Existence of Limit Cycles in a Non Linear Dynamic System With Random Parameters

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Abstract

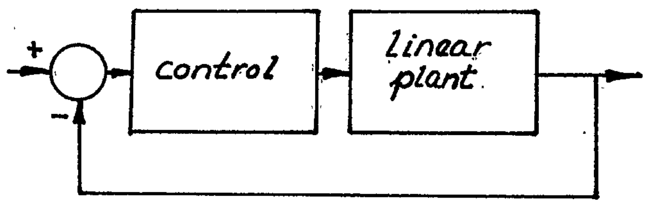
The paper describes an efficient technique for investigating the performance of a nonlinear dynamic system with uncertain parameters. An analysis is made to determine the probability that a nonlinear control system will exhibit limit cycle (sustained periodic oscillation) behavior. The quadrature method of Evans, which provides an efficient computational tool for characterizing the moments of a function of random variables, is used in the analysis.

INTRODUCTION

The modeler is often faced with a situation in which uncertainty manifested in the imprecise knowledge of system parameters, inputs and environmental disturbances plays an important role in the design of a dynamic system. The most common approach when presented with such a problem is to base the simulation on a single set of parameter values--perhaps mean values or values which produce "worst case" conditions. The first of these approaches has the obvious drawback that the simulation will not contain any information concerning the variability of the response; the second assumes that the model builder knows what "worst case" means, and furthermore that the perhaps subtle dependence of the response upon the parameter values is understood--both of these assertions are doubtful in all but the simplest of cases. A more natural approach to this problem is to treat the parameters of the system as random variables and thus the simulated system response becomes a random variable.

Before we proceed to discuss the alternatives for incorporating such random variation into an analysis or system design, let us focus our attention on a simple dynamic system--one which will allow us to more clearly define the problem at hand.

FIGURE 1
CONTROL SYSTEM



In a typical linear position control system, the classical measures of dynamic performance include: static error coefficients, percent overshoot to a step input, rise time, settling time, etc. In an earlier paper [1], the author presented an analysis of a linear position servo in which the moment of inertia, the damping coefficient of the load and the electrical variables of the dc motor were random parameters with known distributions. The system performance was assumed to be characterized by the percent overshoot to a step input. Since the percent overshoot depends upon the parameters of the control system, each of which is a random variable, the performance then becomes a random variable. In the cited reference, the probability that the percent overshoot exceeds some prescribed value is determined.

In the current paper, we consider a more general problem. Here we will assume that the control system is nonlinear. We will restrict ourselves, however, to the case in which the system can be separated into two parts: a linear plant, and a nonlinear decision element. The nonlinear element may be a simple nonlinear function such as a relay or a multilevel quantizer, or it may incorporate memory into the decision process. The nonlinearity introduces many complications into any analysis and subsequent design. The standard measures of performance used in linear control system design are no longer adaquate.

The nonlinear control system can exhibit a self-sustained periodic oscillation. The existence

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of such a limit cycle is usually undesirable and can generally be "designed out" of the system by proper selection of the gain of the control system. One might reasonably ask the following question: If the control system is designed for one set of system parameters, what is the probability that parameter variation away from the design value will cause the system response to degrade and begin to limit cycle?

Such parameter variation can arise from a variety of sources:

1) In fabricating a large number of systems, statistical fluctuation in component values causes differences among a set of samples drawn from a production lot. We wish to predict the statistical parameters which describe the ensemble of all such systems.
2) A single system is to be exposed to a variety of environmental conditions which may (over a long period of time) cause variation in the system components. Although we are not sure exactly what these variations will be, we are willing to make a statistical statement concerning the parameter changes. We then wish to predict how these parameter variations (statistically) affect the system response.
3) In generating the design, the engineer may have only an estimate of the critical parameters. Statistically describing these parameters, the uncertainty in the system response can be quantified.

SYSTEM PERFORMANCE

In the three situations described in the preceeding section, the system response depends upon a set of parameters $(\lambda_1, \lambda_2, \ldots \lambda_n)$; the numerical values of these parameters may describe component values, the system state, environmental disturbances or classes of input functions which force the system. Typical measures of dynamic performance include: peak overshoot, rise time, the value of the response at some time t_1, the time at which the response exceeds a value x_1, the integral of the square of the deviation of the response from the desired value, etc. In this paper we will describe the performance by a scalar quantity J.

$$J = f(\lambda_1, \lambda_2, \ldots \lambda_n) \tag{1}$$

The functional dependence of J upon the parameter set need not be given by an explicit function of λ_k, but rather as a rule for uniquely determining J given a set of values for the parameters. Thus, equation (1) may involve the use of iterative methods, the numerical solution of differential equation, etc.

An acceptable system performance is described as a set of permissible values for J. It is possible to solve the more general problem in which the performance is a vector of indices and acceptable system performance is described as a volume in the n-dimensional performance space. In order to determine the probability that the performance is acceptable, we must be able to find the probabilities associated with a jointly distributed random variable over an n-dimensional space. Since the distributions are generally not gaussian, this is a difficult problem.

The problem at hand is now to determine the density function w(J) and determine the probability that the performance resides in the acceptable zone described above. The method used in the analysis requires that the parameters be independent random variables whose distributions are known.

METHODS OF APPROACH

Several different approaches are possible for the solution of the problem posed above. These include the linear propagation of errors method which is based upon the first order terms in the Taylor series expansion of expression (1), the extended Taylor series method, Monte Carlo methods, and the quadrature or numerical integration method. The linear propagation of errors attack is the best known and most widely used. We will proceed to discuss the several approaches but focus principally on the quadrature method, an efficient computational approach for solving this problem.

Let us consider the case in which (1) is either linear or can be linearized.

$$J = a_1\lambda_1 + a_2\lambda_2 + \ldots + a_N\lambda_N \tag{2}$$

The a_k are static sensitivity coefficients which reflect the impact that parameter variations have upon the index of performance. Since by assumption the λ_k are statistically independent random variables it is well known that

$$\begin{aligned} \text{ave } J &= a_1 \text{ave}\lambda_1 + a_2 \text{ave}\lambda_2 + \ldots + a_N \text{ave}\lambda_N \\ \text{var } J &= a_1^2 \text{var}\lambda_1 + a_2^2 \text{var}\lambda_2 + \ldots + a_N^2 \text{var}\lambda_N \end{aligned} \tag{3}$$

One can usually find justification for assuming that the index J is normally distributed; thus the mean and variance computed from equation (3) can be used to make quantitative statements concerning the probability that J lies within some specified interval. Such an analysis is important in manufacturing; here the λ_k are component values (e.g., resistance, capacitance, etc.) and J is a measure of the response of a system fabricated from these components. In order to justify such probability statements concerning the response, the component tolerances must be given as distributions: e.g., λ_k is uniformly distributed in (a_k, b_k) or λ_k is normally distributed with mean μ_k and standard deviation σ_k, etc. The classical linear propagation of errors technique (based upon retention of the first order terms in the Taylor series expansion for J yields reasonable results; the results are perhaps better than one would expect from experience with linearization in deterministic engineering problems. The reader is referred to a recent three part paper by Evans on the state of the art of statistical tolerancing [2], [3], [4] for further discussion of these points. When a linear analysis is not good enough, three alternate approaches are possible. The nonlinear propagation of errors technique based upon a Taylor series expansion through order five [3], [5], [6], [7] may be used. Although the method is powerful, it poses computational difficulties if the function f is not sufficiently tractable. Monte Carlo methods form a second conceptually simple alternative; the parameter values λ_k are randomly selected from known distributions and used to compute the response index J. The resulting set of values of J is used to compute the statistical nature of the distribution w(J) by ordinary statistical methods. The drawback of Monte Carlo method is the rate of convergence. The final procedure--a quadrature formula for numerical integration--forms the computational basis for the example presented in this paper. The quadrature formula which we will discuss in a subsequent section uses the nonlinear equation relating J to the λ_k; it can be used if J can be evaluated for a given set of parameters. The method is straightforward and easily programmed on a digital computer. Evans [3] compares the methods enumerated above and describes the salient computational aspects of each.

THE QUADRATURE METHOD

The problem we have set forth involves finding the density function for the response, w(J), given the densities of the N independent random variables $w_k(\lambda_k)$. The quadrature method is an approximation technique based upon the fact that distributions are determined by their moments and that the moments are integrals; the integrals are further approximated by numerical integration and the densities are specified by their lower moments.

Let us introduce the following notation for the central moments of the system parameters, λ_k:

$$\begin{aligned} E[\lambda_k] &= \mu_k \\ E[(\lambda_k-\mu_k)^2] &= \sigma_k^2 \\ E[(\lambda_k-\mu_k)^3] &= \gamma_k\sigma_k^3 \\ E[(\lambda_k-\mu_k)^4] &= \Gamma_k\sigma_k^4 \end{aligned} \tag{4}$$

A normal or gaussian distribution is completely specified by the mean μ_k and the variance σ_k. Since the normal distribution is symmetric, the coefficient of skewness $\gamma_k = 0$, while the fourth normalized central moment $\Gamma_k = 3$. For a uniform distribution over the interval [a,b], $\mu_k = (b-a)/2$, $\sigma_k^2 = (b-a)^2/3$, $\gamma_k = 0$ and $\Gamma_k = 9/5$.

The moments of the response J are given by

$$M_i' = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} h_i(\lambda_1,\ldots,\lambda_n) \prod_{k=1}^{N} w_k(\lambda_k)d\lambda_k \tag{5}$$

where

$$h_1 = F(\mu_1,\mu_2,\ldots,\mu_N)$$

$$h_i = [F(\lambda_1,\ldots,\lambda_N)-h_1]^i$$

These are the moments around $J = h_1$; the central moments for J are obtained by application of the moment transfer formulas [8]. The integral for M_i' is approximated by a quadrature formula which is constructed using standard symmetry arguments of multidimensional numerical integration [9], [10], [11]. The quadrature approximation requires that the response index $J = F(\lambda_1,\lambda_2,\ldots,\lambda_N)$ be evaluated at $2N^2 + 1$ prescribed values of the parameter set; these prescribed values are specified in terms of the first four moments of the parameters. All four moments of the response J can be approximated from this single set of $2N^2 + 1$ values.

Since the quadrature technique is an approximation, two important points should be discussed: how precise is the approximation for the moments, and how good are the moments for making probability statements. It has been shown that the precision of the estimates of the moments is $O(\sigma^5)$, in general, and in the special case when the parameter distributions are all symmetric it is $O(\sigma^6)$, where σ is a representative standard deviation from the set $\{\sigma_k\}$; this aspect of the error problem has been discussed [10]. The use of the first four moments to graduate a distribution is a widely used approximation device in statistics; its acceptance is based more on experience than on theory since all of the moments would be required to obtain exact results. The Pearson system [12] is perhaps the oldest and most widely used such device; the system encompasses a wide class of distributions and contains the standard ones, e.g., the normal, the beta, and the gamma or type III distributions. Tables are available [13]. Pearson's third and fourth moments $\sqrt{\beta_1}$ and β_2 are defined exactly as γ and Γ. The reader is cautioned against putting too much reliance on the value for β_2 obtained by quadrature [10] in using these tables.

Evans and Falkenburg [14] have published computer programs which implement the quadrature method described in this section. The user need only write a subprogram which defines the dependence of the performance upon the system parameters.

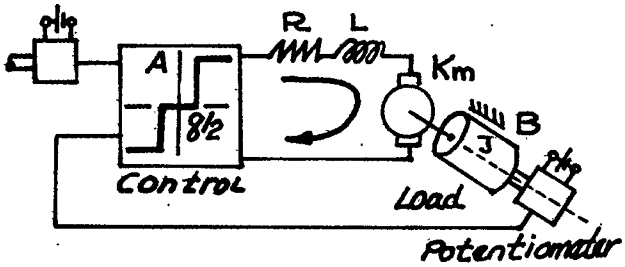
EXAMPLE PROBLEM

In the introduction, we pointed out that a nonlinear control system may exhibit limit cycle behav-

<u>System With Random Parameters</u> (continued)

ior. We propose to examine the probability that such a limit cycle will exist if the system parameters are random variables. Figure 1 represents the structure of a control system pictured in figure 2. The d.c. servo-system is controlled by a three level quantizer.

FIGURE 2
NONLINEAR SERVO



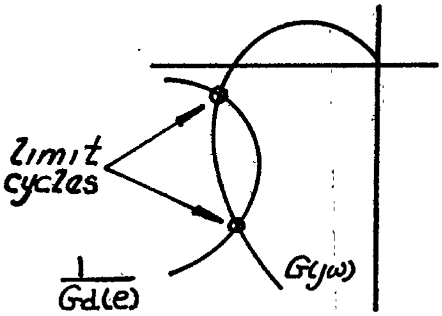
R and L are the resistance and the inductance of the armature circuit, while K_m is the torque constant of the motor. The load is characterized by the moment of inertia J and the viscous damping coefficient B. Finally, the decision element contains two parameters--the voltage output A and the dead zone q.

In order to determine the existence of a limit cycle, the nonlinearity is replaced by the describing function--an equivalent transfer function giving the same attenuation and phase shift at the limit cycle frequency; this approximation neglects higher harmonics produced at the output of the nonlinearity. The condition required for the existence of a limit cycle is given in equation (6).

$$1 + Gd(e)G(jw) = 0 \qquad (6)$$

In order to determine the existence of a limit cycle, two loci are plotted in the complex plane. The first of these is the polar plot of the transfer function G(jw). Each w defines a vector in the complex plane; the locus of all such vectors is the polar plot for the linear plant--it is parameterized by the frequency w. In a similar fashion, the locus of 1/Gd(e) is plotted with the limit cycle amplitude e as a parameter. The intersection of these two curves indicates that a limit cycle exists; the condition for intersection depends upon the parameters. Two such limit cycles are indicated in figure 3.

FIGURE 3
POLAR PLOT AND DESCRIBING FUNCTION



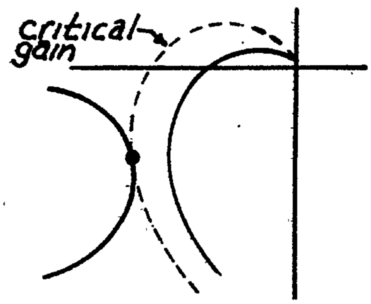
If the loci do not cross, there is no limit cycle. In addition to establishing the existence of a limit cycle, the magnitude and the frequency of the limit cycle can be found by equating magnitudes and angles of both sides of equation (6).

The geometrical problem of determining the intersection of a given G(jw) and 1/Gd(e) locus can be reduced to straightforward calculation under the following set of reasonable assumptions:

1) The gain of the linear plant, if sufficiently large will cause the two loci to intersect. This gain merely scales the G(jw) locus.
2) There is one and only one gain which causes the G(jw) locus to be tangent to the 1/Gd(e) locus.

These two conditions are met by many linear plants and describing functions. They are, however, not always true. In Figure 4 we show such an admissible pair of loci. The dashed G(jw) plot has exactly the minimum gain which will produce a limit cycle. A further reduction of the gain will admit no sustained periodic oscillation. Increases in gain will always produce limit cycles.

FIGURE 4
POLAR PLOT WITH CRITICAL GAIN



Let us introduce a scaling factor M which multiplies the transfer function G(jw). As a consequence of assumption (1), an M can be found which will produce a limit cycle. Let us furthermore define Mcr to be the gain which causes the two loci G(jw) and 1/Gd(e) to be tangent. Mcr then becomes the minimum gain which produces a limit cycle.

Let us choose to describe our system performance in terms of this gain margin Mcr. If Mcr<1 a limit cycle exists, otherwise the system will not sustain a periodic oscillation. In addition to allowing us to determine the existence of a limit cycle, the gain margin also gives a measure of how close the system is to this condition of sustained oscillation. Obviously, Mcr=1.05 corresponds to a situation in which a limit cycle is imminent, while Mcr=10.5 describes a system which is much further from this limit cycle boundary.

$$J = Mcr = f(\lambda_1,\lambda_2,\ldots,\lambda_n) \qquad (7)$$

Again, the λ_k are the system parameters. For the system shown in Figure 2, the linear transfer function and the describing function are given in equations (8) and (9).

$$G(s) = \frac{K_m}{s[(Ls+R)(Js+B)+K_m^2]} \quad (8)$$

$$s = j\omega$$

$$Gd(e) = 4\sqrt{1-(q/2e)^2}/(\pi e) \quad (9)$$

The use of Evans' quadrature method to determine the moments of the distribution of performance function, requires that each parameter in the performance index be statistically independent. Generally, the system gains, the pole and zero locations of the transfer functions are not independent. In most physical systems, these transfer function variables can be expressed in terms of the system parameters which are often independent. In order to apply the quadrature method, it is necessary to describe the function relating the system parameters to the performance--in this case the gain margin Mcr. It was mentioned earlier that such a function need not be given in explicit form, but rather one must specify the rule for calculating the performance. The gain margin can be found by the following two part procedure: First, a function is established relating the, amplitude of a limit cycle and the gain M which will produce this sustained oscillation. The rules describing this function are summarized in the structured flow chart shown in Figure 5 below.

FIGURE 5
GAIN M AS A FUNCTION OF LIMIT CYCLE AMPLITUDE. M = M(e)

Assign initial guess for e--the magnitude of limit cycle
Compute magnitude and angle of 1/Gd(e)
Equate the angles of 1/Gd(e) and G(jw) and solve for the limit cycle frequency. This is the crossing condition.
Compute the gain M which will produce a limit cycle of this magnitude and frequency. Solve $M = 1/(\lvert Gd(e)\rvert \lvert G(jw)\rvert)$.

The second part of the algorithm is to find the minimum gain which produces a limit cycle. In order to accomplish this, the function defined in Figure 5 is imbedded in a minimization routine. The minimum gain M producing a limit cycle is the gain margin Mcr--the system performance.

RESULTS

The problem which is described in the previous section contains seven system parameters. Each of the electrical parameters is assumed to be normally distributed with a known mean value. The standard deviation is selected assuming that the 3σ tolerance interval is known. Such a specification insures that 99.7% of all observations will fall within the $\mu+3\sigma$ tolerance band. Since the distributions are normal the third and fourth central moments are known. In order to illustrate the fact that it is not necessary to assume the parameters are normally distributed random variables, the mechanical parameters--the moment of inertia and the viscous damping coefficient are assumed to be given by a general distribution. It seems most reasonable to assume, for example, that deviations of the viscous damping coefficient above the mean are more likely than those below the mean; such a situation would yield a non-symmetric distribution. We have chosen to represent these statistical fluctuations by a beta distribution [8], and use a methodology common in PERT analysis [15] to find the distribution. This procedure involves making three estimates of the parameter--an optimistic value, a pessimistic value and the most likely value. The procedure for determining the moments for B and J are outlined in [1]. All the parameter values are summarized in Figure 6.

FIGURE 6
MOMENTS OF SYSTEM PARAMETERS AND THE PERFORMANCE

electrical parameters	μ	100%x $3\mu/\sigma$	σ	γ	Γ
R [Ω]	1.620	2%	.0108	0.0	3.0
L [mh]	30.0	2%	.200	0.0	3.0
K_m $\left[\frac{\text{nt-m}}{\text{amp}}\right]$	0.905	2%	.006	0.0	3.0
A [volts]	4.0	5%	.0667	0.0	3.0
q	0.10	10%	.0033	0.0	3.0
mechanical parameters					
J [kg-m^2]	.0513	---	.0050	.3094	2.557
B $\left[\frac{\text{kg-m}^2}{\text{sec}}\right]$	.0529	---	.0067	.4795	2.702
performance Mcr	1.086	---	.0845	.3460	3.067

The results of the preceeding table indicate that the third and fourth central moments are nearly those of the normal distribution. It seems reasonable to attempt to use the tables of the standard normal random deviate to make a probability statement concerning the likelihood of a limit cycle appearing in this control system. Using these standard tables, (1.0855-1)/.0845 = 1.012. Thus, the probability that the system will exhibit such undesired behavior is about .156. Although we used the tables for the normal distribution to make a probability statement in this example, the tables of percentage points of the Pearson curves [13] could be used if such an assumption seems unwarrented.

System With Random Parameters (continued)

SUMMARY

We have presented a computationally efficient way to investigate the performance of a nonlinear dynamic system with random parameters. The method requires us to estimate the moments of the distribution of the system parameters. In addition, we must write a computer subprogram to calculate the performance of the system given a specific set of values of the system parameters. The method produces an estimate of the moments of the distribution of the system performance. Such results can be used by an engineer to assess the acceptability of a proposed design.

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