THE EXPONENTIAL MARKOV STOCHASTIC MODEL

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ABSTRACT

This paper presents a system stochastic model described by Kolmogorov's differential equations which they lead in the steady state to the system balance equations. Applications of this model to some engineering simulation situations are briefly treated.

I. INTRODUCTION

The exponential stochastic model can be described mathematically by stationary Markov process, $\{x(t);\ t \geq 0\}$, where $x(t)$ is the state of the process which will be assumed discrete and range over the positive integers. Actually, $x(t)$ is the number of events that occur in the interval $(0, t)$. The time duration spent by the process in any state before moving into another state is exponentially distributed random variable with mean $1/\lambda$. The static transition of the process from state i at time 0 into state j at time $t + s$; $s, t \geq 0$, is described by the stationary Chapman-Kolmogorov equations:

$$P_{ij}(t + s) = \sum_k P_{ik}(s)\, P_{kj}(t), \text{ all } i \text{ and } j \qquad (1.1)$$

where k is an intermediate state in the path from state i to state j and $P_{ij}(t)$ is the transition probability $P\{x(t) = j \mid x(0) = i\}$ which satisfies the following conditions:

$$0 \leq P_{ij}(t) \leq 1 \quad , \quad \sum_j P_{ij}(t) = 1$$

$$\lim_{t\to 0} P_{ij}(t) = 1, \quad P_{ij}(t) = e^{-\lambda t}\, \frac{(\lambda t)^{j-i}}{(j-i)!} \qquad (1.2)$$

Also, $P_{0j}(t)$ will be denoted by $P_j(t)$ and

$$P_j(0) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{cases} \qquad (1.3)$$

II. THE STOCHASTIC MODEL

The dynamic behavior of the transition is described by infinitesimal transition scheme in a very small time Δt and is represented by the following two differential equations:

$$\dot{P}_{ij}(t) = -q_j P_{ij}(t) + \sum_{k \neq j} q_{kj} P_{ik}(t); \; i, j = 0, 1, \ldots n \qquad (2.1)$$

$$\dot{P}_{ij}(t) = -q_i P_{ij}(t) + \sum_{k \neq i} q_{ij} P_{kj}(t); \; i, j = 0, 1, \ldots n \qquad (2.2)$$

The initial conditions for both equations are:

$$P_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad (2.3)$$

Equation (2.1) for fixed i and equation (2.2) for fixed j are called forward and backward Kolmogorov equations respectively. The quantities q_i and q_{ij}, $i, j = 1, 2, \ldots n$, are the transition rates of the process, and they are related by:

$$\sum_{j \neq i} q_{ij} = q_i \quad , \quad q_{ii} = -q_i \qquad (2.4)$$

The matrix forms of (1.4) and (1.5) are given respectively by:

$$\dot{P}(t) = P(t)\, Q \;,\; P(0) = I \qquad (2.1)'$$

$$\dot{P}(t) = Q\, P(t) \;,\; P(0) = I \qquad (2.2)'$$

Where $P(t)$ is an $(n + 1) \times (n + 1)$ probability transition matrix whose ij entry is $P_{ij}(t)$, Q is the transition rate matrix whose ij entry is q_{ij} and I is the identity matrix. The solution of (2.1)' and (2.2)' is given by:

$$P(t) = e^{Qt} \qquad (2.5)$$

The matrix function e^{Qt} is given by the series

$$e^{Qt} = \sum_{k=0}^{\infty} Q^k \frac{t^k}{k!} \quad , \quad Q^0 = I \qquad (2.6)$$

Although the analytic solution of Kolmogorov equations appears simple, it is very difficult to compute the matrix function e^{Qt}, especially when the number of states ia large. However, a mathematically tractable situation can be achieved in the steady state under the condition of process irreducibility. In this case, the following limit:

$$\lim_{t\to\infty} P_{ij}(t) \to P_j \qquad (2.7)$$

exists and independent of the starting state i. Therefore, the derivative $\dot{P}_{ij}(t)$ vanishes and (2.1) becomes:

$$0 = -q_j P_j + \sum_{k \neq j} q_{kj} P_k, \quad j = 1, 2, \ldots n \qquad (2.8)$$

The above equations may be written in the form:

$$q_j P_j = \sum_{k \neq j} q_{kj} P_k; \quad j = 1, 2 \ldots. \qquad \sum_j P_j = 1 \qquad (2.9)$$

The equations (2.9) are called the balance equations since they balance or equate the rate at which the process enters state j with the rate at which it leaves this state.

III. APPLICATIONS

Two main applications will be presented briefly in this section to demonstrate the potential of the above model in the simulation of some engineering systems. First, the general birth and death process, which is the best representation of congestion processes, will be presented. Then, a basic reliability model for engineering system efficiency will be demonstrated. The transition rates of the general birth and death process are given by:

$$q_{ij} = \begin{cases} \lambda_i & , j = i + 1 \text{ (birth)} \\ u_i & , j = i - 1 \text{ (death)} \\ \lambda_i + u_i & , j = i \neq 0 \text{ (no change)} \\ 1 & , i = j = 0 \text{ (0 state is absorbing)} \end{cases} \qquad (3.1)$$

where λ_i and u_i are the i-th birth and death rates respectively.

If $u_i = 0$, then the process is pure birth process and if besides $u_i = 0$, $\lambda_i = i\lambda$ where λ is constant, then it is yule process. But if $\lambda_i = \lambda$ = constant and $u_i = 0$, the process will turn out to be the familiar Poisson process. In the case where $\lambda_i = \lambda$ and $u_i = u$, where λ and u are appropriate constant rates, then the birth and death process yields the single-server exponential queuing system. The birth and death process will yield s-server exponential queuing system if

$$\lambda_i = \lambda$$

$$u_i = \begin{cases} iu & 1 \leq i \leq s \\ su & i > s \end{cases} \qquad (3.2)$$

where s is the number of servers in the system. The s-server queuing model becomes an infinite server model when $u_i = iu$ for all $i \geq 1$. In the case when $\lambda_i = i\lambda$ and $u_i = iu$, the general birth and death model will clearly become the so called linear birth and death process. Besides, if $\lambda_i = i\lambda + \beta$, where β is an exponential rate of increase from external source such as immigration, the process is called linear birth and death process with immigration β.

Another model which is represented by Kolmogorov equations and which has important relevance to operational efficiency of complex engineering systems is the system reliability model (1). The forward equation for the replacement of failing components of unserviced system is given by:

$$P'_k(t) = -h(t)\, P_k(t) + h(t)\, P_{k-1}(t), \quad k > 0 \qquad (3.3)$$

$$P'_o(t) = -h(t)\, P_o(t)$$

where h(t) is component hazard rate and $P_k(t)$ is the probability of replacement k components in the interval (0, t). If the system has N components and the failure distribution of the k-th component is negative exponential with hazard rate λ_k and service u_k, then the reliability model is represented by birth and death process where the state of the system is the number of failed components. If we assume that at least ℓ components should be working for the system to be operative, then the forward Kolmogorov equations of this model will be (1):

$$P'_k(t) = -(\lambda_n + u_n)\, P_k(t) + \lambda_{k-1}\, P_{k-1}(t) + u_{k+1} P_{k+1}(t), \quad 0 < k \leq N-\ell$$

$$P'_{N-\ell+1}(t) = -u_{N-\ell+1} P_{N-\ell+1}(t) + \lambda_{N-1} P_{N-1}(t)$$

$$P'_o(t) = -\lambda_o\, P_o(t) + u_i\, P_i(t) \qquad (3.4)$$

$$P_o(0) = 1$$

REFERENCES

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