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A NEW APPROACH TO UNBIASED ESTIMATION FOR SDE'S

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ABSTRACT

In this work, we introduce a new approach to constructing unbiased estimators when computing expectations of path functionals associated with stochastic differential equations (SDEs). Our randomization idea is closely related to multi-level Monte Carlo and provides a simple mechanism for constructing a finite variance unbiased estimator with "square root convergence rate" whenever one has available a scheme that produces strong error of order greater than 1/2 for the path functional under consideration.

1 INTRODUCTION

Suppose that we wish to compute an expectation of the form $\alpha = \mathbf{E}k(X)$, where $X = (X(t) : t \geq 0)$ is the solution to the SDE

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB(t), \tag{1}$$

$B = (B(t) : t \geq 0)$ is an m-dimensional standard Brownian motion, $\mu : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $k : C[0,\infty) \to \mathbb{R}$, and $C[0,\infty)$ is the space of continuous functions mapping $[0,\infty)$ into $\mathbb{R}^d$. In general, the random variable (rv) $k(X)$ cannot be simulated exactly, and one typically approximates X via a discrete-time approximation $X_h(\cdot)$ which in turn leads to a biased estimator $k(X_h)$. The traditional means of dealing with the bias is to intelligently select the step size h and number of independent replications R as a function of the computational budget c, so as to maximize the rate of convergence. However, as pointed out by Duffie and Glynn (1995), such biased numerical schemes inevitably lead to Monte Carlo estimators for α that exhibit slower convergence rates than the "canonical" order $c^{-1/2}$ rate associated with Monte Carlo in the presence of unbiased finite variance estimators.

However, Giles (2008) introduced an intriguing multi-level idea to deal with such biased settings that can dramatically improve the rate of convergence and can even, in some settings, achieve the canonical "square root" convergence rate associated with unbiased Monte Carlo. His approach, however, does not construct an unbiased estimator. Rather, the idea is to construct a family of estimators (indexed by the desired error tolerance ε) that has controlled bias. In this work, we show how it is possible, in a similar computational setting, to go one step further and to produce (exactly) unbiased estimators. To the best of the authors' knowledge, this is the first simulation algorithm that is both unbiased and achieves "square root convergence" for d-dimensional SDEs.

2 THE BASIC IDEA

We consider here a sequence $(X_{h_n} : n \geq 0)$ of discrete-time time-stepping approximations to X that are all constructed on a common probability space in such a way that:

i) $\mathbf{E}k(X_{h_n}) = \mathbf{E}k(X) + O(h_n)$ as $h_n \to 0$;
ii) $\mathbf{E}|k(X_{h_n}) - k(X)|^2 = O(h_n^{2r})$ as $h_n \to 0$ for some $r > 0$,

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where $O(f(n))$ represents a function which is bounded by some constant multiple of $f(\cdot)$ as $h_n \to 0$. Under these conditions, we introduce a rv N, independent of B, that takes values in the positive integers and has a distribution with unbounded support (so that $\mathbf{P}(N \geq n) > 0$ for $n \geq 1$). For such a rv N,

$$\begin{aligned}
\mathbf{E}k(X) &= \mathbf{E}k(X_1) + \sum_{n=1}^{\infty} \mathbf{E}(k(X_{2^{-n}}) - k(X_{2^{-(n-1)}})) \\
&= \mathbf{E}k(X_1) + \sum_{n=1}^{\infty} \mathbf{E}\left[(k(X_{2^{-n}}) - k(X_{2^{-(n-1)}}))I(N \geq n)\right]/\mathbf{P}(N \geq n) \\
&= \mathbf{E}\left[k(X_1) + \sum_{n=1}^{N} (k(X_{2^{-n}}) - k(X_{2^{-(n-1)}}))/\mathbf{P}(N \geq n)\right] \\
&\triangleq \mathbf{E}Z.
\end{aligned}$$

i.e., Z is an unbiased estimator for α. This suggests computing α by generating iid replicates of the rv Z. For iid unbiased estimators, "square root convergence rate" ensues if var$Z < \infty$ and if the expected computational effort required per replication of Z is finite; see Glynn and Whitt (1992). The following conditions guarantee the finite variance and the finite expected effort required so that our iid estimator achieves the square root convergence rate; see Rhee and Glynn (2012) for proofs.

- iii) $\mathbf{P}(N \geq i) \sim c2^{-\gamma i}$ as $i \to \infty$, for $0 < \gamma < 2r$ (where $a_i \sim b_i$ means that $a_i/b_i \to 1$ as $i \to \infty$)
- iv) The effort required to compute $k(X_{2^{-i}})$ is $O(2^i)$.
- v) $2r > 1$.

It should be noted that these conditions essentially coincide with the conditions required by multi-level Monte Carlo to converge at the same rate.

3 PRELIMINARY COMPUTATIONAL INVESTIGATION

We implemented our method and compared it to the multi-level Monte Carlo (MLMC) algorithm suggested in Giles (2008). We considered the first moment of Geometric Brownian Motion (GBM) and the Cox-Ingersoll-Ross process (CIR) at a fixed time point with typical parameters used in finance context. The numerical scheme used to solve each of the SDEs was the Milstein scheme; see Rhee and Glynn (2012) for more details.

The results from our method show that the new estimators are indeed unbiased, and computationally comparable to those associated with MLMC despite the fact that we did essentially no tuning to optimize the distribution of N for the purposes of this experiment. In particular, for the CIR example, our estimator appears to require less work for comparable accuracy, while for the GBM example, MLMC is more efficient. In addition, our estimator is (arguably) easier to implement than MLMC, since (in its current form) there are no algorithmic parameters that are estimated "on the fly" within the algorithm (in contrast to MLMC). Thus, the unbiased estimators introduced here offer a promising computational alternative to MLMC in the presence of SDE numerical schemes having a strong order greater than 1/2.

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