AN APPLICATION OF PARAMETRIC TIME SERIES IN SIMULATION MODELING

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ABSTRACT

A parametric time series model is employed to simulate the number of aircraft present in an air traffic control sector. Borrowing from the work of Box and Jenkins, the identification, parameter estimation and associated tests of adequacy are illustrated using data from a low altitude control sector. The fitted model provides a useful algorithm and reflects the time dependency which exists between the number of aircraft measured in consecutive time periods. The simulation is part of a larger air communications system simulation which is under development.

I. INTRODUCTION

The air-ground-air verbal communications system over which information is relayed between pilots and air traffic controllers is currently threatening to restrict the level of air traffic over large metropolitan areas. A fast-time computer simulation model of the communications system is being developed for the Federal Aviation Administration as a tool for studying system responses to changes in variables which relate to the kinds and amounts of information relayed.

While the primary object of this study is not to describe general patterns of traffic flow, some method of modeling the number of aircraft is required. This "aircraft loading" is directly related to the proportion of a controller's time spent in verbal communication. For the purposes of the computer simulation, aircraft loading is treated as an exogeneous variable to be represented by a time dependent stochastic model.

This paper discusses the identification and estimation of a parametric time series model descriptive of the aircraft loading experienced by a particular enroute controller handling traffic over New York during a four-hour period in April of 1969.

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The procedures discussed for modeling the response of a continuous variable over time have been presented by George E.P. Box and Gwilym M. Jenkins in their text Time Series Analysis Forecasting and Control (1970). They discuss the identification and estimation of a particular family of "parametric" time series models called the Autoregressive Integrated Moving Average (ARIMA) models. A brief introduction to ARIMA processes is included for those readers who are unfamiliar with the methodology.

II. GENERAL DISCUSSION OF PARAMETRIC TIME SERIES MODELS

In many experimental situations, where the variability of some quantity over time is of interest, sequential observations are taken at equally spaced intervals of time. Such a set of n observations is called a "discrete time series" and can be simply expressed by

$$y_t = \eta_t + \varepsilon_t \qquad t = 0,1,2,\ldots n$$

where y_t is the observed value of the variable at time t, η_t is the unknown value of the underlying response at time t, and ε_t is a "shock" or "error" at time t. In many cases, η_t can be assumed to be a constant and all the variability explained in terms of disturbances z_t. The estimate of η is given by the average

$$\bar{y} = (\Sigma y)/n$$

and the estimated disturbances by

$$z_t = y_t - \bar{y} .$$

The parametric time series form a family of stochastic models appropriate for describing the non-independence of the successive quantities z_t. The models are described as autoregressive of order p, moving average of order q, operating on the d^{th} difference of the z_t's. In practice, p, d, and q are usually less than or equal to 2.

BACKWARDS OPERATOR:

To elucidate the parametric models we employ a convenient notational operator B, called the "backwards operator". It is used to identify the z_t taken earlier in time. Thus $Bz_t = z_{t-1}$ and $B^2 z_t = z_{t-2}$. Two useful expressions involving the backwards operator are the first difference $(1-B)z_t = z_t - z_{t-1}$ and the second difference $(1-B)^2 z_t = (1-2B+B^2)z_t = z_t - 2z_{t-1} + z_{t-2}$.

WHITE NOISE:

The basic idea behind parametric time series models is that a stochastic process can be described as a dynamic system subject to independent "shocks" a_t. These shocks are assumed to be Normally and independently distributed with zero mean and constant variance. The parametric model is a "linear filter" which transforms this "white noise" into the observed quantities z_t. The general model is thus

$$z_t = \phi(B)a_t$$

where $\phi(B)$ is the parametric model or transfer function of the linear filter.

MOVING AVERAGE MODELS:

The general moving average model of order q expresses the current z_t as a linear function of a current shock a_t and q previous shocks. The expression for an MA(q) process is given by

$$z_t = a_t - \Theta_1 a_{t-1} - \Theta_2 a_{t-2} - \dots - \Theta_q a_{t-q}$$
$$= (1-\Theta_1 B - \Theta_2 B^2 \dots - \Theta_q B^q)a_t$$

which may also be written

$$z_t = \Theta_q(B)a_t$$

where $\Theta_q(B)$ indicates a polynomial of degree q in the backwards operator B. Fitting such a model to an observed time series requires estimating q+2 parameters from the data: the mean level of the series η_t, the q parameters $\Theta_1 \dots \Theta_q$, and the variance σ^2 of the independent shocks a_t.

AUTOREGRESSIVE MODELS:

The general autoregressive model of order p expresses the current z_t as a linear function of a current shock a_t and the p previous values $z_{t-1}, z_{t-2}, \dots$. The expression for an AR(p) process is

$$z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2} - \dots - \phi_p z_{t-p} = a_t$$

or

$$(1-\phi_1 B - \phi_2 B^2 \dots - \phi_p B^p)z_t = a_t$$

which may also be written

$$\phi_p(B)z_t = a_t$$

where $\varphi(B)$ indicates a polynomial of degree p in the backwards operator B. Fitting such a model to an observed time series requires estimating p+2 parameters from the data: η, $\phi_1, \phi_2, \dots, \phi_p$, and the variance σ^2 of the independent shocks a_t.

INTERCHANGEABILITY, MIXED MODELS AND STATIONARITY

The AR and MA models are interchangeable. For example, the AR(1) model $(1-\phi_1 B)z_t = a_t$ can be written as the infinite order MA model $z_t = (1-\phi_1 B)^{-1} a_t$. Similarly a finite MA model can be re-expressed as an infinite order AR model. Since models with the fewest parameters are almost always of greatest value, mixed AR-MA models are often used to minimize the total number of parameters. For example, the mixed AR(2)-MA(1) model is: $(1-\phi_1 B - \phi_2 B^2)z_t = (1-\Theta_1 B)a_t$. To estimate the parameters in a parametric time series model, the series must be stationary, that is, the parameters in the model must remain invarient to the location of the time origin. Nonstationarity in the z_t can be induced by movement in η_t. It is often possible to acquire the attributes of stationarity through the simple device of using either the first or second differences of the z_t in place of the original z_t. A complete family of parametric time series models of order p, d, q, exist; that is, a p^{th} order autoregressive, operating on the d^{th} difference, combined with a q^{th} order moving average. For example, the 1,1,1 model is written: $(1-\phi B)(1-B)z_t = (1-\Theta B)a_t$.

BOX-JENKINS MODELING PROCEDURE

Step 1 - Model Identification:

The first step in employing the methodology of parametric time series models, as proposed by Box and Jenkins, is to identify the model. The most useful tool is the sample autocorrelation function (acf) of the z_t. In practice, the estimated lagged autocorrelation coefficients up to lag k are plotted where k is equal to about onefourth of the number of observations. The k^{th} lagged autocorrelation coefficient r_k is

$$r_k = \sum_{t=1}^{n-k} (z_t-\bar{z})(z_{t+k}-\bar{z}) / \sum_{t=1}^{n} (z_t-\bar{z})^2$$

with variance approximately

$$\text{Variance}(r_k) = \frac{1}{n}(1+2 \sum_{i=1}^{k-1} r_i^2)$$

where n is the number of observations in the original data trace.

Figure 1 shows the <u>theoretical</u> autocorrelation functions for moving average processes and autoregressive processes of order 1 and order 2 respectively. In general, a moving average process of order q will show non-zero autocorrelation coefficients for the first q lags. The autocorrelation coefficients for a first order autoregressive process decay exponentially. The signs of the lagged autocorrelations may or may not alternate in sign. Autocorrelation functions for second order autoregressive processes take the form of a sum of two decaying exponentials or of an exponentially decaying cosine function, as illustrated in Figure 1.

Remembering that <u>estimated</u> autocorrelation functions will not match the theoretical functions exactly, one infers from the shape of the estimated function which model, or models, should be initially entertained and tested.

Autocorrelation functions which do not die out as k increases indicate nonstationarity of the series z_t. In such an event, the first or second differences of the z_t's are taken and their autocorrelations estimated in the hope of identifying a stationary model.

<u>Step 2 - Parameter Estimation</u>:

After the form of time series model has been selected, the parameters must be estimated. Autoregressive parameters can be estimated using ordinary least squares techniques. Moving average parameters require iterative least squares. Computer programs exist which will plot the estimated autocorrelation function and, given a postulated model, will estimate the parameters. (1)

<u>Step 3 - Diagnostics</u>:

Once the parameters of the model have been estimated, various checks of the adequacy of the fitted model must be made. Let z_t^* be the disturbance predicted by the model at time t. Then the discrepancies, $z_t - z_t^*$, between observed and predicted values should be independent and Normally distributed. A test for Normality involving standardized skewness and standardized kurtosis can be used to test the hypothesis that the residuals are Gaussian noise (see pages 86-88 of (2)). Further, the estimated autocorrelation function for the residuals should show no statistically significant non-zero coefficients.

<u>Step 4 - Forecasting and Updating</u>:

The fitted model may be used to forecast. For example, consider the fitted AR(2) model

$$\hat{z}_t = 1.2z_{t-1}-0.3z_{t-2}+a_t$$

and suppose that at time position t it is necessary to forecast the event at time position t+1. The model may then be written

$$z_{t+1} = 1.2z_t - 0.3z_{t-1}+a_{t+1}.$$

Since the oncoming random Normal shock a_{t+1} is unknown at time position t, it is replaced by its expected value (zero) to give the equation

$$\hat{z}_t(1) = 1.2z_t - 0.3z_{t-1}$$

where $\hat{z}_t(1)$ is read "the predicted value of z, one unit ahead in time, made at time t." Predicting two units ahead in time would give

$$\hat{z}_t(2) = 1.2\hat{z}_t(1) - 0.3z_t$$

The extension to predictions ℓ units ahead in time, $\hat{z}_t(\ell)$, is obvious. When a moving average model is employed, the shocks a_t, a_{t-1} etc. are estimated using the discrepancies between the previous observed and predicted values, thus

$$a_{t-1} = z_{t-1}-\hat{z}_{t-1}.$$

The one ahead forecasts, $z_t(1)$, are used to obtain a predicted value for each observed value in the time series. The discrepancies $z_{t+1}-\hat{z}_t(1)$ are then used to check on the adequacy of the fitted model. These discrepancies should have all the attributes of Normally distributed, zero mean, homogeneous variance, independent events.

<u>Step 5 - Simulating a Time Series</u>

Once the steps of model identification, parameter estimation, and tests of adequacy have been completed, the fitted model may be used as an algorithm for generating simulated events z_t. To explain, consider the fitted AR(2) model given above where the a_t are Normally and independently distributed with zero mean with a known standard deviation. As ach

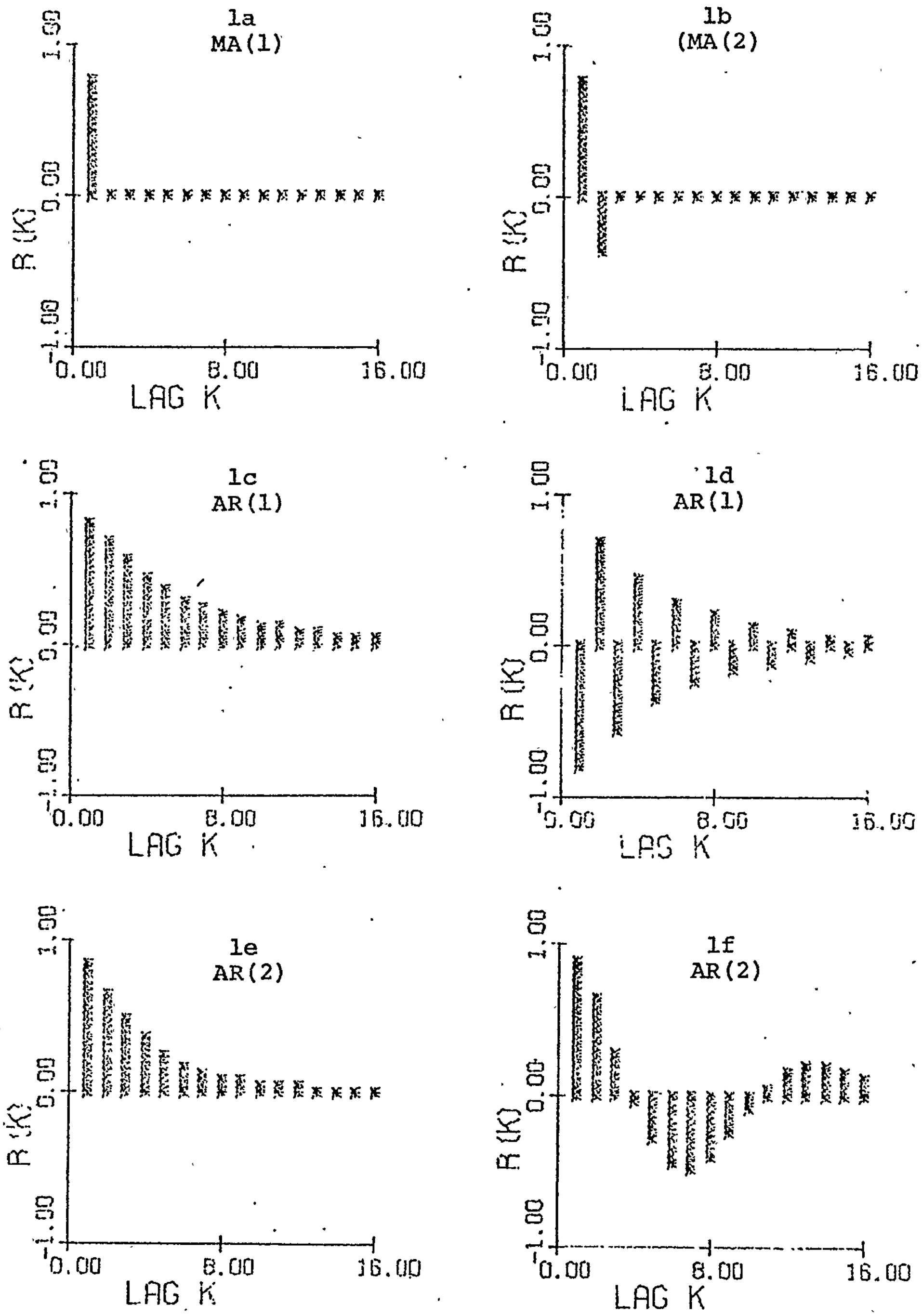

Figure 1

Theoretical Autocorrelation Functions

random Normal shock a_t occurs; the model generates a corresponding value z_t dependent on that shock plus the linear combination of the previous two z_t. The z_t so generated from an AR(2) model can be used to simulate the time series originally employed in determining the fitted model.

III. MODELING AIRCRAFT LOADING

The attempt to model the number of aircraft which a controller is required to handle presented an initial problem since data on the number of aircraft present was recorded every second. It was decided that aircraft loading indices computed on a sixty-second basis would be more useful in the context of the communication simulation being developed. The number of transactions (conversations) between pilots and controllers, the length of such conversations, and the gaps between consecutive communications could all be tied to these loading indices. Also, averages based on sixty one-second observations would eliminate the discreteness and persistence of integer numbers in the raw second-by-second data. Figure 2 is a plot of the sixty-second averages which were computed for a particular low-altitude sector within the New York Air Traffic Control Center.

It is clear from this plot of aircraft loading versus time that consecutive observations are highly correlated and that a parametric time series model might be an effective way of modeling the time dependency. To determine which type of parametric time series model should be considered, the estimated autocorrelation function of the observed z_t about the mean level of the data was plotted and is shown in Figure 3. The plot of the estimated acf is similar to that of a second order autoregressive model with complex roots (see Figure 1f).

The model proposed has the form

$$(y_t-\eta) = \phi_1(y_{t-1}-\eta) + \phi_2(y_{t-2}-\eta)+ a_t.$$

Estimates of four parameters, η, ϕ_1, ϕ_2, and the variance σ^2 of the a_t's, were thus required. An initial estimate of η was given by the average of the y_t's. Estimates of ϕ_1 and ϕ_2 were computed from the first and second estimated lagged autocorrelation coefficients using the Yule-Walker equations (see p. 60 of (1)). Iterative computer algorithms were used to search for better parameter estimates, but in this case only one iteration was performed on the initial estimates and the change was not significant. The fitted model is

$$z_t = 1.24\, z_{t-1} - .34\, z_{t-2} + a_t$$

where

$$z_t = y_t - 3.78 \text{ and } \sigma_a^2 = 0.57.$$

Tests of the adequacy of the fitted model were then performed. If the model fits well, the residuals (i.e., observed disturbances z_t minus the predicted values determined from z_{t-1} and z_{t-2}) should be Normally and independently distributed. The histogram of the residuals, shown in Figure 4, has the general shape associated with the Normal density function. Tests based on the estimated skewness and estimated kurtosis do not contradict the hypothesis that the residuals are Normally distributed at the 99% and 95% confidence levels, respectively. Figure 5 is a plot of the estimated autocorrelation function for the residuals. At all lags, the estimated autocorrelation coefficients fall within two standard deviations of zero. A Chi-Square test performed on the estimated autocorrelations was also not significant at the 95% confidence level. The hypothesis that the residuals are independent is not contradicted. It was thus felt that the fitted model adequately described the stochastic nature of the data.

An illustration of how this model is used to forecast ahead in time is given in Figure 6. The original time series is shown along with the forecasts one unit ahead in time. That is, at each time t an estimate is made of y_t given the previous values of y_{t-1} and y_{t-2}. The forecasts for time t are superimposed on the observed series as plus signs (+). It should be noted how closely the forecasts follow the general pattern of the series.

The next step was to use the fitted model to simulate aircraft loading as required in the model for the verbal communications system. Figure 7 is a simulated time series employing the fitted model, based on a set of Normally distributed random shocks. The general shape of the simulated response does conform well to that of the original series. One additional restriction on the simulated series was necessary, since actual aircraft loading is clearly bounded at zero. Since the probability of the simulated series extending below zero did exist, any random shock which would have sent the series below zero was replaced.

A stochastic model was thus developed which simulated the general aircraft loading pattern exhibited by the particular air traffic control sector under consideration. This model is now being integrated into a larger computer simulation of the communications problem.

IV. GENERAL APPLICABILITY

The use of parametric time series models

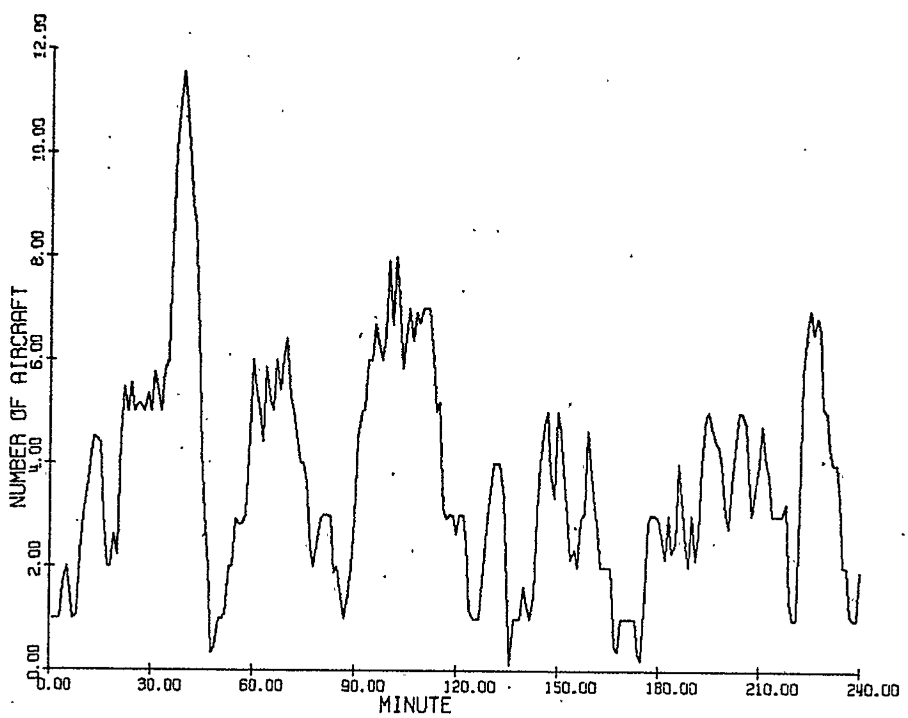

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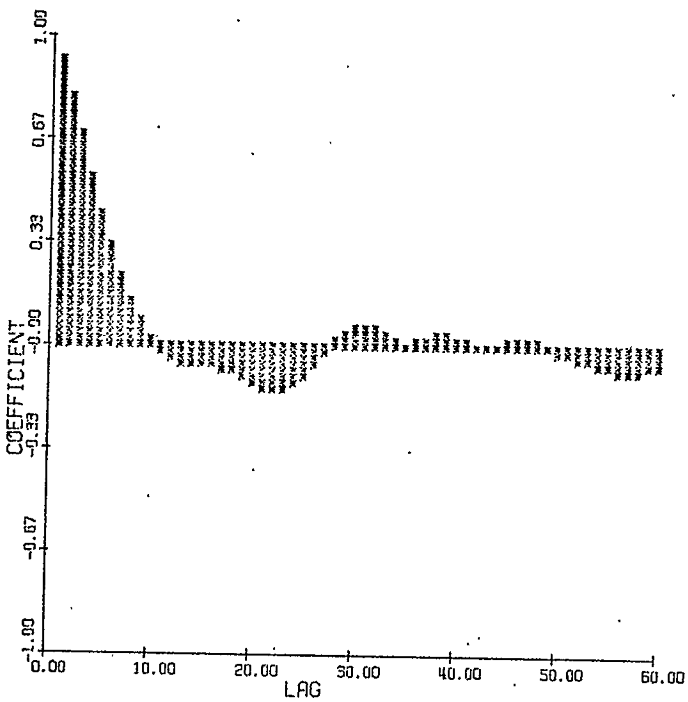
Figure 2



AUTOCORRELATION FUNCTION

Figure 3

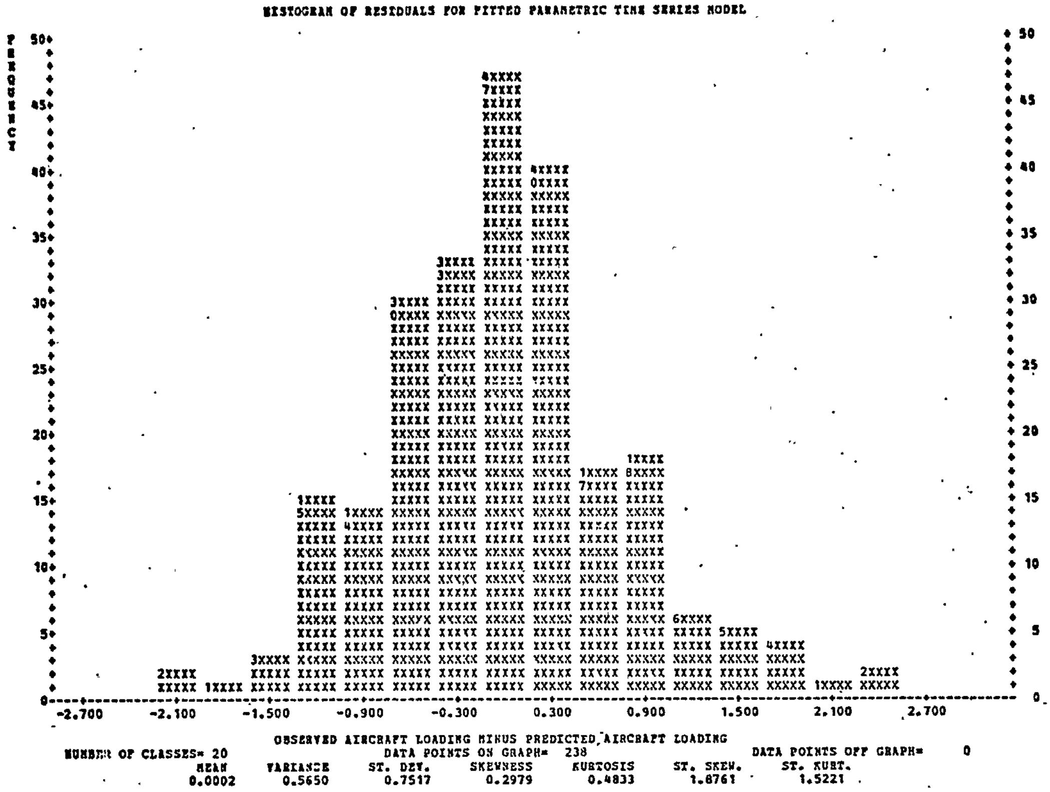


Figure 4

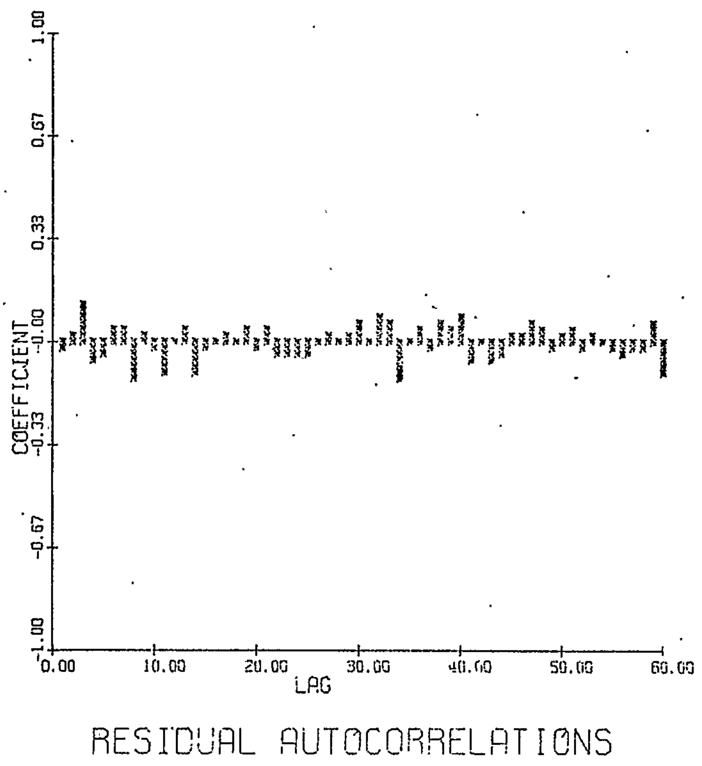


Figure 5

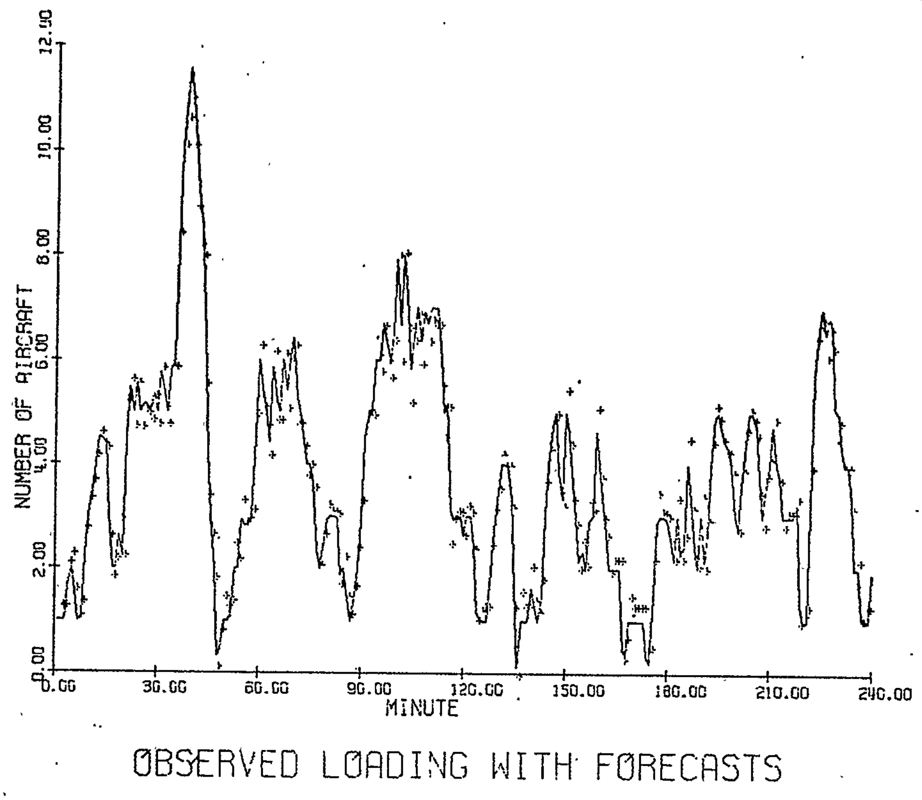


Figure 6

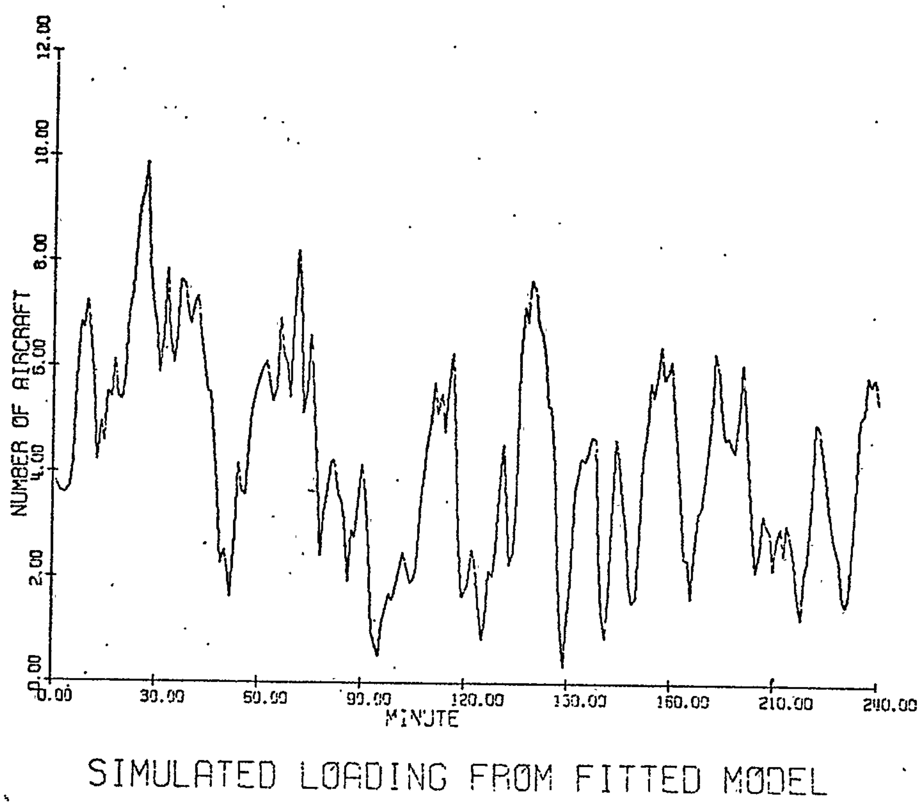


Figure 7

in describing the response of continuous variables over time is no longer an uncommon technique, yet its general applicability in various simulation problems is often overlooked. The modeling of exogeneous variables in many studies might often be more easily accomplished by fitting a parametric time series model rather than by trying to simulate the complex operations which give rise to those variables. The time series approach allowed us, in this instance, to ignore relationships which were not of interest in the context of our larger problem, and provided an acceptable and parsimonious stochastic model for the relevant exogeneous variable. In this way, we have been able to concentrate our efforts on studying the efficiency of the verbal communications system rather than on the detailed mechanisms of air traffic flow.

BIBLIOGRAPHY

1. Box, George E.P. and Gwilym Jenkins (1970), Time Series Analysis Forecasting and Control. Holden-Day, San Francisco.

2. Snedecor, George W. and William G. Cochran (1967), Statistical Methods, sixth ed., The Iowa State University Press, Ames, Iowa.