EFFECTIVE REPRESENTATION OF NON-NORMAL FACTOR VARIABILITY IN RISK ANALYSIS SIMULATION: A COMPARATIVE STUDY

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Introduction

Conventional methods of analyzing investment decisions do not adequately measure investment risk because they typically compute investment return from a single expected value for the investment factors. The risk analysis method of Monte Carlo simulation provides a more accurate measure of variability in investment return because it includes a range of factor probabilities in computing the return and because it determines the approximate probability distribution of expected rates of return.

The probability distributions used to represent factor variability are usually assumed to be normally distributed. This simplifying assumption prohibits any non-normal factor variability that might influence the variability of investment return or risk. The beta distribution can be used to represent non-normal factor variability. However, the beta distribution requires restrictions on possible outcome estimates and is not easily programmed into a simulation model. The triangular distribution can be used to represent both normal and non-normal factor variability. Management can readily provide estimates for the distribution, and it is easily programmed for use in an investment simulation model. The basic purpose of this is to determine the effectiveness of the triangular distribution in representing non-normal investment factor variability.

Probability Distributions

Theoretically, the Monte Carlo method is a useful technique for analyzing risk in a capital investment decision. However, the technique involves obtaining subjective probability distributions for a number of variables. To develop these probability distributions, the decision maker must estimate the possible range of values for each factor, the average, and some idea as to the likelihood that the various possible values will be realized (1), all on the basis of subjective information.

There are many possible distributions that could be used to estimate the subjective probability distributions. However, in selecting the distribution, four criteria should be met. First, the distribution's shape should be flexible. There is no reason to assume that the probabilities will always follow the same pattern. The parameters of the distribution must allow for shifting of central tendency to produce changes in symmetry or skewness. Second, the conditional probability of occurrence of an event during the next small time interval, dt, given that it has not occurred by time t, should be an ever-increasing quantity.

For the third criteria, the distribution should have a discrete range. This is a matter of practical convenience since it is a necessary restriction if there is to be a limit to the number of possible outcomes requiring a probability assignment. Lastly, the distribution should be easy to use. This is necessary if management is to feel comfortable and confident that the results of the analysis are realistic. (2)

Most studies of Monte Carlo risk analysis have assumed a normal type of distribution because they considered forecasting errors to be normally distributed. However, in Allen's study (3),* separate simulations of a hypothetical investment were performed using both the normal distribution and the beta distribution.

These distributions are defined by varying the parameters of the functions. The normal distribution is symmetrical, and different standard deviations change the degree of peakedness for a given mean. Its equation is: (4)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e - (x-m)^2/2\sigma^2 \qquad (1)$$

The shape of the normal distribution is not flexible. There is no way to represent skewness with the normal distribution.

However, the beta distribution has parameters to change both peakedness and skewness. Its equation is: (5)

$$f(x) = \frac{(a + b + 1)}{a!\ b!} x^a(1 - x)^b \qquad (2)$$

With the beta distribution, an extremely vital requirement is that the spread between the pessimistic and optimistic estimator should represent six standard deviations for the actual distribution. Management, therefore, must ignore extremely unlikely windfalls or catastrophic events. (6)

This study employs a third distribution in the simulation of a hypothetical investment: the triangular distribution. This distribution provides a simple and understandable means of portraying the manager's feeling of uncertainty about the variable he is estimating, and it meets the four criteria for the selection of a probability distribution. To use the triangular distribution, the manager need only provide a "most likely" estimate (M), a "most optimistic" estimate (O), and a "most pessimistic" estimate (P). Figure 1 illustrates the triangular distribution.**(7)

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*See also Allen, William B. and Brewerton, F. J., "Variability Assumptions and Their Effect on Capital Investment Risk," 1973 Winter Simulation Conference Proceedings, January, 1973, San Francisco, pp. 481-496.

**For a discussion of the triangular distribution, see Brewerton, F. J., and Gober, R. Wayne, "Management Science Applications of the Triangular Distribution: Some Pros and Cons," Proceedings of Midwest AIDS Conference, April, 1975, Indianapolis, Indiana, pp. 428-431.

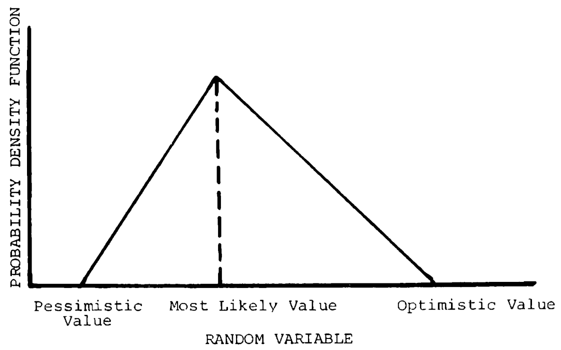

Figure 1

PROBABILITY DENSITY FUNCTION FOR TRIANGULAR DISTRIBUTION

Use of the triangular distribution is convenient mathematically and provides an easily manipulated source of random samples. The equation is:

$$f(x) = \frac{2X}{MO} \qquad P \leq X \leq M \qquad (3)$$

$$f(x) = \frac{-2X}{O\ (O - M)} + \frac{2}{O - M} \qquad M \leq X \leq O \qquad (4)$$

To perform the investment simulation, the triangular density function must be transformed into a cumulative distribution function. Given a random value (RN) from a 0 - 1 uniformly distributed population, the inverse of the cumulative distribution function is then used to compute a corresponding value from the triangular distribution. This involves a two-step sequence: (8)

1. Determine which segment of the inverse function to use. (There are two segments because of the discontinuity in the density function.) The segment is determined by comparing the 0 - 1 random variable to the ratio (M - P)/(O - P). If the random variable is less than this ratio, the first segment is used; otherwise, the second segment is used.

2. Having determined the segment in effect, evaluate the corresponding inverse cumulative function using the 0 - 1 random variable as the argument. The computed value is then used as a sample value.

The inverse cumulative functions which serve as process generators are:

$$P + SQRT\ ((M - P)*(O - P)*RN) \qquad P \text{ to } M \qquad (5)$$

$$O - SQRT\ ((O - M)*(O - P)*(1 - RN)) \qquad M \text{ to } O \qquad (6)$$

Use of the triangular distribution to describe management's expectations of the future is extremely simple. Only the three parameters mentioned need be supplied by management to completely define the probability distribution. In addition, the process generators are easily programmed, which facilitates the implementation of the capital investment model.

The Investment Model

The investment model used in this study to evaluate the triangular distribution is a replicate of the one used in Allen's study, and is shown schematically in Figure 2. It is constructed to reveal the risk involved in capital investment decisions. The model includes 11 investment factors which have a significant effect on the risk of achieving expected after tax profitability. For the purpose of this study, the probability distributions of these factors are assumed to be triangularly distributed.

The model is designed to represent reasonably realistic business conditions but not to parallel any particular capital investment. Important elements of the model are the rate of return function, the basic investment factors of the return function, the interrelationships between these factors, the estimated numerical values for the factors, and the type of variability in the factors.

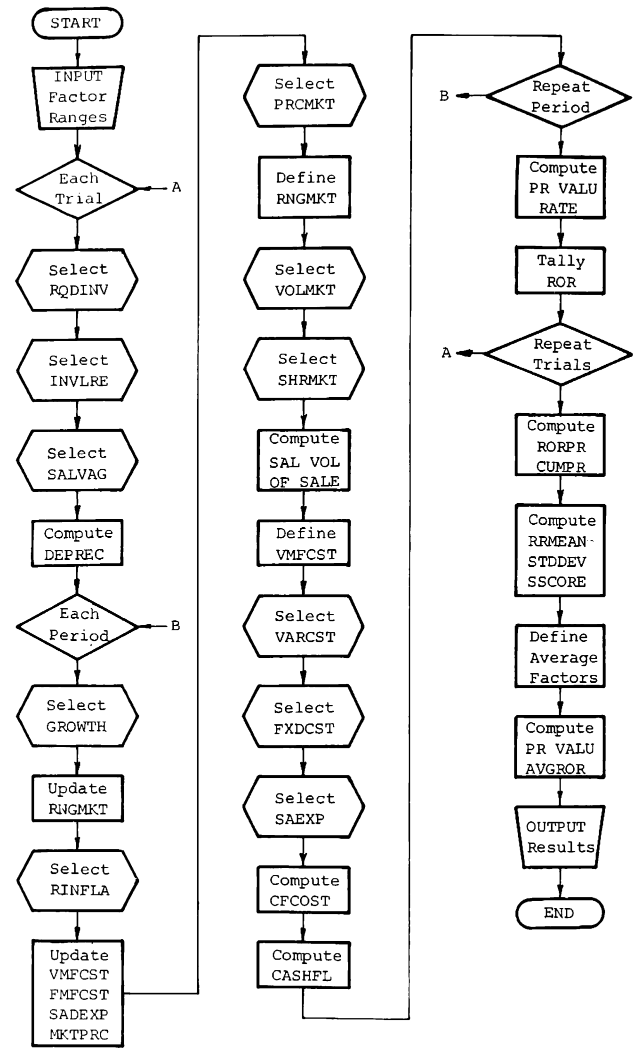

Figure 2

PROGRAM FLOW DIAGRAM

The rate of return function determines the interest rate which equates the sum of the present value of future period net cash flows to the amount of the initial investment. The net cash flow consists of the sum of cash inflows less cash outflows. Cash inflows are the sales revenues each period. Cash outflows are the costs of owning and operating the investment each period as well as taxes on period profits. Taxes are computed using straight time depreciation.

The simulation model utilizes 11 investment factors. These factors are divided into four groups--(1) marketing factors, (2) production factors, (3) investment factors, and (4) dynamic factors. The marketing factors include the dollar sales price of the investment product, the number of product units of total industry sales, and the firm's percentage share of the total industry sales. The production factors are the variable manufacturing costs in dollars per unit, the dollar amount of fixed manufacturing costs, and the dollar amount of selling and administrative expenses. The investment factors are the dollar amount of investment required for starting production, the useful production life of the investment, and the residual value of the investment at the end of its useful life. The dynamic factors include the percentage rate of change in market sales volume and the inflation rate in product prices and production costs and expenses.

Allen's model recognizes two sets of functionally related investment factors, market price-sales volume and sales volume-variable manufacturing costs. The interrelationship between these correlated factors is accomplished by utilizing multiple subjective probability estimates. For each possible value of one variable, there is a range of possible values for the other correlated variable. These estimates represent each factor as statistically independent although the variables themselves are functionally correlated. Each time period of the investment life is also interrelated so that the rates of growth and inflation determine a new range for the probability distribution of each investment factor.

The numerical values for the range estimator for each of the 11 investment factors are arbitrary. The objective of the model is to produce a measurable rate of return and to reveal the effect of factor variability on rate of return variability.

To demonstrate the sensitivity of rate of return variability to the type of factor probability distribution, comparative simulations were performed using five different triangular distributions and five similarly shaped variations of the normal and beta distributions. The triangular variations included a symmetrical triangular, a peaked symmetrical triangular, a flat symmetrical triangular, a left-skewed triangular, and a right-skewed triangular.

Simulation of the investment model uses the unit interval as a point of reference to define these distributions. The first three triangular distributions had a common most likely value (M) of .50. The values of P and O were computed using ± .40 for the symmetrical triangular, ± .30 for the peaked triangular, and ± .60 for the flat triangular distribution. The most likely value was .25 for the right-skewed distribution. The selection of these parameters was intended to allow comparison between the triangular distribution and the normal or beta distributions.

The simulation model is restricted to decisions involving the investment in initial production facilities. The investment consists of a single cash outlay, which produces various revenues at varying costs over a future useful life. The model does not consider any time-lags--an assumption is made that any time-lags are constant over the life of the investment. With this assumption, the investment begins production simultaneously with the investment, and sales occur at the time and rate of production.

Results

Table 1 shows the calculated values for statistical measures for the five rate of return distributions obtained in this simulation and the five distributions utilized in Allen's study. The distributions for the symmetrical, peaked, and flat variability simulations have approximately similar means (152, 153, 144). Their standard deviations bear a relationship corresponding to their assumed type of variability. The standard deviation for the distribution using flat variability is larger than the standard deviation using symmetrical variability (44:30). The distributions from the skewed simulations have means (115:200) that are considerably different from the means for the symmetrical, peaked, and flat variability simulations. The triangular distributions were able to generate rate of return distributions that were similar to the normal and beta distributions.

Referring to Table 2, the distribution for the five simulations varied over the range of returns according to the type of investment factor variability assumed during the simulation.

Table 1

Statistical measures for the distributions of simulated rates of return

Assumed type of variability	Mean	Standard deviation
Symmetrical Triangular	152.5	30.07
Standard Normal	151.5	29.04
Peaked Triangular	153.2	23.91
Peaked Normal	154.5	23.84
Flat Triangular	144.3	44.54
Flat Normal	148.1	37.69
Left Skewed Triangular	204.5	19.51
Left Skewed Beta	195.1	30.39
Right Skewed Triangular	115.8	32.52
Right Skewed Beta	95.4	33.05

Table 2

Rate of return frequencies for the simulation assuming triangular ractor variability

Rate of Return Intervals	Symmetrical	Peaked	Flat	Left Skewed	Right Skewed
0 or less	0	0	2	0	0
1 to 10	0	0	1	0	0
11 to 20	0	0	1	0	0
21 to 30	0	0	4	0	0
31 to 40	0	0	2	0	4
41 to 50	0	0	8	0	7
51 to 60	0	0	12	0	13
61 to 70	0	0	15	0	40
71 to 80	2	0	31	0	77
81 to 90	10	2	28	0	115
91 to 100	21	7	50	0	114
101 to 110	59	28	68	0	109
111 to 120	66	56	86	0	114
121 to 130	101	104	97	0	109
131 to 140	117	117	95	0	95
141 to 150	119	162	84	0	60
151 to 160	135	170	81	5	53
161 to 170	103	134	60	22	36
171 to 180	95	97	69	87	23
181 to 190	67	57	51	170	12
191 to 200	38	44	46	188	9
201 to 210	39	15	38	189	5
211 to 220	13	6	28	140	3
221 to 230	10	1	17	85	1
231 to 240	2	0	10	76	1
241 to 250	3	0	10	29	0
251 to 260	0	0	2	7	0
261 to 270	0	0	0	2	0
Over 270	0	0	4	0	0
Total Trials	1000	1000	1000	1000	1000

The simulation assuming symmetrical variability produced a distribution that was approximately symmetrical over the range of returns. The distribution for the peaked triangular was also symmetrical but more peaked than the symmetrical triangular. The distribution for the flat triangular resulted in a distribution that was also symmetrical but less peaked than the symmetrical triangular. The peakedness of the left-skewed distribution was greater than the symmetrical triangular and was somewhat skewed to the left. The peakedness of the right-skewed distribution was similar to the symmetrical triangular and was skewed to the right. The frequency distributions were quite similar to the normal and beta distributions used in Allen's study (Table 3).

Summary

In summary, the results of the Monte Carlo simulation of the hypothetical investment proposal indicate the

Table 3

Rate of return frequencies assuming normal, peaked normal, flat normal, left-skewed, and right-skewed factor variability

Rate of Return Intervals	Normal	Peaked Normal	Flat Normal	Left Skewed	Right Skewed
0 or less	0	0	0	0	0
1 to 10	0	0	0	0	0
11 to 20	0	0	0	0	3
21 to 30	0	0	0	0	9
31 to 40	0	0	0	0	18
41 to 50	0	0	2	0	37
51 to 60	0	0	3	0	80
61 to 70	0	0	4	0	102
71 to 80	2	0	14	0	113
81 to 90	10	0	31	0	131
91 to 100	24	6	52	1	132
101 to 110	41	21	63	3	97
111 to 120	72	46	81	2	82
121 to 130	100	92	101	15	61
131 to 140	119	145	98	25	37
141 to 150	129	155	107	34	29
151 to 160	126	160	114	62	23
161 to 170	130	140	78	73	25
171 to 180	88	00	65	105	10
181 to 190	61	66	44	128	5
191 to 200	42	33	38	136	4
201 to 210	27	30	41	120	2
211 to 220	15	5	30	91	0
221 to 230	9	2	17	79	0
231 to 240	4	0	11	60	0
241 to 250	1	0	6	36	0
251 to 260	0	0	0	22	0
261 to 270	0	0	0	6	0
Over 270	0	0	0	2	0
Total Trials	1000	1000	1000	1000	1000

Source: Monte Carlo Analysis of Risk in Capital Investment decisions, pp. 49-51.

triangular distribution can be used to represent both normal and non-normal variability. The peaked, flat, and skewed triangular variability give rates of return that were significantly different from the distribution of return from the symmetrical triangular simulation. These distributions also reflect the type of variability distribution assumed for the factors. The use of the triangular distribution provides a simple, easy to use approach to probabilistic financial planning.

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