RANDOM VARIATE GENERATION: A SURVEY[1]

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ABSTRACT: The state of the art of generating random variates on a digital computer is surveyed. General concepts are presented, followed by criteria for comparing algorithms. The literature is surveyed for continuous univariate, discrete univariate, continuous multivariate, and discrete multivariate distributions, as well as for point processes, time series, order statistics and geometrically inspired problems. An extensive list of references is provided.

1. INTRODUCTION

Assuming the existence of a source of independent U(0, 1) observations $u_1, u_2, \ldots$, we survey the state of the art of transforming the uniform random numbers to obtain random variates $x_1, x_2, \ldots$ satisfying specified properties of distribution and/or dependency structure, for use as inputs to stochastic simulation experiments on digital computers.

We assume the U(0, 1) random variables are ideal; that is, they are exactly uniformly distributed over the interval (0, 1) and they are independent. The consequences of this assumption not being entirely true are discussed in Burford and Willis (1978), Chay, Fardo and Mazumdar (1975), Golder and Settle (1976), Monahan (1978) and Neave (1973). Kennedy and Gentle (1980) provide an excellent and up-to-date discussion of U(0, 1) generation.

Note that it is possible, although very uncommon, to use distributions other than U(0,1) as the basic source of randomness. Lünow (1974), for example, discusses using truely random Poisson observations.

We discuss general underlying concepts in Section 2 and criteria for comparing variate generation algorithms in Section 3. Section 4, which surveys the state of the art of specific problems, considers both continuous and discrete random variables and random vectors, as well as processes correlated and changing over time, order statistics, and geometrically inspired problems such as generation of points uniformly distributed on the surface of a sphere and random permutations.

For completeness there are a few references listed at the end of the paper which are not discussed.

2. FUNDAMENTAL CONCEPTS

It is important to distinguish between the fundamental approaches for random variate generation and the resulting algorithms. While occasionally an efficient algorithm results from the direct application of a single concept, more often an algorithm is a combination of more than one concept. As in other fields, such as mathematical programming, the same concepts applied in much the same way can still lead to different algorithms due to changes in data structure and tailoring to specific computer efficiencies.

We discuss four fundamental concepts: (1) inverse transformation, (2) composition, (3) acceptance/rejection, and (4) special properties. Unlike the algorithms discussed in Section 4, these concepts have changed only little since variate generation was first studied. Butler (1956), for example, discusses these concepts. Kennedy and Gentle (1980) discuss both basic concepts and algorithms and provide an extensive recent bibliography. Other general references include Ahrens and Dieter (1974b),

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Chambers (1970), Deák and Bene (1979), Fishman (1973, 1978b), Galliher (1959), Handbook of Mathematical Functions (1964), Hammersley and Handscomb (1964), Halton (1970), Jansson (1966), Kahn (1964), Knuth (1969), Knuth and Yao (1976), P.A.W. Lewis (1972, 1979), P.A.W. Lewis and Learmonth (1973), T.G. Lewis (1975), McGrath and Irving (1973), Newman and Odell (1971), Sowey (1972, 1978), Spanier and Gebhard (1969), Teichrow (1953, 1965), and Tocher (1963).

2.1 The Inverse Transformation

The use of the inverse of the cumulative distribution function (cdf) leads to the most fundamental method for generating random variates. It is applicable to any univariate distribution, whether discrete, continuous, or mixed. The method is to convert the U(0,1) random number u to the value x lying at the u th fractile; that is, $x = F^{-1}(u)$, which is analogous to a percentile test score of u (or 100u) with a corresponding raw score of x.

First consider an arbitrary discrete distribution with cdf F(x). The probability of observing x_i is $F(x_{i+1}) - F(x_i)$ and any method of assigning this probability to x_i is a valid method. However, the most straightforward procedure is to return x_i if and only if $F(x_i) < u \leq F(x_{i+1})$. Some care must be taken on the end points to be sure all values are defined and to avoid round-off error, but otherwise implementation is direct: (1) Generate $u \sim U(0,1)$ and set i=0, (2) set i=i+1, (3) if $u > F_i$, go to 2, (4) otherwise return $x=v_i$. Here two vectors are needed: F_i to store the value of $F(x_i)$ and v_i to store the value of x_i, i=1, 2, ..., n. For many discrete distributions, the explicit use of F_i can be avoided since a recursive relationship can be used to calculate F_i from F_{i-1}. (For example, see the discussion of the Poisson, binomial, and negative binomial distributions below, as well as the geometric distribution which has a closed form inverse transformation.) Likewise, the vector v can be made implicit when simple relationships exist between v_i and i, such as $v_i=i$ or $v_i=i-1$. Chen and Asau (1974) suggest the use of index tables to speed the search for the proper interval.

A similar concept applies to continuous distributions, where now we want $P(a < x \leq b) = F(b) - F(a)$ for all values of a and b. This property is satisfied when $x= F^{-1}(u)$ is used, since the distribution of the random variable Y=F(X) is U(0, 1) for any continuous random variable X. The continuous version can also be obtained by considering the limiting case of the discrete concept as the intervals $x_{i+1} - x_i$ become shorter.

For some distributions the inverse transformation leads to closed form algorithms which may be implemented directly. Examples are x = a + (b-a)u for $X \sim U(a, b)$ and x = -(ln(1-u)/α)**(1/γ) for the Weibull distribution with shape parameter γ and scale parameter α. Note that γ=1 yields the exponential distribution with mean 1/α.

Numerical methods may be used when the inverse transformation is not closed form. Butler (1970) discusses a general, although approximate, method for generating random variates from any continuous distribution via numerical integration of the density function. (See corrections by Proll (1972).) Numerical methods for the normal, gamma, and beta distributions are referenced in Section 4. When the distribution is in the form of a histogram (a mixture of uniform distributions), Barnard and Cawdery (1974) suggest using an approximate but fast algorithm based on approximating the distribution with equally likely uniform distributions and linear interpolation.

In both the discrete and continuous cases, there are several reasons for using the inverse transformation even if slow numerical techniques are involved: (1) Order statistics can be easily generated, as discussed in Section 4, (2) truncated distributions may be generated using $x=F^{-1}(u')$ where u'=a+(b-a)u, resulting in $F^{-1}(a) < x < F^{-1}(b)$, (3) the use of variance reduction techniques is aided, as discussed in Section 3.

2.2 Composition

Composition, or probability mixing, is often used without realizing the generality of the method. For example, the double exponential (LaPlace) distribution is commonly generated by obtaining a negative exponential random variate and assigning a random sign. Another example is mixed distributions, such as rainfall in a particular week, where zero rainfall may occur with probability p_z and the amount of rainfall, conditional on there being some, may follow a gamma distribution. The algorithm is to set x=0 if $u < p_z$ and to generate a gamma variate x otherwise. However, composition is useful in many situations where the concept is not so intuitively applied.

Composition, like the inverse transformation, has both a discrete and continuous form. However, the type of composition is independent of the type of random variable; discrete random variables can be mixed continuously and vice versa. We first consider discrete mixing.

Let f(x) denote the density function if X is a continuous random variable or the probability of observing x if X is discrete. Then discrete composition is applicable when f(x) is written as

$$f(x) = \sum_{i=1}^{n} p_i f_i(x)$$

where $\sum^n p_i = 1$ and n may be infinite. The generation of random variates from f(x) simply requires generating a variate x from $f_i(x)$ with probability p_i. The selection of i is usually via the discrete inverse transformation and the generation from $f_i(x)$ may use any algorithm. In the double exponential example, $f_1(x) = \lambda \exp(-\lambda x) I(x)_{(0,\infty)}$ and $f_2(x) = \lambda \exp(\lambda x) I(x)_{(-\infty,0)}$, where $I(x)_{(a,b)}=1$ if $a < x < b$ and zero otherwise, and $p_1 = p_2 = .5$. In the rainfall example, $p_1 = p_z$, $p_2 = 1 - p_z$, $f_1(x) = I(x)_{[0,0]}$ and $f_2(x)$ is the gamma density function.

Note that the linear combination of random variables $X = \sum_{i=1}^{n} a_i X_i$ is a convolution and the proper generation procedure is to generate each of the n random variates x_i and to combine them as indicated in the linear combination. Do not confuse convolution and composition.

Discrete composition has an intuitive geometric interpretation in terms of the density function f(x), in that the area under the density may be partitioned in any way to form the n subdensities $f_i(x)$. In the case of the double exponential, the partition between the two subdensities is vertical. A horizontal partition may be used to partition a trapezoidal shaped density function into a uniform (rectangular) subdensity and a triangular subdensity. The area of each subdensity $f_i(x)$ is p_i.

Many of the fastest algorithms for univariate continuous distributions use discrete composition. See Marsaglia (1961c) who discusses the concept and applies it to the normal distribution. Other applications are discussed in Section 4.

One of the most important advances in the generation of discrete random variables is due to Walker (1974a, 1974b, and 1977), who describes the concept of "aliasing" for distributions having a finite number of possible values. Walker noted that any discrete distribution having a finite number of outcomes can be expressed as a mixture of n distributions each having exactly two outcomes and each having coefficient $p_i = 1/n$. This yields the very fast discrete composition algorithm (1) Generate $u \sim U(0, 1)$, set $u = u\,n$, set $i = INT(u) + 1$, set $u = i - u$, (2) if $u \leq F_i$, return $x = i$, (3) otherwise return $x = A_i$, where it is assumed that x = 1, 2, ..., n are the possible values of x. Here i has a discrete uniform distribution over the range 1, 2, ..., n; F_i is the probability that $x=i$ and $1-F_i$ is the probability that the alias value $x=A_i$ is returned. Kronmal and Peterson (1979) discuss the calculation of F_i and A_i for i=1, 2, ..., n and also prove that the method is applicable for all distributions with range x = 1, 2, ..., n. Of course the use of an additional vector analagous to v in the discrete inverse transformation allows generation from any discrete distribution with a finite number of outcomes.

Tabling discrete values, which results in very fast algorithms at the expense of rounding the probabilities and/or using large tables, is a composition method. Marsaglia (1963) discusses an ingenious modification to reduce the table size. See also Norman and Cannon (1972).

The continuous composition algorithm can be used when f(x) is expressed as $f(x) = \int_{-\infty}^{\infty} f_{X|y}(x)\, dF_Y(y)$, where Y is a continuous random variable mixing conditional density functions or discrete mass functions $f_{X|y}(x)$. Variate generation proceeds in two steps: (1) Generate a continuous random variate y having cdf $F_Y(y)$ and (2) generate a random variate x from $f_{X|y}(x)$. Distributions which can be handled in this way are called compound distributions. Examples include the beta-binomial, where the probability of success p in the binomial distribution is a random variable with a beta distribution. Less intuitive is that a Pearson type IV distribution can be generated as a gamma(α, $1/\beta$) with β being a gamma(δ, γ) random variate, where gamma(a, b) denotes the gamma distribution with shape parameter a, scale parameter b, and mean ab. Another example is the negative binomial discussed below. For other examples of compound distributions, see Johnson and Kotz (1969).

Note that since a χ^2 random variable is the square of a standardized normal random variable, it is not unreasonable to consider generating a normal variate using a χ^2 variate. The problem arises when it is noted that either of the two roots of the χ^2 variate corresponds to normal variates. Due to symmetry, it seems reasonable to use each root with probability .5, which is correct. Michael, Schucany and Haas (1976) derive the correct multinomial probabilities for selecting one of multiple roots, leading to a simple composition algorithm for the inverse Gaussian distribution, as an example.

2.3 Acceptance/Rejection

The acceptance/rejection concept is to generate variates from one distribution and discard (reject) some of them in such a way that the remaining variates have the desired distribution. Although until the last few years the acceptance/rejection concept has been used almost exclusively with univariate continuous distributions, it is valid for either discrete or continuous and univariate or multivariate distributions.

Let $f(x)$ denote the density function of X if X is a continuous random variable or the mass function if X is a discrete random variable. Here X may be either univariate or multivariate. Let $t(x)$ be any majorizing function of $f(x)$; that is, we require that $t(x) \geq f(x)$ for all values of x. Let $g(x) = t(x)/c$ denote the density function proportional to $t(x)$ if X is continuous (in which case $c = \int_{-\infty}^{\infty} t(x)\,dx$) or the mass function proportional to $t(x)$ if X is discrete (in which case $c = \Sigma_x t(x)$). The algorithm is (1) generate $x \sim g(x)$, (2) generate $u \sim U(0, 1)$, (3) if $u > f(x)/t(x)$, then go to step 1, (4) otherwise return x.

The algorithm's execution time depends on three factors: (1) The time to generate x in step 1, (2) the time to perform the comparison in step 3, and (3) the expected number of interations, c, to return x. The selection of the majorizing function $t(x)$ plays a major role in all three factors, making it crucial to the development of efficient algorithms. In elementary textbook discussions of the acceptance/rejection algorithm, $t(x) = \max_x f(x)$ is usually used, as originally discussed by von Neumann (1951). Step 1 is then to generate a uniform variate over the range of X, which is fast, but the expected number of iterations, c, is often unacceptably large, such as for the beta distribution over the interval (0, 1) as the shape parameters p and/or q become large, as discussed in detail in Section 4. Many recent algorithms use acceptance/rejection.

The basic concept can be made more efficient by adding some logic between steps 2 and 3. Since step 3 often requires slow exponential type operations, preliminary comparisons using simple one-sided approximations to $f(x)/t(x)$ can speed up an algorithm by accepting or rejecting x before $f(x)/t(x)$ is calculated. This modification has been termed the "squeeze" method by Marsaglia (1978). Marsaglia (1970) discusses one-sided approximations.

It is also common to apply two "tricks" to step 3. First, $f(x)$ is rescaled to avoid having to calculate normalizing constants which tend to involve hard to compute constants such as gamma and beta functions. Since the shape of the density function does not depend on these normalizing constants, other constants can be substituted. Setting the normalizing constant to 1 sometimes causes numerical problems, however. Ahrens and Dieter (1974) rescale the gamma distribution so that $\max_x f(x) = 1$, thereby avoiding the gamma function as well as numerical problems. The second "trick" is to compare $\ln(u)$ to $\ln(f(x)/t(x))$ in step 3, since this also helps to avoid numerical problems, often eliminates some exponential calculations, and special methods exist to generate $\ln(u)$ directly (as the negative of an exponential random variate).

Schmeiser and Lal (1980), Schmeiser and Babu (1980) and Tadikamalla (1978), for example, use acceptance/rejection to generate variates from subdensities in composition algorithms. Kronmal and Peterson (1979b, 1979c) and Kronmal, Peterson and Lundberg (1978) combine the concepts of acceptance/rejection, aliasing, and discrete composition. Jeswani and Sikdar (1978) appear to have recently rediscovered the acceptance/rejection concept.

2.4 Special Properties

Sometimes the distribution from which random variates are to be generated has one or more special properties which can be used, leading to methods of generation which are specific to that distribution. Three topics are discussed in this section: transformations from nonuniform distributions, generation of trigonometric functions with random arguments, and von Neumann's comparison method.

<u>Transformations from nonuniform distributions</u>

Many of the classical methods for generating random variates from common distributions are based on generating some intermediate nonuniform random variates $y_1, y_2, \ldots, y_n$ and then calculating the desired variate as $x = f(y_1, y_2, \ldots, y_n)$. For $n = 1$, examples are nonstandard normal via $x = \mu + \sigma z$, where z is a standard normal variate; $U(a, b)$ via $x = a + (b-a)u$; and lognormal variates via $x = \exp(y)$, where y is the appropriate normal variate. There are many examples for $n > 1$. These include Erlang as the sum of k exponential variates ($x = -\ln(\prod^k u_i)$), beta as a ratio of gammas, Student's t via standardized normal and chi-square, F via chi-squares, chi-squares via normals, binomial as a sum of Bernoulli trials, negative binomial as a sum of geometric random variates, and on and on. Many of these are excellent approaches. One very common example that is not good, because it is a rather crude approximation, is the approximation of the normal distribution by the sum of twelve uniform variates, $x = u_1+u_2+\cdots+u_{12}-6$. The kurtosis is 2.9 rather than 3 and the tails are truncated at $\pm$ 6. If a very simple generation algorithm is needed, such as when using a hand calculator, an easier and more accurate approximation is

$x = (u^{.135} - (1-u)^{.135})/.1975$, as discussed in Schmeiser (1980).

An important special case of transformations from intermediate random variates is the ratio-of-uniforms method of Kinderman and Monahan (1976, 1977). They suggest defining a region R so that conditional on $\underline{v} = (v_1, v_2)$ being uniformly distributed over R, then $x = v_1/v_2$ is a random variate from the distribution of interest. While any method may be used to generate $\underline{v}$, commonly two dimensional acceptance/rejection is used, where $f(\underline{v}) = 1/\int_R d\underline{v}\, I(\underline{v})_{(R)}$ and $t(\underline{v}) = 1/\int_R d\underline{v}\, I(\underline{v})_{(S)}$, where S is the smallest rectangle enclosing R. A well-known particular example is the generation of Cauchy variates where R is the unit circle.

Trigonometric functions with uniformly distributed arguments

Some standard "tricks" are available for generating random values of trigonometric functions having uniformly distributed arguments. They are suggested for two dimensions by von Neumann (1951) and extended to n dimensions by Cook (1957).

The problem is to generate values of sin(Y), cos(Y) and tan(Y) when $Y \sim U(0, 2\pi)$. There is no conceptual problem with generating the intermediate random variate $y = 2\pi u$ and calculating the trigonometric function directly, but the following method is faster and eliminates the need for the subprogram call.

Let (v_1, v_2) be a point uniformly distributed over the unit circle; that is, $f(v_1, v_2) = 1/\pi$ if $v_1^2+v_2^2 \leq 1$ and $f(v_1, v_2) = 0$ otherwise. Such points may be generated using the two dimensional acceptance rejection concept discusses immediately above. Let α denote the angle between the positive v_1 axis and the vector defined by the origin and (v_1, v_2). Clearly, $\alpha \sim U(0, 2\pi)$. Let $r = (v_1^2+v_2^2)^{1/2}$, the distance of (v_1, v_2) from the origin. Then $\cos(\alpha) = v_1/r$, $\sin(\alpha) = v_2/r$, and $\tan(\alpha) = v_2/v_1$ can be used to generate the trigonometric functions. Improvements can still be made in the sin and cos which involve the square root calculation of r. Note that (1) $\sin(\alpha)$ and $\cos(\alpha)$ have the same distribution, (2) $\cos(\alpha)$ and $\cos(2\alpha)$ have the same distribution, and (3) $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$, which yields $(v_2^2-v_1^2)/(v_1^2+v_2^2)$ as random values for either $\sin(\alpha)$ or $\cos(\alpha)$.

This idea is used to generate Cauchy random variates as $x=v_2/v_1$ and by Knop (1973) for the dipole distribution. The polar method for generating normal random variates, as given in Marsaglia and Bray (1964) is also based on these concepts.

von Neumann's comparison method

von Neumann (1951) gave a method for generating exponential random variates which involves only comparing uniform random numbers and no exponential level calculations. Forsythe (1972) extended the ideas, based on a comment at the end of von Neumann's paper, to the normal distribution and any others satisfying the differential equation $f'(x) + b(x)f(x) = 0$ for $0 \leq x < \infty$. The concept has been used by Ahrens and Dieter (1973), Dieter and Ahrens (1973), and Brent (1974) and extended further by Monahan (1979).

3. CRITERIA FOR ALGORITHM COMPARISON

Before we discuss algorithms for specific distributions, we list here some criteria which are useful both when developing algorithms and when selecting an algorithm for a particular situation.

1. Accuracy
    - 1.a Theoretical
    - 1.b Error induced by U(0, 1) numbers not being random
    - 1.c Error induced by computer arithmetic -- Monahan (1977)
2. Execution speed
    - 2.a Set-up time -- Apostolopoulos and Schuff (1979)
    - 2.b Marginal execution time -- Greenwood (1976)
3. Ease of implementation
    - 3.a Number of lines of code
    - 3.b Support routines required
    - 3.c Bit manipulation required
4. Portability -- Greenwood (1977)
5. Memory requirements
6. Interaction with variance reduction techniques -- Franta (1975)

This list is in no particular order of importance. In fact, an important point is that the criteria to be used differ from application to application, making it impossible to order criteria in order of importance. This makes it impossible to select a "best" algorithm except in the very uncommon case where an algorithm is better than all others in terms of every criterion. On the other hand, many published algorithms are dominated by other algorithms in that there is no situation where the algorithm is the best choice. However, even then, a poor algorithm may be the best selection because it is already implemented.

4. STATE OF THE ART

Having discussed the fundamental concepts for generating random variates in Section 2 and criteria for evaluating algorithms in Section 3, we now discuss the state of the art in each of several specific areas: continuous univariate distributions, discrete univariate distributions, continuous multivariate distributions, discrete multivariate distributions, point processes, time series, order statistics, and geometric problems.

4.1 Continuous Univariate Distributions

Without a doubt, continuous univariate distributions have received more attention in the literature than any of the other topics considered here. About half of the references of this paper fall in this category. Within the family of univariate continuous models, the normal, gamma, and beta distributions are the most common topics, in that order.

The normal distribution

The first exact method for generating normal variates exactly, given by Box and Müller (1958), yields pairs of independent standard normal variates using $r = (-2\ln(u_1))^{1/2}$, $\alpha = 2\pi u_2$, $x_1 = r\sin(\alpha)$, and $x_2 = r\cos(\alpha)$. The validity of the algorithm can be shown directly via change of variables. A more intuitive explanation is to note that (r, α) are the polar coordinates of (x_1, x_2). If X_1 and X_2 are independent standardized normal random variables, the bivariate density function is symmetric about the origin, implying that α is a $U(0, 2\pi)$ variate, and implying that the squared distance from the origin $r^2 = x_1^2+x_2^2$ has a chi-square distribution with two degrees of freedom. Noting that this chi-square distribution is the exponential distribution with mean 2 yields the algorithm from the point of view of necessary conditions for x_1 and x_2 to be independent standardized normal variates.

Marsaglia and Bray (1964) mention an improvement to the Box-Müller algorithm which was developed in Marsaglia (1962) and based on the trigonometric results descussed in Section 2.4. Noting also that $v_1^2+v_2^2 \sim U(0, 1)$ conditional on $v_1^2+v_2^2 \leq 1$ yields the algorithm (1) generate (v_1, v_2) uniformly distributed over the circle with unit radius centered on the origin, (2) set $s = v_1^2+v_2^2$, (3) set $c = (-2\ln(s)/s)^{1/2}$, (4) set $x_1 = c\, v_1$ and (5) set $x_2 = c\, v_2$.

While these two early algorithms are based on special properties of the normal distribution, later algorithms have been primarily composition and acceptance/rejection based. At the assembler language level, where bit manipulation is easy, the composition based algorithm of Marsaglia, MacLaren and Bray (1964) is very fast. Not as fast, but requiring no bit manipulation, is the composition algorithm of Kinderman and Ramage (1976). The are many algorithms which are easy to implement, but not as fast.

Marsaglia (1961c, 1964), Kinderman and Monahan (1976) and Schmeiser (1980) present algorithms for random variates from the tails of the distribution. Tail variates may also be obtained using the inverse transformation, which is considered in Abramowitz and Stegun (1964), Beasley and Springer (1977), Burr (1967), Hill and Davis (1973), Müller (1958), Odeh and Evans (1974), Page (1977), Ramberg and Schmeiser (1972), Schmeiser (1980), and Wetherill (1965).

Other references on normal variate generation include Ahrens and Dieter (1972, 1973), Bell (1968), Best (1979), Brent (1974), Burford and Willis (1978), Butcher (1961), Chay, Fardo and Mazumdar (1975), Chen (1971), Dieter and Ahrens (1973), Forsythe (1972), Gates (1978), Gebhardt (1964), George (1976), Kinderman and Monahan (1977), Kinderman, Monahan and Ramage (1975), Kronmal (1964), Marsaglia (1961c), Marsaglia, Ananthanarayanan and Paul (1976), Miklich and Austin (1976), Moritsas (1973), Müller (1959b), Payne (1977), Pike (1965), Pullin (1980), Sakasegawa (1978), Shafer (1962), Shepherd and Hynes (1976), Sibuya (1962), Swick (1974), Tadikamalla (1978c), Tadikamalla and Johnson (1977), and C.S. Wallace (1976).

The state of the art of normal variate generation is very good. No matter what criteria are applicable, there are algorithms which are satisfactory. This is not surprising since the normal distribution has only one shape, thereby allowing variates to be generated with no overhead for setting-up constants. The simple transformation of multiplying by the standard deviation and adding the mean yields all possible normal distributions. Gamma and beta variate generation are more difficult because the shape of the distribution changes as a function of the parameters.

The gamma distribution

The gamma distribution with shape parameter $\alpha>0$ has density function $f(x)=x^{\alpha-1}e^{-x}/\Gamma(\alpha)\ I(x)_{(0,\infty)}$. Multiplying by the scale parameter $\beta>0$ yields a mean of $\alpha\beta$ and variance $\alpha\beta^2$. Several other distributions are special cases: The exponential with mean β when $\alpha=1$, the Erlang when α is integer, the chi-square with n degrees of freedom when $\alpha=n/2$ and $\beta=2$, and the normal in the limit as $\alpha \to \infty$. We discuss the exponential, Erlang and chi-square distributions before we consider the general gamma distribution.

The classic method of generating exponential variates is the inverse transformation $x = -\beta\ln(1-u)$. Other methods include the rectangle, wedge, tail algorithm of MacLaren, Marsaglia and Bray (1964), the comparison method of von Neumann (1951) discussed in Section 2.4, modifications to the comparison method by Ahrens and Dieter (1972), Marsaglia (1961a) with a modification by Sibuya (1962), and polynomial sampling in Ahrens and Dieter (1972). The Monte Carlo results in Ahrens and Dieter (1972) show their algorithm SA to be the fastest available in assembler language and the inverse transformation to be the fastest in FORTRAN. This author, in unpublished Monte Carlo results, found a slightly faster FORTRAN level algorithm on a CDC CYBER 72 in 1978 to be (1) set $y = -\ln(u_1 u_2)$, (2) set $x_1 = u_3 y$ and (3) set $x_2 = y - x_1$. Here u_3 partitions the Erlang (with mean 2) variate y into two independent exponential variates. In terms of computational comparison to the inverse transformation, it trades a U(0, 1) generation for a logarithm computation. With the additional overhead of the pointers necessary to keep track of the two exponential variates, this new algorithm is about 10% faster than the inverse transformation. A more general algorithm studied was to partition an Erlang (with mean k) variate y by k-1 U(0, 1) order statistics to obtain k independent exponential variates with mean 1, but k=2 proved to be the fastest and easiest to implement.

Erlang variates with mean k have classically been generated using the special property that the sum of k exponential variates have the desired distribution. Using the inverse transformation and some algebra yields $x = -\ln(\Pi_1^k u_i)$. This is an excellent algorithm for small values of k, but execution time grows linearly with k, making the use of the more general gamma algorithms discussed below faster for large k.

The classical method of generating chi-square random variates with n degrees of freedom has been $x = y+z^2$ where y is an Erlang variate with k the largest integer less than or equal to n/2 and z is standard normal if n is odd and is zero if n is even. The special case of n=2 is the exponential distribution with mean 2. For large values of n, the general algorithms for the gamma distribution are faster.

The earliest exact method for generating a gamma variate for any $\alpha > 0$ is due to Jöhnk (1964), which is written in German. Fishman (1973) discusses the algorithm, which is $x = y + wz$, where y is an Erlang k variate, w is an exponential variate with mean 1, and z is a beta variate with parameters γ and $1-\gamma$, where k is the integer portion of α, and γ is the fractional portion. Again the dependence on Erlang variates makes this algorithm inefficient for large values of α, making the general algorithms discussed below faster.

As late as the mid 1970's, approximate algorithms where being published, since exact methods were unacceptably slow for large values of α. These include Phillips (1971), Phillips and Beightler (1972), Ramberg and Schmeiser (1974), Ramberg and Tadikamalla (1974), and Wheeler (1974, 1975). See also Bowman and Beauchamp (1975). All are approximations to the inverse cdf and should not be considered in light of the current state of the art. Approximations yielding machine accuracy inverse transformations may be found in Best and Roberts (1975) and Bhattacharjee (1970). Since the evaluation of the inverse transformation is usually performed by iteratively evaluating the cdf, Gautschi (1979) is of interest. See also Narula and Li (1977).

Exact algorithms which execute in time relatively insensitive to α are now plentiful. Schmeiser and Lal (1980) give algorithm G4PE which has the smallest execution time per variate for large values of α, but its set-up time makes it not fastest when only one variate is needed. Best (1978b) gives a simple algorithm with almost no set-up time. There are many algorithms which provide a continuum in tradeoff between set-up time and marginal execution time between these two algorithms. When a very fast normal generator is available, Marsaglia's (1977) algorithm RGAMA is very fast. Most, but not all, recent algorithms are valid for $\alpha > 1$, since Jöhnk's (1964) algorithm is quite acceptable for $\alpha < 1$.

Other references include Ahrens and Dieter (1974), Atkinson (1977), Atkinson and Pearce (1976), Cheng (1977), Cheng and Feast (1979), Dagpunar (1978), Dieter and Ahrens (1974), Fishman (1976), Franklin and Sen (1975), Greenwood (1974), Kinderman and Monahan (1978), McGrath and Irving (1973), Popescu (1974), Tadikamalla (1978a, 1978b), C.S. Wallace (1976), N.D. Wallace (1974), Whittaker (1974), Berman (1971), and Locks (1976). Takahashi (1959), in Japanese, may also be of interest.

The beta distribution

The beta distribution with shape parameters $p > 0$ and $q > 0$ has density function

$$f(x) = x^{p-1}(1-x)^{q-1} / \beta(p, q) \quad I(x)_{(0, 1)},$$

where $\beta(p, q)$ is the beta function. The mean is $p/(p+q)$ and the variance is $pq/((p+q)^2(p+q+1))$. Special cases include the uniform distribution when $p = q = 1$, the arcsin distribution when $p = q = \frac{1}{2}$, the gamma distribution in the limit as $p \to \infty$, $q \to \infty$, and p/q remains constant; and the normal distribution in the limit as $p \to \infty$, $q \to \infty$, and $p = q$. When p and q are both less than 1, the density function is U shaped, with the density function infinite at $x = 0$ and $x = 1$. When exactly one of p and q are less than 1, the distribution is J shaped, and when both p and q are greater than 1, the distribution is unimodal. This diversity of shapes makes the beta distribution an important model of real world phenomena (often after rescaling to the interval (a, b)), but this same diversity makes development of beta variate generation algortihms difficult. Most algorithms consider only one shape of the beta distribution, requiring the use of a combination of algorithms to obtain variates efficiently for all parameter values.

As with gamma generation, early algorithms dealt with special cases. Fox (1963) suggested the use of U(0, 1) order statistics when p and q are integer. A classical general technique is $x = w/(w+y)$ where $w \sim$ gamma (α=p) and $y \sim$ gamma (α=q), which results in a reasonable algorithm when good gamma generators are used. Jöhnk (1964) gave an algorithm valid for any parameter values, but which has execution times which grow rapidly with p and/or q.

Interest in beta variate generation was spurred by Ahrens and Dieter (1974a) who used a normal majorizing function with mean $p/(p+q)$ truncated at zero and one to obtain algorithm BN. Execution time is least when p and q are close to the limiting normal case of large and equal values. Execution time in the limiting case as the beta approaches the gamma (p and q large and unequal) is asymptotically infinite since the heavier tails of the gamma distribution force a poor fit by the normal majorizing function.

The first algorithm which executes in finite time for all parameter values $p > 1$ and $q > 1$ is BB, which is developed in Cheng (1978). Algorithm B4PE developed in Schmeiser (1980) has marginal times about half of those of BB, but the set-up time is longer and B4PE requires more lines of code.

Atkinson and Whittaker (1976, 1979) consider J shaped beta distributions having one parameter less than and one parameter greater than 1.

Other references are Arnason (1972), Atkinson (1979c), Bankövi (1964), Békéssey (1964), Best (1978a), Dieter and Ahrens (1974), Locks (1976), and Schmeiser and Shalaby (1980). Majumder and Bhattacharjee (1973) consider the inverse transformation.

Other continuous distributions

Other continuous univariate distributions have received considerably less attention. Often only a single paper has been written for a particular distribution. We simply list the relevant references here.

Inverse Gaussian (Wald) distribution: Michael, Schucany and Haas (1976).
von Mises distribution: Best and Fisher (1979).
Ansari-Bradley W statistic: Dinneen and Blakesley (1976).
Weibull distribution: Léger (1973).
Exponential power distribution: Johnson (1979) and Tadikamalla (1980).
Stable distribution: Bartels (1978) and Chambers, Mallows and Stuck (1976).
Lognormal distribution: Chamayou (1976).
Student's t distribution: Kinderman and Monahan (1978), Kinderman, Monahan and Ramage (1975, 1977) and Best (1978a).
Pearson family: Cooper, Davis and Dono (1965) and McGrath and Irving (1973).
Dipole distribution (a generalization of the Cauchy): Knop (1973).
Cauchy distribution: Arnason (1974), Monahan (1979), and Robinson and Lewis (1975).
Kolmogorov-Smirnov statistic: Devroye (1980c).
Burr and Pareto distributions: Popescu (1977).
Extreme value distribution: Goldstein (1963).
Generalized (four parameter) gamma distribution: Tadikamalla (1979a).
Weibull, normal, gamma and beta tails: Schmeiser (1980)
Devroye (1980a) considers variate generation when only the characteristic function is known.

Ramberg (1975), Ramberg and Schmeiser (1972, 1974), Burr (1942, 1973), N.L. Johnson (1947), Ramberg, Tadikamalla, Dudewicz and Mykytka (1979), Johnson, Tietjen and Beckman (1980), Schmeiser (1977), and Schmeiser and Deutsch (1977) discuss various general families of distributions for which variate generation is straightforward.

4.2 Univariate Discrete Distributions

The Poisson and binomial distributions have received the most attention of all the univariate discrete distributions. Johnson (1974) developes a unifying theory for discrete variate generation.

The Poisson distribution

The Poisson distribution has mass function $f(x) = e^{-\mu}\mu^{x}/x!$ for $x = 0, 1, 2, \ldots$, where μ is the mean and the variance of X.

There are three approaches appropriate when μ is small. The inverse transformation, implemented using the recursion $f(x) = f(x-1)\,\mu/x$ almost always dominates the equally easy to implement algorithm based on simulating a homogeneous Poisson process with rate 1 for μ time units, which is discussed in Kahn (1956) and Schaffer (1970). Both methods require a set-up involving $\exp(-\mu)$. When the mean changes often, a "thinning" algorithm which requires no set-up is faster. To generate variates for changing mean in the range $(0, \gamma)$, generate a Poisson variate with mean γ and reject each event with probability $1-(\mu/\gamma)$, which is equivalent to using the product of a Poisson variate y with mean γ and a binomial variate with parameters $n=y$ and $p=\gamma/\mu$. Thinning algorithms are further discussed in the section on point processes.

Generating Poisson variates when μ is large has posed a more substantial problem over the years. Ahrens and Dieter (1974a) give a composition algorithm with execution time increasing with $\sqrt{\mu}$ and a method based on gamma variates with time increasing in $\ln(\mu)$. Fishman (1976b) surveys the Poisson variate literature and gives algorithm PIF which sets-up quickly and has low marginal execution times for moderate values of μ, although execution time increases with $\sqrt{\mu}$.

Akinson (1979a) gives the first algorithm which is exact and has execution time which does not go to infinity as $\mu \to \infty$. Algorithm PA is based on acceptance/rejection, used a logistic majorizing function, and evaluates $\ln(x!)$ by tabling values for x through 200. Using Stirling's approximation for $\ln(x!)$ with enough terms to provide machine accuracy, rather than the tabled values, allows PA to be used for any large value of μ.

Devroye (1980e) gives algorithm IP which is based on composition. The inverse transformation is used for the left tail. The body of the distribution is handled via acceptance/rejection and a normal majorizing function. The right tail is handled with an exponential majorizing function. Evaluation of $x!$ is performed explicitly via $x(x-1)(x-2)\cdots(2)$, but is seldom necessary due to the use of preliminary acceptance and rejection comparisons. Execution times are very stable as $\mu \to \infty$.

Schmeiser and Kachivichyanukul (1980) give algorithm P2PE using composition. Each of three subdensities are handled via acceptance/rejection. The tails have exponential majorizing functions and the body of the distribution has a uniform majorizing function. Using the Kinderman and Ramage (1976) normal generator with IP, P2PE requires about half the marginal execution time and PA is about half again slower. For one variate, IP and P2PE require about the same time, due to P2PE taking longer to set-up. For more than one variate, P2PE is preferred. However, if a very fast assembler language normal generator is used, IP will perform better than with the FORTRAN level normal generator.

Other Poisson references include Atkinson (1979b), Bolshev (1965), Hufnagel and Kerr (1969), Molenaar (1970), Pak (1975), Snow (1968), and Tadikamalla (1979b).

<u>The binomial distribution</u>
The binomial distribution has mass function $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \ldots, n$. The mean is np and the variance is $np(1-p)$.

When np is small, the inverse transformation with recursion $f(x) = f(x-1)\,(n-x+1)\,(p/(1-p))\,/\,x$ is good. When n is small, summing n Bernoulli trials each having probability of success p works well.

For moderate values of n, the use of Chen and Asau's (1974) index table for searching the inverse cdf is fast, but as n goes to infinity, either the size of the table or the execution time becomes infinite, as does the set-up time. Similarly for Walker's (1977) alias method. Norman and Cannon's (1972) tabling procedure works well if rounding the probabilities is acceptable.

Relles (1972) and Ahrens and Dieter (1974a) give algorithms whose execution times increase only slowly with the mean, based on the binomial distribution's relationship with the beta distribution.

There are two exact algorithms which have finite execution time as n and np go to infinity. Fishman (1979) suggests using an acceptance/rejection algorithm with a Poisson majorizing function. Using any of the three Poisson algorithms requiring finite time yields a finite time binomial generator. The other algorithm is Devroye and Naderisamani (1980).

<u>The negative binomial distribution</u>
The negative binomial distribution has mass function $f(x) = \binom{n+x-1}{x} p^n (1-p)^x$ for $x = 0, 1, \ldots$. The mean is $n(1-p)/p$ and the variance is $n(1-p)/p^2$. It is also called the Pascal distribution when n is integer, in which case it can be viewed as the sum of n geometric random variables with probability of success p and x is the number of failures before n successes. The geometric distribution is the special case of n=1.

Geometric random variables may be generated directly using the inverse transformation $x = \lfloor \ln(1-u)/\ln(1-p) \rfloor$, where $\lfloor y \rfloor$ denotes the largest integer less than or equal to y. Of course summing n geometric variates results in execution times which increase linearly with n.

As suggested by Devroye and Naderisamani (1980), a negative binomial variate for any n and p can be generated in a reasonable amount of time by generating a gamma (α=n, β=(1-p)/p) variate y and then generating a Poisson variate x with mean y, which is an example of continuous composition for a discrete random variable. Léger (1973) also discusses the negative binomial distribution.

4.3 MULTIVARIATE DISTRIBUTIONS

The generation of random vectors $(X_1, X_2, \ldots, X_n)$ having specified properties is substantially harder than the generation of univariate random variates. The marginal distributions of the X_i's need to be correct while at the same time some form of dependence between the variables must be established. Schmeiser and Lal (1980a) survey multivariate input models for simulation, including continuous and discrete random vectors, point processes, time series, and order statistics.

<u>Continuous multivariate distributions</u>
A common problem is to need to have the marginal distributions and dependence structure specified by the joint density function $f(x_1, x_2, \ldots, x_n)$. Composition, acceptance/rejection, and conditional distribu-

tions are applicable, the first two being straightforward extensions of the univariate concepts. The use of conditional distributions reduces the multivariate problem to n univariate problems by using the algorithm (1) generate x_1 from $f_1(x)$, (2) generate x_2 from $f_2(x_2|x_1)$, (3) generate x_3 from $f_3(x_3|x_1, x_2)$, and so on. While it is very general, the use of conditional distributions is often intractable.

Often in simulation input modeling, however, the data can be used to estimate the conditional distributions directly, making generation via conditional distributions straightforward. See Kottas and Lau (1978), Eilon and Fowkes (1973), and Johnson (1976).

The multivariate normal distribution has been the subject of more papers than any other multivariate topic considered here: Barr and Slezak (1972), Bedall and Zimmerman (1976), Deák (1978, 1979a, 1979c), Hurst and Knop (1972), Jansson (1964), Page (1974), Scheuer and Stoller (1962) and Schmeiser and Ali (1978). Franklin (1965) discusses the related topic of Gaussian processes.

Several authors have considered various multivariate gamma distributions. Mitchell, Paulson and Beswick (1977) generate bivariate exponential random vectors with any positive correlation and some negative correlations. (The paper says that any correlation between -2.5 and 1 can be obtained, but this is obviously a misprint.) Ronning (1977) and Prékopa and Szántai (1978) present multivariate gamma distributions and generation methods for nonnegative correlations. Schmeiser and Lal (1979) give a family of algorithms for bivariate vectors having any gamma marginal distributions and any correlation consistent with the marginal distributions, including negative correlations.

Macomber and Myers (1978) consider multivariate beta distributions. Arnason (1972) considers the Dirichlet distribution, which has all beta marginal distributions.

Chalmers (1975) and Dempster, Schatzoff and Wermuth (1977) generate random correlation matrices. Chambers (1970) and Smith and Hocking (1972) consider generation of Wishart matrices and Gleser (1976) generates noncentral Wishart distributions. Odell and Feiveson (1966) generate sample covariance matrices.

Coleman and Saipe (1978), Gārgano and Tenenbein (1977), Johnson and Ramberg (1977b), and Johnson and Tenenbein (1979) discuss bivariate distributions having U(0, 1) marginal distributions. Multivariate uniform distributions are important primarily because $(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n))$ has exactly the specified marginal distributions and by modifying the correlation structure of the uniform random vector, various correlation structures can be obtained in the multivariate distribution of interest. The major problem with this approach is that the correlation between x_i and x_j must be determined via numerical integration.

Although not in the context of random variate generation, Kimeldorf and Sampson (1975a, 1975b) provide the basis for a wide range of multivariate uniform distributions. They advocate the study of multivariate distributions via the distribution of $(F_1(X_1), F_2(X_2), \ldots, F_n(X_n))$. Since each $F_i(X_i)$ has a U(0, 1) distribution, analysis of the correlation structure is easier after this transformation. This suggests an algorithm of the following type: (1) generate $(z_1, z_2, \ldots z_n)$ from any n-dimensional multivariate distribution (the multivariate normal being the obvious choice), and (2) calculate $x_i = F_i^{-1}(\Phi(z_i))$ for $i = 1, 2, \ldots, n$, where $\Phi(\cdot)$ denotes the cdf of the normal distribution. Still the problem remains that the correlation between x_i and x_j must be determined via numerical integration.

Hull (1977) uses this method (although there is no indication that he was influenced by Kimeldorf and Sampson's work) to approximate the correlation by matching points on the regression curve $E(X_1|x_2)$. Johnson (1976) discusses direct transformation from one multivariate distribution to another. Mardia (1970) offers a good discussion of bivariate distributions. Moran (1967) and Whitt (1976) contain good discussions of the correlations which are theoretically possible for given marginal distributions. Other references include Arnold (1967), Friday (1976), Johnson (1949), Johnson and Ramberg (1977a), McArdle (1976) and Pearson (1925).

<u>Discrete multivariate distributions</u>
Little work has appeared on discrete multivariate distributions. Fishman (1978a) and Ho, Gentle and Kennedy (1979) discuss the multinomial distribution. Kemp (1976) and Kemp and Loukas (1978a, 1978b) consider the generation of bivariate dicrete distributions in general. Boyett (1979) gives an algorithm for generating R x C contingency tables. See also Wakimoto (1976).

<u>Point Processes</u>
Generation of point processes is most commonly encountered when providing arrivals of customers to a system. The simplist case is that of independent Poisson arrivals with constant rate μ, which is most commonly handled by generating exponential interarrival times with mean $1/\mu$ and adding the time to the time of the last arrival. Complications arise when the interevent times are not exponential or the rate varies as a function of time or the state of the system.

A Poisson point process with rate $\mu(t)$ which varies with time is called a nonhomogeneous Poisson point process (NHPP). A NHPP can be generated using the inverse transformation, composition, acceptance/rejection, and special properties.

Çinlar (1975) gives the inverse transformation, which he terms the time scale transformation. Let

$$\Lambda(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} \mu(t)\, dt,$$

which is the expected number of Poisson arrivals between times t_{i-1} and t_i. The cdf of the time of the next arrival T_i, conditional on the time of the last arrival t_{i-1}, is

$$F_{T_i|t_{i-1}}(t_i) = 1 - \exp(-\Lambda(t_{i-1}, t_i)).$$

Since T_i is a continuous random variable, $F_{T_i|t_{i-1}} \sim U(0,1)$. Setting $F_{T_i|t_{i-1}}(t_i|t_{i-1}) = u$ and solving for t_i yields the inverse transformation algorithm, which for many simple NHPP's is closed form. For example, if $\mu(t) = 2ct$, the inverse transformation algorithm is $t_i = (t_{i-1}^2 - \ln(1-u)/c)^{\frac{1}{2}}$. Kaminsky and Rumpf (1977) also discuss the inverse transformation.

The special property is that Poisson processes, like Poisson random variables, can be added. Consider n NHPP's having rate functions $\mu_i(t)$, for i = 1, 2, ..., n. Then merging the events from the n independent processes yields a NHPP with rate function $\mu(t) = \Sigma_i\, \mu_i(t)$.

The acceptance/rejection concept in the context of NHPP's is commonly termed "thinning." Here events from one NHPP are accepted or rejected to obtain events from another NHPP. Let $\mu'(t) \geq \mu(t)$, where $\mu'(t)$ is chosen so that the inequality is close and events from the NHPP having rate function $\mu'(t)$ are easy and fast to generate. The thinning concept is to generate events with rate $\mu'(t)$ and to accept each event with probability $\mu(t)/\mu'(t)$, where t is the time of the event. See Lewis and Shedler (1979b).

Lewis and Shedler (1976) discuss generating events when $\mu(t) = \exp(\mu_0 + \mu_1 t)$ and Lewis and Shedler (1979a) consider $\mu(t) = \exp(\mu_0 + \mu_1 t + \mu_2 t^2)$ for NHPP's.

Jacobs and Lewis (1977) and Laurance and Lewis (1977) discuss point processes having correlated exponential interevent times. Fishman and Kao (1977) discuss parameter estimation and generation of interevent times using a harmonic function to model the expected interevent time conditional on t_{i-1} to obtain nonhomogeneity. They also consider nonexponential interevent times. Kimbler, Davis and Schmidt (1980) consider estimating and generating point processes when the data is in the form of counts and are nonPoisson.

Time series

Time series having normal marginal distributions were studied by Franklin (1965). Coleman and Saipe (1977) note a correct method for generating time series having lognormal marginal distributions. Gaver, Lavenberg and Price (1973), Lawrance and Lewis (1977, 1978), Jacobs and Lewis (1977) and Schmeiser and Lal (1979) consider time series having gamma marginal distributions. Price (1976) and Hoffman (1979) generate binary time series. Fraker and Rippy (1974), Kaplan and Orr (1976), Nawathe and Rao (1979), Polge, Holliday and Bhagavan (1973) and Yagil (1963) consider various related problems, as do Li and Hammond (1975) who provide some additional references.

Order statistics

We briefly review some results for generating order statistics. Schmeiser (1978a) gives a complete survey.

Let $x_{(i)}$ denote the i th largest observation from a sample of n (not necessarily independent) observations. Then $x_{(i)}$ is the i th order statistic. The minimum observation is $x_{(1)}$, the maximum is $x_{(n)}$ and the median is $x_{((n+1)/2)}$ when n is odd. The need for random order statistics arises in many contexts; reliability is a common example. Clearly the direct method of generating $x_1, x_2, \ldots, x_n$ and sorting is always valid. However, when n is large or not all order statistics are needed, considerable savings are possible using the methods discussed here.

First consider the case of independent U(0, 1) random variables $U_1, U_2, \ldots, U_n$. Schucany (1972) showed that the following algorithm is valid for generating the order statistics directly without sorting:

(1) Generate $v_1, v_2, \ldots, v_n$ independent U(0, 1).

(2) Set $u_{(n)} = v_1^{1/n}$

(3) Set $u_{(n-i)} = u_{(n-i+1)}\, v_{i+1}^{1/(n-i)}$ for i = 1, 2, ..., n-1.

The algorithm can be terminated after k iterations to obtain only the top k order statistics. Execution time is linear in k, whereas sorting algorithm times increase faster than linearly. The intuitive thought behind Schucany's algorithm is that conditional on knowing $u_{(n-i+1)}$, the distribution of the remaining n-i order statistics is that of n-i independent U(0, $u_{(n-i+1)}$) random variables, thus permitting the recursion. Lurie and Hartley (1972) published a similar algorithm, the difference being that they generate the order statistics in the reverse order:

(1) Generate $v_1, v_2, \ldots, v_n$ independent U(0, 1).

(2) Set $u_{(1)} = 1 - v_1^{1/n}$

(3) Set $u_{(i)} = 1 - (1 - u_{(i-1)})\, v_i^{1/(n-i+1)}$ for i = 2, 3, ..., n.

Lurie and Mason (1973), Mason and Lurie (1973) and Rabinowitz and Berenson (1974) consider these ideas further. Ramberg and Tadikamalla (1978) suggest using $u_{(i)} \sim$ beta (i, n-i+1) to allow the recursion in either algorithm to begin anywhere, rather than only the top or bottom. (Note the relationship to Fox (1963) who used the same relationship to generate beta variates.)

These algorithms for U(0, 1) order statistics are more general than they first appear, since $x_{(i)} = F_X^{-1}(u_{(i)})$ is a valid method for obtaining random variates for the i th order statistic for any random variable X. The validity follows from F_X^{-1} being a monotonic function.

Devroye (1980d) considers the case of $u_{(n)}$ when n is so large that numerical problems make the use of $u_{(n)} = v_1^{1/n}$ impossible. Schmeiser (1978) considers the generation of $x_{(1)}$ or $x_{(n)}$ when the observations are not indentically distributed, but $F_{X_i}^{-1}$ is available.

In other cases, some kind of sorting is required. The use of a histogram provides an approximate sort in time proportional to the number of observations. Good sorting algorithms require execution time proportional to n ln(n), although for small samples n^2 sorts are reasonable. When only some of the order statistics are required, the partial sorts of Chambers (1971, 1977) and Floyd and Rivest (1975) are useful.

A final point is that when order statistics are being generated, the use of exact algorithms for generating x_i is important. An insignificant error in the tail of the distribution under regular sampling can be magnified into a serious problem with order statistics, since extreme observations become more likely.

Geometric problems

Many random generation problems have geometric interpretations, the most common being points uniformly distributed on a sphere and random permutations (card shuffling).

Müller first considered the generation of a point uniformly distributed on an n-dimensional sphere. Let $z_1, z_2, \ldots, z_n$ be independent standardized normal random variates. Then if $x_i = z_i^2/(\Sigma_i z_i^2)$ for i = 1, 2, ..., n; $(x_1, x_2, \ldots, x_n)$ is a point uniformly distributed on the n-dimensional sphere with radius one centered on the origin. Execution time grows linearly with n.

Acceptance/rejection from an n-dimensional unit cube looks appealling at first, but the ratio of the volume of the sphere to the cube goes to zero quickly as $n \to \infty$. See, for example, Schmeiser and Ali (1978).

Other references include Cook (1959), Deák (1979b), Hicks and Wheeling (1959), Marsaglia (1972), Sibuya (1964), and Yoshihiro (1977).

Algorithms for generating random permutations may be found in Boyett (1979), Eisen (1964), Page (1967), and Rao (1961).

Crain (1978) considers generation of random polygons and Hsuan (1979) generates uniform polygonal random pairs. Knop (1970) and Schrack (1972) discuss generation of random vectors distributed over a solid angle. Heiberger (1978) considers random orthogonal matrices.

5. SUMMARY

The state of the art of random variate generation has changed greatly in the last ten years. Fast, exact and easy to implement algorithms are available for most common univariate distributions. Order statistics and nonhomogeneous Poisson point processes are much more tractable than they were a few years ago. Multivariate gamma vectors with any correlation structure can now be generated, although as with many multivariate generation problems numerical integration is involved. Several families of distributions which are much more general than the commonly used distributions have been developed.

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JSCS -- Journal of Statistical Computation and Simulation
CACM -- Communications of the ACM
JASA -- Journal of the American Statistical Association
JRSS -- Journal of the Royal Statistical Society