ADDITIVE VS MULTIPLICATIVE UNIFORM PSEUDO-RANDOM NUMBER GENERATORS IN THE GENERATION OF ERLANG VARIATES

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ABSTRACT

This paper presents the comparative results of several test runs involving the use of several unit uniform random number generators with an inverse transform to generate Erland random variates with $\lambda=1$ and k = 3, 4, 5, and 6. The results show that two commonly used generators (a multiplicative and a mixed one) produce unacceptable results. Two others (an additive one and a multiplicative one) produce acceptable results. The computer system used is an IBM 370/3033.

INTRODUCTION

The objective of this paper is to present some results which have been obtained recently as a result of a continuing process of testing which has been conducted for some time (4). The tests reported on here concern the comparative effectiveness of several unit uniform random number generators for the generation of Erlang variates using the sum of inverse exponential transforms of unit uniform variates. The unit uniform generators tested include an additive congruential generator ADRAND (2,3,4); two multiplicative generators, IBM's RANDU and Lewis and Learmont's Naval Post Graduate School generator (10); and a mixed congruential generator proposed by MacLaren and Marsaglia (13). The Erlang transform used was the inverse transform

$$x = -\frac{1}{\lambda} \ln \left(\prod_{i=1}^{k} u_i \right)$$

where the u_i are the unit uniform numbers generated by the above four uniform generators, $\lambda = 1$ and k = 3, 4, 5, and 6.

THE TEST APPROACH

The tests which were run were all chi-square goodness of fit tests. Each test involved generating 1,000,000 Erlang variates and k= 3, 4, 5, and 6 ($\lambda=1$). Each test was replicated three times for each unit uniform generator and for each value of k.

THE TEST RESULTS

The results of the tests which were run are shown in Tables 1 through 6. While the test results must be considered somewhat tentative, subject to additional tests, the results in Tables 1 through 6 are interesting and quite revealing.

It is clear that for k=3 RANDU did not perform well at all. M & M's performance is better than RANDU's for k=3 but not as good as either L & L or ADRAND. In terms of the goodness of fit for k=3 the four generators rank from best to worst as follows: ADRAND, L & L, M & M, and RANDU.

According to Table 2, the relative rankings of the four generators are the same for k=4 as for k=3 in Table 1.

Table 3 indicates that neither RANDU nor M & M performed well for k=5, L & L performed better than either of the others. Table 4, however, indicates that ADRAND, when computed as $r_i = (r_{i-1} + r_{i-2} + r_{i-3} + r_{i-4}) (\text{Mod } 10^8)$, blew up totally. In Table 4 this generator is referred to as ADRAND(4). Also in Table 4 are the results for ADRAND(5), defined as $r_i = (r_{i-1} + r_{i-2} + r_{i-3} + r_{i-4} + r_{i-5}) (\text{Mod } 10^8)$. While ADRAND(4) clearly did not perform well for k=5, ADRAND(5) performed approximately the same as L & L.

Tables 5 and 6 contain the results of tests for k=6. Clearly RANDU and M & M failed to perform satisfactorily. The performance of L & L was relatively satisfactory. Neither ADRAND(4) nor ADRAND(5), however, came close to a satisfactory performance. On the other hand, ADRAND(6) defined as $r_i = (r_{i-1} + r_{i-2} + r_{i-3} + r_{i-4} + r_{i-5} + r_{i-6}) (\text{Mod } 10^8)$, performed approximately the same as L & L.

CONCLUSION

It seems clear from the data in Tables 1 through 6 that neither RANDU nor M & M can generate satisfactory Erlang variables from

the transform used here. The results of these tests also suggest that L & L performs reasonably well as a generator of Erlang variates and that ADRAND(m), where $m \geq k$, also performs well.

While no definitive timing statistics were collected, the CPU times required on an IBM 370/3033 system for the tests run are shown in Tables 1 through 6. Clearly L & L required less time than either of the other generators and ADRAND required the most time. However, the absolute differences in times required were small in each case.

TABLE 1

Chi-Square Values, By Interval, For Erlang Variates (λ=1, k=3), By Generator And Run, n = 1,000,000

	Adrand			Randu			M & M			L & L		
X	1	2	3	1	2	3	1	2	3	1	2	3
0.5	1.30	.00	2.03	1.07	.07	2.25	2.15	.81	.04	1.73	.50	.69
1	.50	4.05*	.62	.35	2.16	1.69	.60	1.70	.14	.02	.12	1.57
1.5	.31	.46	.10	.04	3.18+	.06	.80	.44	.05	.86	3.26+	.29
2	.08	.07	.07	4.73*	.06	.22	.01	.37	1.10	.81	1.16	3.24+
2.5	2.90+	.77	.29	1.51	.06	1.30	1.48	.18	.52	.00	.07	1.39
3	.00	.44	.04	4.20*	14.05**	13.46**	2.22	2.16	.03	.03	.28	.21
3.5	.55	1.56	.04	14.45**	9.79**	5.54*	2.57	7.01**	.61	.19	1.52	.10
4	.00	.91	.82	.05	.02	.52	1.34	1.74	1.13	2.24	.24	.01
4.5	1.56	.07	.71	3.71+	6.87**	2.75+	4.97*	.98	3.91*	2.84+	.08	4.47*
5	1.52	.32	.35	.64	.01	1.38	.27	.58	.07	7.81**	.80	.29
5.5	.08	.00	.35	5.24*	.87	.37	.96	.89	.04	.06	.04	1.84
6	.74	1.01	.72	.29	.03	.10	.31	2.06	.87	.15	1.06	1.02
6.5	2.60	.00	.05	5.38*	.59	.15	.12	.16	.08	.13	.00	2.35
7	1.20	.58	.12	.14	.44	.97	.01	2.06	.00	.37	.62	1.17
7.5	2.43	.32	.41	2.56	.29	.15	4.52*	1.56	.46	.11	5.54*	.03
8	.86	.73	.59	.71	.35	.12	1.03	.96	.67	1.24	.01	.61
>8	1.53	.53	.67	1.01	1.39	6.20*	1.79	.11	.24	1.79	.19	4.19*
Overall χ^2	18.18	11.86	7.98	46.09**	40.24**	37.22**	25.16+	23.77+	9.97	20.37	15.49	23.47

Times	Adrand	Randu	M & M	L & L
Run 1	1:03.28	0:52.96	0:57.02	0:40.85
2	1:03.25	0:53.01	0:57.07	0:40.92
3	1:03.26	0:52.98	0:57.04	0:40.77

Critical Chi-Square

$\chi^2_{16}(.10) = 23.54$

$\chi^2_{16}(.05) = 26.30$

$\chi^2_{16}(.01) = 32.00$

\+ Significant at 0.10
* Significant at 0.05
** Significant at 0.01

TABLE 2

Chi-Square Values, By Interval, For Erlang Variates (λ=1,k=4), By Generator And Run
n = 1,000,000

	Adrand			Randu			M & M			L & L		
X	1	2	3	1	2	3	1	2	3	1	2	3
0.5	3.66+	.02	.87	6.30*	8.51**	22.18**	5.16*	1.06	11.69**	.17	.25	1.16
1	2.27	.16	.04	.06	1.17	.17	.10	9.56**	.20	.11	.25	.23
1.5	1.45	.26	.07	.51	.01	.50	.54	.03	.19	.04	.76	1.17
2	.16	.11	.01	2.08	2.56	.11	.85	.19	1.63	3.31+	1.46	3.46+
2.5	.63	.40	1.12	1.95	.44	2.45	1.36	2.58	3.38+	.43	.65	.23
3	.19	.23	.01	2.67	3.11+	.18	2.37	.02	.56	.02	3.52+	2.60
3.5	.24	.04	.50	1.07	.24	.02	.02	1.15	.12	1.95	1.47	.10
4	.00	.09	.36	3.32+	.77	.28	6.87**	.06	.46	1.95	1.13	.70
4.5	.01	1.03	4.51*	1.49	.20	.85	.17	.12	.93	.69	.01	.62
5	.04	.19	3.04+	3.89*	1.51	.02	.08	.01	.53	.30	.44	.09
5.5	.92	2.64	.00	.60	.05	9.56**	.03	3.37+	1.41	.16	.21	1.89
6	.23	.06	.71	1.40	.59	.80	.25	3.97*	1.85	.37	1.17	.01
6.5	2.23	.15	.18	.94	.02	.65	.04	1.31	1.43	.00	3.92*	.00
7	.09	.01	.27	1.91	.02	.02	.46	.02	.68	2.02	3.40+	.01
7.5	.67	.31	1.31	.79	.11	.06	3.25+	.02	.03	2.87+	.26	.00
8	.57	.66	.14	.10	.00	4.88**	1.15	.68	.70	.00	.02	2.58
>8	.27	2.57	.11	.50	4.76*	1.35	.05	.50	5.71*	.02	.38	1.31
Overall χ^2	13.64	8.94	13.25	29.61*	24.05+	44.08**	22.75	24.65+	31.50*	14.41	19.32	16.18

Times

Run 1	1:19.67	1:05.72	1:11.52	0:49.22
Run 2	1:19.60	1:05.78	1:11.55	0:49.18
Run 3	1:19.69	1:05.75	1:11.53	0:49.24

Critical Chi-Squares

$\chi^2_{16(.10)} = 23.54$

$\chi^2_{16(.05)} = 26.30$

$\chi^2_{16(.01)} = 32.00$

\+ Significant at 0.10
* Significant at 0.05
** Significant at 0.01

TABLE 3
Chi-Square Values, By Interval, For Erlang Variates (λ=1,k=5), By Generator And Run
n = 1,000,000

	Randu			M & M			L & L		
X	1	2	3	1	2	3	1	2	3
0.5	15.91**	14.71**	15.30**	17.15**	9.41**	11.39**	.29	.05	.01
1	2.22	.84	8.58**	.02	1.10	1.21	.03	.84	.00
1.5	.47	.18	2.12	3.27+	.41	1.82	4.09*	1.10	2.10
2	.69	.53	4.51	.75	1.53	.01	.94	2.64	.00
2.5	2.51	1.60	.10	.07	.05	1.26	1.09	.62	.03
3	1.31	.52	13.78**	.06	.58	.13	.67	3.29*	.97
3.5	1.79	.43	1.04	.00	.05	.22	.45	2.24	.31
4	.00	.39	.44	.18	.32	.12	.09	.03	1.03
4.5	.01	4.30*	.00	.01	1.79	.03	.09	.00	1.92
5	.10	1.07	.19	.53	.00	.58	1.81	.87	.17
5.5	.35	1.27	.24	.49	.13	3.25+	.91	.01	.41
6	.06	3.80*	4.85*	1.84	.14	.30	.24	2.79+	.11
6.5	.00	.36	.09	.07	.68	3.36+	3.91*	.31	.16
7	3.08+	5.21*	.81	.06	1.90	1.51	.44	2.62	.11
7.5	.57	.00	.20	.13	.00	.03	.03	.18	.03
8	3.73+	2.92+	.94	.17	3.54+	.03	2.47	.93	.02
8.5	.01	.14	.01	.03	.08	1.41	.00	.15	.89
9	1.52	.00	.06	1.61	.22	.46	.01	.42	.00
9.5	.11	.03	1.79	.50	.87	3.70+	8.38**	7.41**	.00
10	.07	1.49	.22	.14	.53	2.88+	.15	.58	.25
> 10	9.10**	3.18+	2.35	.20	7.88**	.01	.88	.28	.02
Overall									
χ^2	43.62**	42.98**	57.62**	27.29	31.22+	33.70*	26.96	27.39	8.58

Time

Run			
1	1:20.41	1:27.48	0:58.58
2	1:20.50	1:27.09	0:58.52
3	1:20.40	1:27.52	0:58.56

Critical Chi-Squares

$\chi^2_{20}(.10) = 28.41$

$\chi^2_{20}(.05) = 31.41$

$\chi^2_{20}(.01) = 37.57$

\+ Significant at 0.10
* Significant at 0.05
** Significant at 0.01

TABLE 4

Chi-Square Values, By Interval, For Erlang Variates (λ=1,k=5), By Generator And Run n = 1,000,000

	Adrand(4)			Adrand(5)		
	1	2	3	1	2	3
0.5	15.91	8.94	17.15	2.35	8.49**	3.39+
1	1613.50	1543.56	1599.93	.96	.04	.31
1.5	365.12	360.75	335.40	.31	.51	.01
2	2744.61	2861.05	2919.89	.47	.02	.03
2.5	99.75	88.14	114.59	.30	.85	2.99+
3	3320.72	3380.38	3285.26	.94	.48	2.27
3.5	186.88	207.49	195.93	.03	.22	.82
4	518.04	614.26	521.56	.01	1.53	.02
4.5	552.07	469.71	592.52	.00	1.79	.56
5	160.96	201.55	136.95	.29	.63	1.21
5.5	1.03	.79	4.00	1.73	3.20+	.46
6	76.99	97.94	82.77	1.70	.02	.43
6.5	60.36	64.25	42.01	2.68	.05	.08
7	25.51	22.05	34.28	2.05	.05	1.20
7.5	7.05	6.66	.03	.75	3.13+	.55
8	.00	5.13	1.78	1.28	1.20	.05
8.5	.05	.03	1.43	.41	.46	.37
9	.00	.90	.94	.06	.93	1.19
9.5	6.71	6.20	1.90	.00	.06	5.76*
10	1.35	.91	.91	.26	.61	.00
>10	2.66	3.72	.24	2.92+	.01	4.94*
Overall χ^2	9798.23**	9943.75**	9889.40**	19.51	24.27	26.65

Time				
Run 1	1:37.71		1:39.65	
Run 2	1:37.73		1:39.74	
Run 3	1:37.73		1:39.60	

Critical Chi-Square

$\chi^2_{20(.10)} = 28.41$

$\chi^2_{20(.05)} = 31.41$

$\chi^2_{20(.01)} = 37.57$

+Significant at 0.10
*Significant at 0.05
**Significant at 0.01

TABLE 5

Chi Square Values, By Interval, For Erlang Variates (λ=1,k=6), By Generator And Run
n = 1,000,000

	Randu			M & M			L & L		
	1	2	3	1	2	3	1	2	3
0.5	10.00**	28.90**	14.40**	8.10**	10.00**	22.50**	.10	.40	4.90*
1	1.26	.01	13.35**	.25	.11	.08	.00	.11	.02
1.5	3.03+	12.46**	8.76**	.00	.07	1.37	1.61	.11	1.69
2	1.71	1.25	1.67	3.27+	.52	.57	2.62	1.76	.36
2.5	4.25*	.02	.86	.07	.01	.61	.03	.01	1.87
3	4.05*	.47	2.13	6.01*	1.13	.34	.00	1.87	2.91+
3.5	.01	1.69	.01	3.56+	.00	.23	.23	.49	.57
4	.61	.41	.10	.08	.06	.04	.00	2.45	1.77
4.5	.20	2.45	.47	1.47	.21	.29	3.33+	.07	.08
5	.28	1.40	.00	.93	2.04	1.22	.46	.67	.14
5.5	.36	.58	.57	1.61	1.62	.00	.23	.62	2.22
6	1.84	.11	3.27+	2.47	1.40	.78	2.70	1.93	.67
6.5	.20	.02	.17	.00	.61	.56	3.06+	.00	.59
7	3.08+	.01	1.43	.44	4.28*	.00	1.99	.72	1.48
7.5	.02	1.95	.00	.48	.04	1.75	.00	1.80	5.56*
8	.05	.71	1.33	1.21	2.20	1.37	2.67	.04	.47
8.5	.00	1.95	.38	1.14	.20	.32	4.61*	1.15	8.59**
9	3.99*	4.69*	.93	.42	.03	.42	1.66	.01	.32
9.5	.64	1.23	.56	1.10	.00	5.86*	.07	2.07	.21
10	.65	.76	.98	.73	4.14*	.00	.00	.07	1.12
10.5	.15	.29	3.28*	4.53*	.72	.25	.63	.92	.17
11	.01	.10	.44	2.78+	.00	1.80	.39	3.87*	.03
11.5	.09	4.00*	2.17	3.65+	.00	1.10	.36	.32	2.78+
12	.01	.00	2.22	3.34+	.03	.08	.03	.66	.55
12.5	.23	.28	.11	1.92	.40	.97	.94	.35	.47
13	2.64	5.07*	.01	1.64	1.89	5.57*	.25	.08	2.25
13.5	1.41	.34	2.35	1.45	.51	3.47+	.36	.01	1.73
14	2.56	3.13+	2.98+	2.98+	5.11*	.41	.28	.01	.10
14.5	3.54+	.77	.39	.01	.82	.11	.16	.25	.02
15	2.54	2.93+	1.46	.59	.15	.00	.02	.01	.50
>15	.06	.79	3.23+	6.44*	2.97+	8.83**	.09	.49	1.56
Overall χ^2	49.47*	78.79**	70.02**	62.64**	41.25+	60.91**	28.89	23.32	45.72*
Time	1:33.93	1:33.85	1:33.89	1:42.39	1:42.34	1:42.35	1:07.91	1:07.87	1:07.82

Critical Chi-Square

$\chi^2_{30\ (.10)} = 40.26$

$\chi^2_{30\ (.05)} = 43.79$

$\chi^2_{30\ (.01)} = 50.89$

\+ Significant at 0.10
* Significant at 0.05
**Significant at 0.01

TABLE 6

Chi Square Values, By Interval, For Erlang Variates (λ=1,k=6), By Generator And Run
n = 1,000,000

	Adrand(4)	Adrand(5)	Adrand(6) 1	2	3
0 .5	.10	1.60	8.10**	6.40*	8.10**
1	409.88	403.89	.00	.14	.99
1.5	715.88	987.47	1.19	.22	7.29**
2	2341.68	496.08	.61	.01	2.95+
2.5	1057.57	1060.42	.00	2.02	.09
3	585.02	1742.65	.36	.00	.32
3.5	3225.97	905.67	.00	.13	2.70
4	157.43	1040.93	.56	.67	.00
4.5	280.30	43.05	.22	1.12	2.01
5	686.89	306.73	.35	.15	2.10
5.5	365.33	323.33	.10	1.17	.00
6	26.25	29.08	1.90	.00	2.45
6.5	4.14	15.36	4.74*	.71	.34
7	9.72	52.17	.73	2.78+	.43
7.5	20.12	24.87	.21	.05	.29
8	26.21	11.57	.08	.85	6.09*
8.5	20.99	.10	1.41	.82	.46
9	12.46	.08	.01	.20	3.06+
9.5	10.26	.37	.03	.39	1.96
10	2.58	.69	2.60	.76	.06
10.5	3.48	4.59	3.30+	.20	.25
11	.04	2.72	1.01	.70	2.41
11.5	.50	4.90	2.65	.76	.09
12	.08	.04	.17	1.33	1.67
12.5	.86	1.19	.37	4.64*	.43
13	.05	.08	1.07	.00	.32
13.5	1.59	.32	1.32	5.37*	4.33*
14	.49	.02	.36	3.21+	1.14
14.5	6.42	3.44	.05	.00	.12
15	.03	1.61	.20	.03	.42
> 15	20.13	.34	.00	.04	.39
Overall χ^2	9991.42**	7465.33**	33.69	34.89	53.27**
Time	1:54.54	1:57.07	1:59.01	1:58.09	1:59.08

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