# SIMULATION METHODS FOR POISSON PROCESSES IN NONSTATIONARY SYSTEMS

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ABSTRACT

The nonhomogeneous Poisson process is a widely used model for a series of events (stochastic point process) in which the "rate" or "intensity" of occurrence of points varies, usually with time. The process has the characteristic properties that the number of points in any finite set of nonoverlapping intervals are mutually independent random variables, and that the number of points in any of these intervals has a Poisson distribution. In this paper we first discuss several general methods for simulation of the one-dimensional nonhomogeneous Poisson process; these include time-scale transformation of a homogeneous (rate one) Poisson process via the inverse of the integrated rate function, generation of the individual intervals between points, and generation of a Poisson number of order statistics from a fixed density function.

We then state a particular and very efficient method for simulation of nonhomogeneous Poisson processes with log-linear rate function. The method is based on an identity relating the nonhomogeneous Poisson process to the gap statistics from a random number of exponential random variables with suitably chosen parameters. This method can also be used, at the cost of programming complexity and some memory, as the basis for a very efficient technique for simulation of nonhomogeneous Poisson processes with more complicated rate functions such as a log-quadratic rate function.

Finally, we describe a simple and relatively efficient new method for simulation of one-dimensional and two-dimensional nonhomogeneous Poisson processes. The method is applicable for any given rate function and is based on controlled deletion of points in a Poisson process with a rate function that dominates the given rate function. In its simplest implementation, the method obviates the need for numerical integration of the rate function, for ordering of points, and for generation of Poisson variates. The thinning method is also applicable to the generation of individual intervals between points, as is required in many programs for discrete-event simulations.

## 1. INTRODUCTION

The one-dimensional nonhomogeneous (nonstationary) Poisson process (see e.g., [5, pp. 28-29; 3, pp. 94-101]) has the characteristic properties that the numbers of points in any finite set of nonoverlapping intervals are mutually independent random variables, and that the number of points in any interval has a Poisson distribution. The most general nonhomogeneous Poisson process can be defined in terms of a monotone, nondecreasing, right-continuous function $\Lambda(x)$ which is bounded in any finite interval. Then the number of points in any finite interval, for example $(0,x_0]$, has a Poisson distribution with parameter $\mu_0 = \Lambda(x_0) - \Lambda(0)$. In this paper we assume that $\Lambda(x)$ is continuous, but not necessarily absolutely continuous. The right derivative $\lambda(x)$ of $\Lambda(x)$ is the rate function for the process; $\Lambda(x)$ is called the integrated rate function and has the interpretation that for $x \geq 0$, $\Lambda(x) - \Lambda(0) = E[N(x)]$, where $N(x)$ is the total number of points in $(0,x]$. Note that $\lambda(x)$ may jump at points at which $\Lambda(x)$ is not absolutely continuous. In contrast to the homogeneous Poisson process, i.e., $\lambda(x)$ a constant (usually denoted by $\lambda$), the intervals between the points in a one-dimensional nonhomogeneous Poisson process are neither independent nor identically distributed.

Applications of the one-dimensional nonhomogeneous Poisson process include modelling of the incidence of coal-mining disasters [5], the arrivals at an intensive care unit [12], transaction processing in a data base management system [15], occurrences of major freezes in Lake Constance [23], and geomagnetic reversal data [22]. The statistical analysis of trends in a one-dimensional nonhomogeneous Poisson process, based on the assumption of an exponential polynomial rate function, is discussed by [4, 5, 12 and 15].

One-dimensional nonhomogeneous Poisson processes are often used as models for event streams when there is gross inhomogeneity in a system, e.g., time of day effect or long-term growth in use of a facility. It is important to be able to simulate these processes since analytic results are difficult to obtain. This is particularly true in the context of queueing systems; see e.g., [19]. The methods given here for simulation of the one-dimensional nonhomogeneous Poisson process have application, for example, to study of the length of a queue at a toll booth at a time corresponding to the peak traffic time, or to study of the arrivals at an intensive care unit where the probability of a bed being free at a time corresponding to the peak of arrivals from afternoon operations is of interest. Note that in simulations of nonhomogeneous systems of this kind, estimates of measures of system behavior will be based on multiple replications.

The two-dimensional homogeneous Poisson process (of rate $\lambda > 0$) is defined by the properties that the numbers of points in any finite set of nonoverlapping regions having areas in the usual geometric sense are mutually independent, and that the number of points in any region of area $A$ has a Poisson distribution with mean $\lambda A$; see e.g., [10, pp. 31-32]. Note that the number of points in a region $R$ depends on its area, but not on its shape. The homogeneous Poisson process arises as a limiting two-dimensional point process with respect to a number of limiting operations; cf., [7,8]. Properties of the process are given in [18]. Applications of the two-dimensional homogeneous Poisson process to problems in ecology and forestry have been discussed in [24] and [9]. The model also arises in connection with naval search and detection problems.

The two-dimensional nonhomogeneous Poisson process is characterized by a (continuous) positive rate function $\lambda(x,y)$. Applications of the two-dimensional nonhomogeneous Poisson process include problems in forestry as well as naval search and detection. The detection and statistical analysis of trends in the two-dimensional nonhomogeneous Poisson process is discussed in [21].

## 2. SIMULATION OF THE ONE-DIMENSIONAL NONHOMOGENEOUS POISSON PROCESS

There are a number of methods for simulating one-dimensional nonhomogeneous Poisson process which we review briefly. Time-scale transformation of a homogeneous (rate one) Poisson process via the inverse of the integrated rate function $\Lambda(x)$ constitutes a first general method; cf. [3, pp. 96-97]. This method is based on the result that $X_1, X_2, \ldots$ are the points in a nonhomogeneous Poisson process with continuous integrated rate functions $\Lambda(x)$ if and only if $X_1' = \Lambda(X_1)$, $X_2' = \Lambda(X_2), \ldots$, are the points in a homogeneous Poisson process of rate one. The time-scale transformation method is a direct analogue of the inverse probability integral transformation method for generating (continuous) nonuniform random numbers. For many rate functions, inversion of $\Lambda(x)$ is not simple and must be done numerically; cf., [6] and [20]. The resulting algorithm for simulation of the nonhomogeneous Poisson process may be far less efficient than simulation based on other methods.

A second general method for simulating a one-dimensional nonhomogeneous Poisson process with integrated rate function $\Lambda(x)$ is to generate the intervals between points individually, an approach which may seem more natural in the event scheduling approach to simulation. Thus, given the points $X_1 = x_1, X_2 = x_2, \ldots, X_i = x_i$, with $X_1 < X_2 < \cdots < X_i$, the interval to the next point, $X_{i+1} - X_i$, is independent of $x_1, \ldots, x_{i-1}$ and has distribution function $F(x) = 1 - \exp[-\{\Lambda(x_i + x) - \Lambda(x_i)\}]$. It is possible to find the inverse distribution function $F^{-1}(\cdot)$, usually numerically, and generate $X_{i+1} - X_i$ according to $X_{i+1} - X_i = F^{-1}(U_i)$, where $U_i$ is a uniform random number on the interval $(0,1)$. Note, however that this not only involves computing the inverse distribution function for each interval $X_{i+1} - X_i$, but that each distribution has different parameters and possibly a different form. An additional complication is that $X_{i+1} - X_i$ is not necessarily a proper random variable, i.e. there may be positive probability that $X_{i+1} - X_i$ is infinite. It is necessary to take this into account for each interval $X_{i+1} - X_i$ before the inverse probability integral transformation is applied. The method is therefore very inefficient with respect to speed, more so than the time-scale transformation method.

In a third method, simulation of a non-homogeneous Poisson process in a fixed interval $(0, x_0]$ can be reduced to the generation of a Poisson number of order statistics from a fixed density function by the following result; cf., [5, p. 45]. If $X_1, X_2, \ldots, X_n$ are the points of the nonhomogeneous Poisson process in $(0, x_0]$, and if $N(x_0) = n$, then conditional on having observed $n\ (> 0)$ points in $(0, x_0]$, the $X_i$ are distributed as the order statistics from a sample of size n from the distribution function $\{\Lambda(x) - \Lambda(0)\}/\{\Lambda(x_0) - \Lambda(0)\}$, defined for $0 < x \leq x_0$. Simulation of the nonhomogeneous Poisson process based on order statistics is in general more efficient (with respect to speed) than either of the previous two methods. Of course, a price is paid for this greater efficiency. First, it is necessary to be able to generate Poisson variates, and second, more memory is needed than in the interval-by-interval method to store the sequence of points. Enough memory must be provided so that with very high probability the random number of points generated in the interval can be stored. Recall that the number of points in the interval $(0, x_0]$ has a Poisson distribution with mean $\mu_0 = \Lambda(x_0) - \Lambda(0)$. Memory of size, e.g., $\mu_0 + 4\mu_0^{1/2}$ will ensure that overflow will occur on the average in only 1 out of approximately 40,000 realizations. This probability is small enough so that in case of overflow, the realization of the process generally can be discarded.

We now summarize several recently developed methods for simulating one- and two-dimensional nonhomogeneous Poisson processes. These methods are discussed in greater detail in [14, 16, 17].

## 3. SIMULATION USING GAP STATISTICS

In a previous paper [14], we have considered the simulation of nonhomgeneous Poisson processes with degree-one exponential polynomial rate function, i.e., $\lambda(x)$ of the form

$$\lambda(x) = \exp\{\gamma_0 + \gamma_1 x\} = \lambda \exp\{\gamma_1 x\},\ \gamma_1 \neq 0. \quad (1)$$

The rate function (1) is the simplest of a general family of log-linear rate functions, i.e., rate functions whose logarithms are linear in the coefficients [4; 12] which are useful in analyzing nonhomogenerous Poisson processes. The rate function (1) represents a situation in which the rate is monotonically increasing or decreasing depending on whether $\gamma_1$ is greater than or less than zero, with $\gamma_1$ equal to zero giving a homogeneous Poisson process. The case where $\gamma_1$ is less than zero and the case where $\gamma_1$ is greater than zero are quite distinct; in the first situation $\lambda(x) \to 0$ as $x \to \infty$, and in the second, $\lambda(x) \to \infty$ as $x \to \infty$. Moreover, when $\gamma_1$ is less than 0, the intervals between events are not proper random variables since there is a nonzero probability that there is no event after any fixed point $x$.

In [14] a method for simulating the nonhomogeneous Poisson process is given based on an identity relating the nonhomogeneous Poisson process with rate function (1) to the gap statistics from a random number of exponential random variables with suitably chosen parameters. This method avoids costly ordering and taking of logarithms required by direct simulation methods and is more efficient than time-scale transformation of a homogeneous Poisson process via the inverse of the integrated rate function $\Lambda(x)$.

Simulation of the one-dimensional nonhomogeneous Poisson process in a fixed interval $(0,x_0]$ is more natural than simulation for a fixed number of events since time, not serial number, is the basic parameter of the inhomogeneity discussed here. Thus the gap statistics algorithm for simulation of the nonhomogeneous Poisson process generates the sequence of times-to-events in a fixed interval. Although such a method requires more memory than successive generation of individual times until the next event, it is far more efficient.

We now state the method of Lewis and Shedler [14] for simulation via gap statistics of the one-dimensional nonhomogeneous Poisson process with rate function (1). This scheme, which is particular to the degree-one exponential polynomial rate function, can use standard packages for exponential random numbers (e.g., [13]) and obviates the need for ordering of the random numbers. It is based on the result (see [25]) that the gap process associated with a Poisson distributed (parameter $-\lambda/\gamma_1 > 0$) number of exponential (parameter $\beta = -\gamma_1 > 0$) gap statistics is a nonhomogeneous Poisson process with rate function $\lambda(x) = \lambda \exp(\gamma_1 x)$ on $(0,\infty)$. Efficient methods for generation of Poisson random numbers for which the generation time does not increase proportionally with the mean are given by [1,2] and [11].

Assuming the availability of a source of unit exponential random numbers $E_1, E_2, \ldots$, obtained by logarithms or by other methods, the resulting algorithm for generating the events in the nonhomogeneous Poisson process is as follows.

Algorithm 1. Gap Statistics Technique ($\gamma_1 < 0$).

1. Generate $m$ as a Poisson random number with parameter $-\lambda/\gamma_1$. If $m = 0$, exit; there are no events in $(0,x_0]$.
2. For $m > 0$, if $E_1/(\beta m)$ is greater than $x_0$, exit; there are no events in $(0,x_0]$. Otherwise, set $E_1/(\beta m)$ equal to $X_1$.

3. If $E_2/\{\beta(m-1)\} + X_1 > x_0$, then return $X_1$ and exit. Otherwise, set it equal to $X_2$.
4. Continue, possibly for $m$ times. If $E_m/\beta + X_{m-1} > x_0$, return $X_1, X_2, \ldots, X_{m-1}$ and exit. Otherwise, set this equal to $X_m$, return $X_1, X_2, \ldots, X_m$ and exit.

The case $\gamma_1 > 0$ is handled in the same way as $\gamma_1 < 0$ by using a time-reversal technique, as follows. Simulate according to Algorithm 1 with $\lambda(x) = \lambda^{\#} \times \exp\{\gamma_1^{\#} x\}$, where $\lambda^{\#} = \exp\{\gamma_0 + \gamma_1 x_0\}$ and $\gamma_1^{\#} = -\gamma_1$. The output of Algorithm 1 is a sequence $X_1^{\#}, X_2^{\#}, \ldots, X_n^{\#}$. Then set $X_1 = x_0 - X_n^{\#}$, $X_2 = x_0 - X_{n-1}^{\#}, \ldots, X_n = x_0 - X_1^{\#}$. These $X_i$ are the required events in the nonhomogeneous Poisson process for $\gamma_1 > 0$.

Lewis and Shedler [16] consider the simulation in a fixed interval of the nonhomogeneous Poisson process with degree-two exponential polynomial rate function

$$\lambda(x) = \exp\{\alpha_0 + \alpha_1 x + \alpha_2 x^2\}, \quad \alpha_2 \neq 0 \qquad (2)$$

the case $\alpha_2 = 0$ giving the degree-one exponential polynomial rate function. Again, the case where $\alpha_2$ is less than zero and the case where $\alpha_2$ is greater than zero are distinct. The simulation method given is based on representation of the process as a superposition of two independent nonhomogeneous Poisson processes, one of which has a fitted rate function of the form (1); simulation of the latter process is accomplished via the gap statistics algorithm [14]. A rejection-acceptance technique is used to generate the other, more complex, nonhomogeneous Poisson process. The resulting algorithm is more efficient than time-scale transformation of a homogeneous Poisson process; see [20]. This method can be improved by using the thinning algorithm given in the next section to simulate the second nonhomogeneous Poisson process. The method can also be extended to more complex rate functions than the degree-two exponential polynomial.

## 4. SIMULATION OF NONHOMOGENEOUS POISSON PROCESSES BY THINNING

In this section we describe a new method [17] for simulating a nonhomogeneous Poisson process. The method is not only conceptually simple, but is also computationally simple and relatively efficient. In fact, at the cost of some efficiency, the method can be applied to simulate the given nonhomogeneous Poisson process <u>without the need for numerical integration or routines for generating Poisson variates</u>. Used in conjunction with the special methods given by Lewis and Shedler [14,16], the method can be used to simulate quite efficiently nonhomogeneous Poisson processes with rather complex rate function, in particular, combinations of long-term trends and fixed-cycle effects. The method is also easily extended to the problem of simulating the two-dimensional nonhomogeneous Poisson process (see Section 5), and of simulating conditional and doubly stochastic Poisson processes.

Simulation of a nonhomogeneous Poisson process with general rate function $\lambda(x)$ in a fixed interval $(0, x_0]$ can be based on thinning of a nonhomogeneous Poisson process with rate function $\lambda^*(x) \geq \lambda(x)$. The main result [17, Theorem 1] is that if $X_1^*, X_2^*, \ldots, X_{N^*(x_0)}^*$ are the points of the process with rate function $\lambda^*(x)$ in the interval $(0, x_0]$ and if the point $X_i^*$ is deleted with (independent) probability $1 - \lambda(X_i^*)/\lambda^*(X_i^*)$, then the remaining points form a nonhomogeneous Poisson process with rate function $\lambda(x)$ in the interval $(0, x_0]$.

This result is the basis for the method of simulating one-dimensional nonhomogeneous Poisson processes on an interval $(0, x_0]$ given by Algorithm 2.

<u>Algorithm 2</u>. One-Dimensional Nonhomogeneous Poisson Process (Thinning)

1. Generate points in the nonhomogeneous Poisson process $\{N^*(x)\}$ with rate function $\lambda^*(x)$ in the fixed interval $(0,x_0]$. If the number of points generated, $n^*$, is such that $n^* = 0$, exit; there are no points in the process $\{N(x)\}$.
2. Denote the (ordered) points by $X_1^*, X_2^*, \ldots, X_{n^*}^*$. Set $i = 1$ and $k = 0$.
3. Generate $U_i$, uniformly distributed between 0 and 1. If $U_i \le \lambda(X_i^*)/\lambda^*(X_i^*)$, set $k$ equal to $k+1$ and $X_k = X_i^*$.
4. Set $i$ equal to $i+1$. If $i \le n^*$, go to 3.
5. Return $X_1, X_2, \ldots, X_n$, where $n = k$, and also $n$.

The method of thinning of Algorithm 2 is essentially the obverse of the conditional method of Section 2 using conditioning and acceptance-rejection techniques to generate the random variables with density function $\lambda(x)/\{\Lambda(x) - \Lambda(0)\}$ [16, Algorithm 3]. The differences are subtle, but computationally important. In the acceptance-rejection method, it is first necessary to generate a Poisson variate with mean $\mu_0 = \Lambda(x_0) - \Lambda(0)$, and this involves an integration of the rate function $\lambda(x)$. Then the Poisson$(\mu_0)$ number, $n$, of variates generated by acceptance-rejection must be ordered to give $X_1, X_2, \ldots, X_n$.

In the simplest form of the method of thinning, $\lambda^*(x)$ is taken equal to $\lambda^* = \max_{0 \le x \le x_0} \lambda(x)$, so that, for instance the points $X_1^*, X_2^*, \ldots, X_{n^*}^*$ can be generated by cumulating exponential$(\lambda^*)$ variates until the sum is greater than $x_0$ (cf. [14, Algorithm 1]. The generated points are then thinned. <u>No ordering, no integration of $\lambda(x)$ and no generator of Poisson variates is required</u>. Of course for both algorithms to be efficient, computation of $\lambda(x)$ and $\lambda^*(x)$ must be easy relative to computation of the inverse of $\Lambda(x)$.

For the thinning algorithm (as well as the algorithm based on conditioning and acceptance-rejection) efficiency as measured by the number of points deleted is proportional to

$$\mu_0/\mu_0^* = \{\Lambda(x_0) - \Lambda(0)\}/\{\Lambda^*(x_0) - \Lambda^*(0)\};$$

this is the ratio of the areas between 0 and $x_0$ under $\lambda(x)$ and $\lambda^*(x)$. Thus, $\lambda^*(x)$ should be as close as possible to $\lambda(x)$ consistent with ease of simulating the nonhomogeneous Poisson process $\{N^*(x): x \ge 0\}$.

It is important to note that the method of thinning can be used to generate individual intervals between events occurring in $(0,x_0]$ if $\lambda(x)$ is bounded on $(0,x_0]$. The resulting algorithm is not only useful in the event scheduling approach to simulation, but also in the generation of conditional Poisson processes. Informally, the "one at a time" thinning algorithm is as follows. If $\lambda^*(x) = \lambda^* \ge \max_{0 \le x \le x_0} \lambda(x)$, then the ith interval $X_i - X_{i-1}$ is obtained by generating and cumulating exponential$(\lambda^*)$ random numbers $E_{i,1}^*, E_{i,2}^*, \ldots$ until, <u>for the first time</u>,

$$U_{i,j} \le \lambda(X_{i-1} + E_{i,1}^* + \cdots + E_{i,j}^*)/\lambda^*,$$

$$i = 1,2,\ldots; \quad j = 1,2,\ldots,$$

where the $U_{i,j}$ are independent, uniform (0,1) random numbers.

## 5. <u>SIMULATION OF TWO-DIMENSIONAL HOMOGENEOUS POISSON PROCESSES</u>

Recall that the two-dimensional homogeneous Poisson process (of rate $\lambda > 0$) has the characteristic properties that the numbers of points in any finite set of nonoverlapping regions having areas in the usual geometric sense are mutually independent, and that the number of points in any region of area $A$ has a Poisson distribution with mean $\lambda A$.

In considering the two-dimensional homogeneous Poisson process, projection properties of the process depend quite critically on the geometry of the regions considered. These projection properties are simple for rectangular and circular regions, and make simulation of the homogeneous process quite easy. We consider here the case of a rectangular region. The following result forms the basis for simulation of the two-dimensional homogeneous Poisson process of rate $\lambda$ in a fixed rectangle $R = \{(x,y): 0 \leq x \leq x_0, 0 < y \leq y_0\}$. If $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_N,Y_N)$ denote the positions of the points of the process in $R$, labelled so that $X_1 < X_2 < \cdots$, then $X_1,X_2,\ldots,X_N$ form a one-dimensional homogeneous Poisson process on $0 \leq x \leq x_0$ of rate $\lambda y_0$; if the points are relabelled $(X_1',Y_1')$, $(X_2',Y_2')$, ... , $(X_N',Y_N')$ so that $Y_1' < Y_2' <\cdots< Y_N'$, then $Y_1',Y_2',\ldots,Y_N'$ form a one-dimensional homogeneous Poisson process on $0 < y \leq y_0$ of rate $\lambda x_0$.

We state next conditional properties of the Poisson process in a rectangle. The important thing to note is that although the processes obtained by projection of the points onto the $x$ and $y$ axes are not independent, there is a type of conditional independence which makes it easy to simulate the two-dimensional process. Thus, conditional on having observed $n > 0$ points $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_n,Y_n)$ in $R$, labelled so that $X_1 < X_2 <\cdots< X_n$, the $X_1,X_2,\ldots,X_n$ are uniform order statistics on $0 \leq x \leq x_0$, and the $Y_1,Y_2,\ldots,Y_n$ are independent and uniformly distributed on $0 < y \leq y_0$, independent of the $X_i$.

From these two results, the following simulation procedure is obtained.

Algorithm 3. Two-Dimensional Homogeneous Poisson Process in a Rectangle

1. Generate points in the one-dimensional homogeneous Poisson process of rate $\lambda y_0$ on $(0,x_0]$. If the number of points generated, $n$, is such that $n=0$, exit; there are no points in the rectangle.
2. Denote the points generated by $X_1 < X_2 <\cdots< X_n$.
3. Generate $Y_1,Y_2,\ldots,Y_n$ independent, uniformly distributed random numbers on $(0,y_0]$.
4. Return $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ as the coordinates of the two-dimensional homogeneous Poisson process in the rectangle, and $n$.

Note that generation of the points $X_1,X_2,\ldots,X_n$ in steps 1 and 2 can be accomplished by cumulating exponential$(\lambda y_0)$ random numbers. Alternatively, after generating a Poisson random number $N = n$ (with parameter $\lambda x_0 y_0$), $n$ independent, uniformly distributed random numbers on $(0,x_0]$ can be ordered; see [14, p. 502].

The basis for another algorithm for simulation of the two-dimensional homogeneous Poisson process in a rectangle is the following corollary. Denote the Poisson points by $(X_1,Y_1),(X_2,Y_2),\ldots$, where the index does not necessarily denote an ordering on either axis. Conditionally, the pairs $(X_1,Y_1)$, $(X_2,Y_2),\ldots,(X_N,Y_N)$ are independent random variables, and furthermore, for each pair $(X_i,Y_i)$, $X_i$ is distributed uniformly between 0 and $x_0$, independently of $Y_i$, which is uniformly distributed between 0 and $y_0$.

Direct generation of homogeneous Poisson points in noncircular or nonrectangular regions is difficult. The processes obtained by projection of the points on the two axes are nonhomogeneous Poisson processes with complex rate functions determined by the geometry of the region. However, the conditional independence which is found in circular and rectangular regions for the processes on the two axes is not present. In particular, given that there are $n$ points $(X_1,Y_1),\ldots,(X_n,Y_n)$ in a nonrectangular region, the pairs $(X_i,Y_i)$ are mutually independent, but $X_i$ is in general not independent of $Y_i$, $i=1,2,\ldots,n$. Therefore it is simpler to enclose the region in either a circle or a rectangle, generate a homogeneous Poisson process in the enlarged area, and subsequently exclude points outside of the given region.

## 6. SIMULATION OF TWO-DIMENSIONAL NONHOMOGENEOUS POISSON PROCESSES

The two-dimensional nonhomogeneous Poisson process $\{N(x,y):x\geq 0,\ y\geq 0\}$ is specified by a positive rate function $\lambda(x,y)$ which for simplicity is assumed here to be continuous. Then the process $\{N(x,y)\}$ has the characteristic properties that the numbers of points in any finite set of nonoverlapping regions having areas in the usual geometric sense are mutually independent, and that the number of points in any such region $R$ has a Poisson distribution with mean $\Lambda(R)$; here $\Lambda(R)$ denotes the integral of $\lambda(x,y)$ over $R$, i.e., over the entire area of $R$.

The basic result of Section 4 on thinning of one-dimensional nonhomogeneous Poisson processes generalizes to two-dimensional nonhomogeneous Poisson processes. Thus, suppose that $\lambda(x,y)\leq\lambda^*(x,y)$ in a fixed rectangular region of the plane. If a nonhomogeneous Poisson process with rate function $\lambda^*(x,y)$ is thinned according to $\lambda(x,y)/\lambda^*(x,y)$ (i.e., each point $(X_i,Y_i)$ is deleted independently if a uniform (0,1) random number $U_i$ is greater than $\lambda(X_i,Y_i)/\lambda^*(X_i,Y_i)$), the result is nonhomogeneous Poisson process with rate function $\lambda(x,y)$.

The nonhomogeneous Poisson process with rate function $\lambda(x,y)$ in an arbitrary but fixed region $R$ can be generated by enclosing the region $R$ either in a circle [17] or a rectangle and applying Algorithm 3. The following procedure assumes that the region $R$ has been enclosed in a rectangle $R^*$, and that $\lambda^*=\max\{\lambda(x,y):x,\ y\in R\}$ has been determined; here the bounding process is homogeneous with rate $\lambda^*$ in the rectangle $R^*$.

<u>Algorithm 4.</u> Two-Dimensional Nonhomogeneous Poisson Process (Thinning).

1. Using Algorithm 2, generate points in the homogeneous Poisson process of rate $\lambda^*$ in the rectangle $R^*$. If the number of points, $n^*$, is such that $n^*=0$, exit; there are no points in the nonhomogeneous Poisson process.
2. From the $n^*$ points generated in 1, delete the points that are not in $R$, and denote the remaining points by $(X_1^*,Y_1^*),\ (X_2^*,Y_2^*),\ldots,\ (X_m^*,Y_m^*)$ with $X_1^*<X_2^*<\cdots<X_m^*$. Set $i=1$ and $k=0$.
3. Generate $U_i$ uniformly distributed between 0 and 1. If $U_i\leq\lambda(X_i^*,Y_i^*)/\lambda^*$, set $k=k+1$, $X_k=X_i^*$ and $Y_k=Y_i^*$.
4. Set $i$ equal to $i+1$. If $i\leq m^*$, go to 3.
5. Return $(X_1,Y_1),\ (X_2,Y_2),\ldots,\ (X_n,Y_n)$, where $n=k$ and $n$.

## 7. CONCLUSION

We have summarized previously known general methods for simulating nonhomogeneous Poisson processes in one dimension. In addition, we have described the simple and efficient new methods of Lewis and Shedler for simulating nonhomogeneous Poisson processes in one and two dimensions. Extensions of the thinning algorithm to the simulation of homogeneous or nonhomogeneous conditional or doubly stochastic Poisson processes will be described elsewhere.

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