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DISCRETIZATION ERROR OF REFLECTED FRACTIONAL BROWNIAN MOTION

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ABSTRACT

The long-range dependence and self-similarity of fractional Brownian motion make it an attractive model for traffic in many data transfer networks. Reflected fractional Brownian Motion appears in the storage process of such a network. In this paper, we focus on the simulation of reflected fractional Brownian motion using a straightforward discretization scheme and we show that its strong error is of order h^H, where h is the discretization step and $H \in (0,1)$ is the Hurst index.

1 INTRODUCTION

Fractional Brownian motion (fBm), and more generally fractional stochastic differential equations (SDEs), are popular mathematical models for long-range dependent stochastic systems. Intuitively, long range dependence (or long memory) is observed when observations that are far apart in time (or space) are strongly correlated. As documented in the literature, long-range dependent models are used to describe phenomena in a variety of disciplines, such as hydrology (noisy rainfall data or water accumulation in river gauges (Hurst 1951)), biophysics, economics (Granger 1980), finance (Comte and Renault 1998), (Chronopoulou and Viens 2010), traffic networks (Leland, Taqqu, Willinger, and Wilson 1994)), and computer vision (Beran, Sherman, Taqqu, and Willinger 1995).

Reflected fractional Brownian motion (reflected fBm), similarly to reflected Brownian motion, behaves as a standard fractional Brownian motion in the interior of its range $(0,\infty)$, but when it reaches its boundary the sample path reflects and returns to the interior. In the literature, reflected fBm has gained attention mainly due to its applications in queuing theory (Norros 1994, Piterbarg 2001, Hu and Lee 2013, Zeevi and Glynn 2000, Delgado 2007). In particular, it plays a central role in the heavy-traffic approximation theory for queuing systems (Whitt 2002). Another application of reflected fBm is in the sequential detection of an abrupt change in the drift of fBm (Chronopoulou and Fellouris 2013). Indeed, the popular CUSUM detection (stopping) rule in this context turns out to be a first hitting time of a reflected fBm. However, the implementation of this rule in practice requires discretizing the reflected fBm; in this context it is very useful to understand the discretization step that can approximate the continuous-time process in a satisfactory way. Motivated by this problem, our main objective in this paper is to study the asymptotic behavior of the discretization error of reflected fBm as the discretization step goes to 0.

The remainder of the paper is organized as follows: In Section 2, we provide the necessary background information on both fBm and reflected fBm and we formulate the problem. In Section 3, we present the main result of this work, while in Section 4 this is illustrated using simulation experiments. Finally, in Section 5 we conclude and we discuss future work.

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2 MATHEMATICAL BACKGROUND

In this section we review the main ingredients necessary for the remainder of the paper.

Definition 1 A Fractional Brownian motion (fBm), $\{B_H(t); t \geq 0\}$, with Hurst parameter $H \in (0,1)$ is a centered, Gaussian process with covariance structure

$$\mathbb{E}\Big[B_H(t)B_H(s)\Big] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big) \tag{1}$$

and almost surely continuous sample paths.

The Hurst index, H, determines the correlation structure between increments of the process and its value determines both pathwise as well as distributional properties of the process. The case $H = 1/2$ corresponds to a standard Brownian motion. Based on (1), one can easily derive the autocovariance of the increments of the process (Norros 1994):

$$\gamma(n) = \mathbb{E}\Big[B_H(1),\, B_H(n+1) - B_H(n)\Big] = H(2H-1)\, n^{-2(1-H)} + \mathscr{O}\left(n^{-3-2H}\right),\ n \in \mathbb{N}. \tag{2}$$

From (2) it is easy to see that for $H > 1/2$, the increments of fBm are positively correlated, thus the process is long-range dependent, that is $\sum_{n=1}^{\infty} \gamma(n) = \infty$. On the other hand, when $H < 1/2$, the increments of fBm are negatively correlated and the process exhibits medium memory, that is $\sum_{n=1}^{\infty} \gamma(n) < \infty$. Apart from the memory parameter, the Hurst index is also the self-similarity parameter of fBm. Indeed, based on the autocovariance structure, (2), it is easy to see that fBm is H self-similar and has H self-similar increments. This means that for any $a > 0$, $B_H(at) \overset{\mathscr{D}}{\sim} a^H\, B_H(t)$, $t \geq 0$. A detailed exposition of the properties of fBm can be found in Samorodnitsky and Taqqu (1994), Beran (1994).

The main focus of this paper is on the reflected fractional Brownian Motion (rfBm) which is formally defined as follows:

Definition 2 Let $\{B_H(t); t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then, the reflected fractional Brownian motion $\{X_H(t); t \geq 0\}$ is defined as

$$X_H(t) = \sup_{0 \leq s \leq t}\Big(B_H(t) - B_H(s)\Big) = B_H(t) - \inf_{0 \leq s \leq t} B_H(s).$$

There are several results for the behavior of the tail probabilities of reflected fBm (Hüsler and Piterbarg 1999, Debicki and Rolski 2002, Albin and Samorodnitsky 2004, Caglar and Vardar-Acar 2013, Hashorva, Ji, and Piterbarg 2013, Debicki and Kosinski 2014), as well as bounds on its return time (Lee 2011). However, an expression for the distribution of reflected fBm is not available.

3 PROBLEM FORMULATION AND MAIN RESULT

The problem we consider in this paper is the simulation of a sample path of reflected fBm in a finite interval, which without loss of generality we let it be $[0,1]$. Specifically, we consider the simulation of this process at a a sequence of n equidistant points in $[0,1]$, i.e., $\{h, 2h, \ldots, 1\}$, where $h = 1/n$ is the mesh of the partition. There are several procedures in the literature that can be used for the exact simulation of fBm, that is without introducing discretization bias. The fastest algorithm was initially proposed by Davis and Harte (1987) and later generalized by Wood and Chan (1994) and is of order $n \log n$. The algorithm is based on the fact that the covariance matrix of a stationary discrete-time Gaussian process can be embedded in a so-called circulant matrix. For the algorithm to work, the latter matrix should be positive, which is indeed the case for the stationary increments of fBm, also known as fractional Gaussian noise. The constructed circulant matrix can be diagonalized explicitly, and the computations are done efficiently with the so-called Fast Fourier Transform (FFT) algorithm. For a review of different simulation techniques, as well as a study of their rates of convergence, we refer to Dieker (2002).

To sum up, it is well understood how to simulate fBm exactly in a grid of points. The case of *reflected* fBM however is far more challenging due to the presence of the reflecting barrier, which generates discretization bias. We should emphasize here that such discretization bias is not inflicted when $H = 1/2$. Indeed, it is well known that *standard* reflected Brownian motion has the same distribution as the absolute value of a standard Brownian motion, and the latter can be easily simulated at a discrete grid as a random walk with normally distributed increments. Unfortunately, a similar argument does not apply when $H \neq 1/2$. Thus, an approximation scheme is needed for the simulation of reflected fBm.

To be more specific, let $X_H^{(n)}$ be the discretized reflected fBm and denote $B_H^{(n)}(j) := B_H(j/n)$, for $j = 0,\dots,n$. Then $X_H^{(n)}$ is defined at the grid points as follows:

$$X_H^{(n)} := B_H^{(n)}(n) - \min_{0\le j\le n} B_H^{(n)}(j), \tag{3}$$

where it is understood that fBM, B_H, is simulated exactly at grid points. Then, the error of this approximation takes the form

$$-\varepsilon^{(n)} := X_H - X_H^{(n)} = \min_{0\le j\le n} B_H^{(n)}(j) - \inf_{0\le s\le 1} B_H(s). \tag{4}$$

Our goal in this paper is to quantify how fast this error goes to 0 as the number of points in the grid increases. This is the content of the following theorem.

Theorem 1 For any given Hurst parameter $H \in (0,1)$, the discretization error in reflected fBm is of order H, in the sense that

$$\mathbb{E}\left[\varepsilon^{(n)}\right] = \mathscr{O}(n^{-H}) \quad \text{as} \quad n\to\infty. \tag{5}$$

Before we prove this result, let us first make some comments. To our knowledge, this result is new when $H \neq 1/2$. In the case of the standard reflected Brownian motion ($H = 1/2$), this result follows from Asmussen, Glynn, and Pitman (1995). However, the proof in this case relies on the Markovian structure of standard Brownian motions, and does not extend to the (highly, when $H > 1/2$) non-Markovian fBM.

Moreover, it is worth noting that the rate of convergence depends on H; the closer the H is to 1, the faster the convergence of the discretization error to 0. A similar phenonomenon has also been observed for the Euler and Milstein discretization schemes for fractional SDEs. In general, the dependence of the convergence rate on H is not surprising, as this parameter determines the pathwise and distributional properties of fBM.

We now proceed with the proof of the Theorem.

Proof. First of all, note that due to the symmetry of fBM, $\varepsilon^{(n)}$ has the same distribution as

$$\sup_{0\le s\le 1} B_H(s) - \max_{0\le j\le n} B_H^{(n)}(j),$$

thus, it suffices to study the convergence rate of the latter random variable.

Since $\max_{0\le j\le n} B_H^{(n)}(j) \uparrow \sup_{0\le s\le 1} B_H(s)$ a.s., by the Monotone Convergence Theorem, we obtain

$$\lim_{n\to\infty} \mathbb{E} \max_{0\le j\le n} B_H^{(n)}(j) \to \mathbb{E} \sup_{0\le s\le 1} B_H(s).$$

Let $t^* = \arg\sup_{0\le s\le 1} B_H(s)$ be the point at which the true maximum is obtained and let k_n^* be the closest grid point to the right of t^*. Then, it is clear that

$$\sup_{0\le s\le 1} B_H(s) - \max_{0\le j\le n} B_H^{(n)}(j) \le B_H(t^*) - B_H^{(n)}(k_n^*),$$

since $\max_{0\le j\le n} B_H^{(n)}(j)$ is not smaller than the value of fBM at any given point on the grid. Now using a finer partition to approximate the maximum and taking expectations to obtain

$$\begin{aligned}\mathbb{E}\left|\sup_{0\le s\le 1} B_H(s) - \max_{0\le j\le n} B_H^{(n)}(j)\right| &\le \mathbb{E}\left|\sup_{0\le s\le 1} B_H(s) - B_H^{(n)}(k_n^*)\right| \\ &= \lim_{m\to\infty} \mathbb{E}\left|\max_{0\le j\le 2^m} B_H^{(n2^m)}(k_n^* 2^m + j) - B_H^{(n)}(k_n^*)\right| \\ &= \lim_{m\to\infty} \mathbb{E}\left|\max_{0\le j\le 2^m} B_H^{(n2^m)}(k_n^* 2^m + j) - B_H^{(n2^m)}(k_n^* 2^m)\right|\end{aligned}$$

From the self-similarity of the increments of fBM it follows that

$$\left\{B_H^{(n2^m)}(k_n^* 2^m + j) - B_H^{(n2^m)}(k_n^* 2^m)\right\}_{0\le j\le 2^m} \overset{D}{=} \left\{n^{-H} B_H^{(2^m)}(j)\right\}_{0\le j\le 2^m}.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left|\sup_{0\le s\le 1} B_H(s) - \max_{0\le j\le n} B_H^{(n)}(j)\right| &\le \lim_{m\to\infty} \mathbb{E}\left|\max_{0\le j\le 2^m} n^{-H} B_H^{(2^m)}(j)\right| \\ &\le n^{-H}\mathbb{E}\left|\sup_{0\le s\le 1} B_H(s)\right|,\end{aligned}$$

since

$$\max_{0\le j\le 2^m}\left\{B_H^{(2^m)}(j)\right\} \le \sup_{0\le s\le 1} B_H(s).$$

It is well known that $\sup_{0\le s\le 1} B_H(s) \in L^1$, which implies that there exists a constant $C(H)\in\mathbb{R}$ such that

$$\mathbb{E}|\sup_{0\le s\le 1} B_H(s)| \le C(H).$$

Based on the above, we conclude that

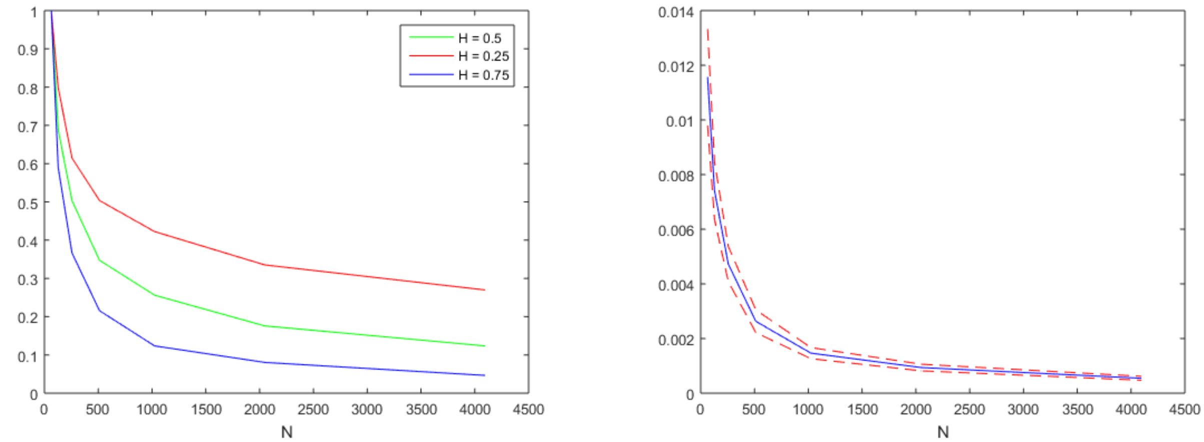
$$\mathbb{E}\left[\varepsilon^{(n)}\right] \le \mathbb{E}\left[B_H(t^*) - B_H^{(n)}(k_n^*)\right] \le C(H)n^{-H},$$

which completes the proof. □

4 NUMERICAL SIMULATIONS

In this section we illustrate our main result with a simulation study. Thus, we simulate 100 samples of reflected fBm with 10^6 time steps, which we consider as a proxy for the "true" continuous process, and compute the average discretization error for various step sizes. In Figure 1(a) we plot the average discretization error as a function of the number of points in the partition for three different values of the Hurst index, H. As we expected from our theoretical result, the closer H is to 0 (resp. 1), the slower (resp. faster) the convergence. In Figure 1(b) we provide a 95% confidence interval for the error of the approximation when $H = 0.75$. We can see that, even for 100 replications, this is a rather tight interval.

Finally, in Figure 2, we plot the scaled discretization error, that is $n^H\varepsilon^{(n)}$. Our graphs suggest that the scaled discretization error converges weakly to a proper limit that is not in general Gaussian, as it is quite skewed when H is close to 1.



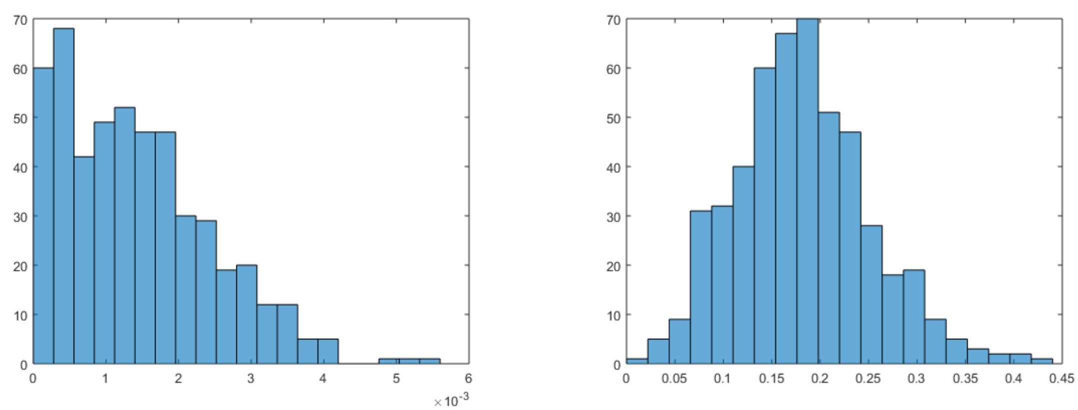
(a) Error for different values of H.

(b) 95% Confidence Interval for the error ($H = 0.75$).

Figure 1: Strong discretization error as function of number of time steps (N).

5 DISCUSSION

In this paper we studied the discretization error of reflected fBm, and we were able to derive the order at which it converges to zero for any given values of the Hurst index, $H \in (0,1)$. One direction of future research is to extend the analysis of this work for other reflected, not necessarily Gaussian, fractional processes. A more challenging problem is to derive the asymptotic distribution of the error. This has been achieved in the case of reflected Brownian motion ($H = 1/2$) by Asmussen, Glynn, and Pitman (1995), again using techniques that do not readily generalize when $H \neq 1/2$.



(a) $H = 0.75$.

(b) $H = 0.25$.

Figure 2: Histograms of the asymptotic distribution of the re-scaled error for two values of H.

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