined for spikes or peaks corresponding to the assigned factor frequencies.

Non-linear response surfaces can be detected by virtue of the following elementary trigonometric identity:

$$\cos\alpha \cos\beta = .5[\cos(\alpha - \beta) + \cos(\alpha + \beta)].$$

It can be seen that product terms of factors would show up as spikes in the spectrum at the sum and difference of the component factor frequencies. An example would be a quadratic term, which would be identified by observing a spike at double the input frequency. A cubic term would be identified by observing a spike at triple the input frequency, and so on.

The procedure is complicated by two problems. The first is called aliasing, and is an artifact of discrete sampling. Because we are observing a discrete sample of points from a continuous process, the highest frequency we can observe is .5 cycles/observation. Higher frequencies will appear at alias frequencies in the range [0,.5]. This alters the problem of selecting frequencies to one of selecting frequencies and aliases which are uniquely identifiable.

The second problem is that the system itself is a filter of unknown characteristics. It may either amplify or attenuate the response at different frequencies. The amount by which the response is altered at different frequencies is called the gain function of the system. There are several ways gain can be accounted for. One possibility is to try and estimate it using white noise (uncorrelated random signals) as input. A second approach is to assign more than one frequency per factor and compare spectra for the same factor to test for gain. A third possibility is to treat gain as a nuisance factor and use an experimental design which blocks to eliminate it. This is the method used by Schruben and Cogliano. The noise run can be incorporated with the signal runs with this design. It allows identification of k factors using k+1 runs in the presence of system gain.

Sanchez [11,12] has extended this work by considering the use of discrete valued functions as the basis for spectral estimation. Several discrete function sets were considered, and Walsh functions were selected as having the most desirable properties.

Walsh functions are two-valued functions which constitute a complete ortho-normal basis. They are similar to sine and cosine functions in that they are grouped in pairs with even and odd symmetry, designated as Cal and Sal functions, respectively. A major difference is that their variations are not periodic. Instead of being grouped by frequency, they are grouped by the average number of variations on a fixed interval. This is called the sequency of the function.

Because of their completeness, orthogonality, and discrete nature, Walsh functions can be used to efficiently represent systems with discrete behavior in either the input factors or the response. They can also be used to evaluate continuous systems. It is interesting to note that using Walsh inputs will yield an optimal experiment for detecting linear effects. This is because the design matrix for a 2^k factorial experimental design is a k x k matrix of Walsh functions.

Non-linear effects and cross-terms can be detected using the following Walsh identity:

$$WAL(\alpha,T)\ WAL(\beta,T) = WAL(\alpha \oplus \beta,T)$$

where:

- α, β are integers
- WAL(α,T) denotes the Walsh function with α variations on an interval of length T.
- $\oplus$ denotes a bitwise exclusive or operation on the binary representations of α and β.

The interested reader will find more comprehensive descriptions of Walsh functions and their characteristics in the materials referenced.

The design of a Walsh spectral experiment proceeds exactly as with a Fourier spectral experiment. The experimenter must select sequencies, make signal and noise runs, and compute the ratios of the spectral estimators.

Both types of spectral estimators will have χ^2 distributions if the errors are assumed to have the form of Gaussian white noise. Their ratios will have an F distribution if the signal and noise runs were independently generated. This fact can be used to construct statistical tests of the hypothesis that a given factor has no effect on the model response.

There are relative merits to both procedures. Fourier analysis is fairly familiar to most simulation practitioners,

and is insensitive to phase shifts in the output. However, it cannot be used with systems which have discrete factors.

Walsh analysis is less familiar, and many sequencies are extremely sensitive to phase shifts (although not all are). However, it can be used for discrete, continuous, and mixed models. The practitioner should be warned, though, that implementing discrete factor spectral tests may present problems which require policy definitions to be clearly specified. For example, what should be done to people in a multi-server queueing system when the server they were waiting for shuts down? One might a) distribute them to other lines, b) let the server finish off the line but allow no new entries, or c) make them disappear. Different options might be useful depending on characteristics of the original system being modelled, but they will clearly yield different model behavior.

The author will present a number of different simulation models in the oral part of this paper. The results of traditional, Fourier, and Walsh sensitivity analyses will be compared.

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