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SENSITIVITY ESTIMATION OF SABR MODEL VIA DERIVATIVE OF RANDOM VARIABLES

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ABSTRACT

We derive Monte Carlo simulation estimators to compute option price sensitivities under the SABR stochastic volatility model. As a companion to the exact simulation method developed by Cai, Chen and Song (2011), this paper uses the sensitivity of "vol of vol" as a showcase to demonstrate how to use the pathwise method to obtain unbiased estimators to the price sensitivities under SABR. By appropriately conditioning on the path generated by the volatility, the evolution of the forward price can be represented as noncentral chi-square random variables with stochastic parameters. Combined with the technique of derivative of random variables, we can obtain fast and accurate unbiased estimators for the sensitivities.

1 INTRODUCTION

The ubiquitous existence of the volatility smile and skew poses a great challenge to the practice of risk management in fixed and foreign exchange trading desks. In foreign currency option markets, the implied volatility is often relatively lower for at-the-money options and becomes gradually higher as the strike price moves either into the money or out of the money. Traders refer to this stylized pattern as the *volatility smile*. In the equity and interest rate markets, a typical aspect of the implied volatility, which is also known as the *volatility skew*, is that it decreases as the strike price increases. The fact that different options are corresponding to different implied volatilities entails that we have to model these smiles accurately in order to achieve a stable hedging.

The stochastic alpha-beta-rho (SABR) model, introduced by Hagan et al. (2002), draws popularity in the financial industry to model implied volatilities in foreign exchange and interest rate markets. It requires a small handful set of parameters but fits the volatility smiles very well. Hagan et al. (2002) develop an asymptotic expansion for the implied volatility of European call options using the singular perturbation technique. This closed-form expression admits clear interpretation of model parameters and yields a convenient way to calibrating the model to the market. In addition, this model is also capable of generating correct co-movements between the underlying and its smile curve, which overcomes a salient drawback of other conventional local volatility models such as Dupire (1994) and Derman and Kani (1994).

Despite the above attractive features of the SABR model, computation involving this model is very challenging. As shown in the next section, we can see that the model consists of two stochastic differential equations (SDE) to describe the respective evolution of the underlying price and its volatility. The volatility process is governed by a geometric Brownian motion. The price follows a constant-elasticity-variance (CEV) type diffusion, coupled with the stochastic volatility. Furthermore, the two processes entangle with each other via two correlated Brownian motions. Except for some trivial examples, the complicated structure of the model — nonlinearity of the CEV type diffusion, coupled price and volatility processes, and correlated Brownian motions — prevents us from obtaining closed form solutions to European option pricing.

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The literature, so far, mainly relies on partial-differential-equation (PDE) based approaches to find various asymptotic expansions for option prices under the SABR model. One may refer to Hagan et al. (2002), Berestycki, Busca and Florent (2004), Henry-Labordère (2005), Hagan, Lesniewski and Woodward (2005), Obłój (2008), and Wu (2010) for further reference on this approach. However, it can only lead to approximations at most, which perform well only under a crucial assumption that the time-to-maturity of options is small. Thus the expansion is not the true option price under the SABR model. As shown by Benaim and Friz (2009), the behavior of the expansion formula obtained by Hagan et al. (2002) is not consistent with the arbitrage-free prices for options with extreme strike prices. Cai, Chen and Song (2011) also report discrepancies between this approximation and the true option price through some numerical experiments.

In light of the absence of closed-form option pricing formula and the drawback of PDE-based asymptotic expansions, Cai, Chen and Song (2011) construct a simulation scheme to generate samples from the exact distribution of the SABR model. It initiates a probabilistic approach to tackle the computational issues of the model. The aim of this paper is to investigate an accompanying method to estimate the price sensitivities under the model. We use the pathwise derivative (PD) method mainly to generate unbiased sensitivity estimates. As illustrated by Glasserman (2004), this method is convenient in the settings, such as the Black-Scholes model, where the transition density of the underlying price process is known and sampling from the exact distribution is possible. However, we do not have closed-form representations for the probability density of the SABR model, which imposes a significant obstacle to the application of the above two methods. The key idea of this paper is that by appropriately conditioning on the path generated by the volatility processes, the probability law of the underlying price can be represented in terms of noncentral chi-square random numbers, making it possible to apply the PD method. In addition, Cai, Chen and Song (2011) use intensively the inverse transform method to generate random variables in the exact simulation procedure. This makes a special form of PD — derivative of random variables (see, e.g., Fu (2006)) — become very useful in the derivation.

This paper is organized as follows. In Section 2, we introduce the SABR model and present its relationship with the time-changed squared Bessel process. On the basis of this theoretical foundation, we review the exact simulation algorithm proposed by Cai, Chen and Song (2011). Section 3 develops the PD estimators for the SABR model. Due to the page limitation, we only consider the sensitivity with respect to vol of vol when the process is with an absorbing boundary. We show the merits of our algorithm through some numerical examples in Section 4.

2 THE SABR MODEL AND ITS EXACT SIMULATION SCHEME

2.1 The Model

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, on which two independent standard Brownian motions $\{(W_t^1, W_t^2) : t \geq 0\}$ are defined. Assume that $\mathscr{G}^1$ and $\mathscr{G}^2$ are the natural σ-algebra filtrations generated by them, respectively, and let $\mathscr{F} = \mathscr{G}^1 \otimes \mathscr{G}^2$. The SABR model describes the dynamics of an asset's forward price and its volatility. Denote F_t and α_t to be their respective values at time t, $0 \leq t \leq T$. The model is then given by the solution to the following SDEs:

$$dF_t = \alpha_t F_t^{\beta} \cdot [\sqrt{1-\rho^2} dW_t^1 + \rho dW_t^2], \ F_0 = f, \tag{1}$$

$$d\alpha_t = \nu \alpha_t \cdot dW_t^2, \ \alpha_0 = \alpha, \tag{2}$$

where β and ν are two positive constants and $\beta \in [0,1]$. Apparently, the volatility α_t follows a geometric Brownian motion and F_t is governed by a CEV-type diffusion. Except for such cases as $\beta = 0$ or $\beta = 1$, the nonlinearity in $\alpha_t F_t^{\beta}$ distinguishes this model from the class of affine stochastic volatility models proposed by Duffie, Pan and Singleton (2000). Therefore we cannot apply the exact simulation scheme of Broadie and Kaya (2004) and Broadie and Kaya (2006) directly for this model.

By using the singular perturbation techniques in PDEs, Hagan et al. (2002) prove that we still can use the celebrated Black formula to evaluate a vanilla European call on the forward price F struck at K under the SABR model (1-2). The option price at time 0 equals

$$c(f,\alpha) = \mathbb{E}[(F_T - K)^+ | F_0 = f, \alpha_0 = \alpha] = [fN(d_+) - KN(d_-)], \tag{3}$$

where T is the time to the maturity of the option, $N(\cdot)$ is the cumulative probability distribution function of a standard normal, and

$$d_\pm = \frac{\log(f/K) \pm \frac{1}{2}\sigma_{im}^2 T}{\sigma_{im}\sqrt{T}}.$$

The most intriguing contribution of their paper is that they find a closed-form approximation to the implied volatility σ_{im}. By fine-tuning the values of parameters ρ, β, ν, and α, one can easily fit the curve of σ_{im} to the observed implied volatility curves very well under various market environments. The model also demonstrates high stability in the fitted parameter values even in the presence of large market noises. These attractive features win popularity for the SABR model among practitioners in the financial industry, especially in the interest rate and foreign exchange markets. However, as mentioned in the introduction, the PDE-based expansion methods suffer from some discrepancies in calculation, which motivates us to pursue this work.

Cai, Chen and Song (2011) investigate a Monte Carlo recipe to simulate the SABR model exactly from its marginal distribution. The theoretical foundation of our algorithm lies in the relationship between the SABR model and noncentral chi-squared random variables, which is summarized in Theorem 1. A subtle issue will arise around the behavior of the forward price process F at the boundary 0. The paper shows that F_t is always nonnegative for all $t \geq 0$. However, under some parameter ranges, it can reach 0 and the characterization of (1-2) alone is not sufficient to determine the process uniquely. Additional specifications about its behavior at 0 are hence needed to complete the description. They consider two major ways to specify the boundary conditions for Y in this paper: *absorbing* boundary and *reflecting* boundary. Roughly speaking, the former one forces the process to stay at 0 once it reaches the boundary; the latter is to "bump" F back to the positive part immediately after it hits 0. More rigorous mathematical treatments on the boundary classification for a general SDE can be found in Borodin and Salminen (2002). From now on, we focus on the absorbing boundary specification only, although the simulation methods by Cai, Chen and Song (2011) and this paper are applicable to deal with the reflecting boundary specification.

Introduce a parameter

$$\delta := 1 - \frac{\beta}{(1-\beta)(1-\rho^2)}.$$

It determines the behavior of F around the boundary 0. Let $Q(x;\mu,\lambda)$ and $f(x;\mu,\lambda)$ denote, respectively, the cumulative distribution function and the probability density function of a noncentral chi-square random variable with μ degrees of freedom and noncentrality parameter λ. In the meanwhile, denote $q(x;\mu)$ to be the cumulative distribution function of a central chi-square random variable with μ degrees of freedom. Closed-form expressions of both functions are given in the appendix. We have

Theorem 1 (Theorem 2.1 of Cai, Chen and Song (2011)) Fix $T > 0$ and suppose that α_0, α_T, $\int_0^T \alpha_u^2 du$ are simulated. Let

$$A = \frac{1}{(1-\rho^2)\int_0^T \alpha_u^2 du}\left(\frac{F_0^{(1-\beta)}}{1-\beta} + \frac{\rho}{\nu}(\alpha_T - \alpha_0)\right)^2.$$

The transitional distribution of F_T can be expressed in terms of noncentral chi-square distributions. More precisely, when $\delta \geq 0$ and F has an absorbing boundary at 0 or $\delta < 0$,

$$P\left[F_T = 0 \middle| F_0, \alpha_0, \alpha_T, \int_0^T \alpha_s^2 ds\right] = 1 - q(A; 2-\delta);$$

and

$$P\left[F_T \le u \middle| F_0, \alpha_0, \alpha_T, \int_0^T \alpha_s^2 ds\right] = 1 - Q\left(A; 2-\delta, C(u)\right),$$

for any $u > 0$, where

$$C(u) = \frac{1}{(1-\rho^2)\int_0^T \alpha_s^2 ds} \cdot \frac{u^{2(1-\beta)}}{(1-\beta)^2}.$$

2.2 Review of Exact Simulation of the SABR

On the basis of the distribution decomposition presented by Theorem 1, Cai, Chen and Song (2011) propose an exact simulation scheme for the SABR model to generate sample pairs of F_T and α_T. It takes the following three steps:

2.2.1 Sampling from α_T

Since the volatility process α_t is governed by a geometric Brownian motion, it follows that

$$\alpha_T = \alpha_0 \exp\left[-\frac{1}{2}\nu^2 T + \nu W_T^2\right].$$

Note that $W_T^2 \sim N(0,T)$. Accordingly, to sample from α_T, we can generate a standard normal random variable Z and let

$$\alpha_T = \alpha_0 \cdot \exp\left[-\frac{1}{2}\nu^2 T + \nu\sqrt{T}Z\right]. \tag{4}$$

2.2.2 Sampling $\int_0^T \alpha_s^2 ds$ for given α_T

Given α_T, the conditional distribution of $\int_0^T \alpha_s^2 ds$ is related to the Hartman-Waston distribution shown by Cai, Chen and Song (2011), which is highly unstable in numerical performance. To circumvent this difficulty, they suggest a Laplace transform inversion-based approach to simulate $\int_0^T \alpha_s^2 ds$.

Let $h(x) = 1/x$. Denote

$$F_h(y) := P\left[h\left(\int_0^T \alpha_s^2 ds\right) \le y \middle| \alpha_0, \alpha_T\right]$$

for $y \ge 0$ and its Laplace transform

$$\widehat{F}_h(\theta) := \int_0^{+\infty} e^{-\theta u} F_h(u) du$$

for $\theta > 0$. Cai, Chen and Song (2011) prove that:

Theorem 2 (Proposition 3.2 of Cai, Chen and Song (2011)) The Laplace transform of $F_h(u)$ is given by

$$\widehat{F}_h(\theta) = \frac{1}{\theta}\exp\left\{-\frac{[\phi_{\ln(\alpha_T/\alpha_0)}(\theta\nu^2/\alpha_0^2)]^2 - [\ln(\alpha_T/\alpha_0)]^2}{2\nu^2 T}\right\},$$

where

$$\phi_x(\lambda) = \operatorname{arcosh}(\lambda e^{-x} + \cosh(x)).$$

Through some numerical inversion algorithms such as Abate and Whitt (1992), we can obtain F_h from $\widehat{F}_h$. Set

$$V = F_h^{-1}(U),\ U \sim U(0,1). \tag{5}$$

It is straightforward to show that $h^{-1}(V)$ follows the same distribution as $\int_0^T \alpha_s^2 ds$. Finding V defined in (5) amounts to solving an equation $F_h(V) = U$ for a given U. We can solve it numerically.

2.2.3 Generate F_T

We can simulate F_T from its conditional distribution law shown in Theorem 2. The idea is to apply the inverse transform method instead. Recall that there is an atom in the distribution of F_T at 0. Therefore we conduct the following procedure to accomplish the simulation: first generate $U \sim U(0,1)$; if

$$U \leq 1 - q(A; 2-\delta),$$

then set $F_T = 0$; otherwise, we use some numerical methods to find $\hat{U}$ which solves

$$1 - Q\left(A; 2-\delta, C(\hat{U})\right) = U, \tag{6}$$

and then set $F_T = \hat{U}$.

3 PATHWISE ESTIMATOR FOR SABR MODEL

In this section we start to construct unbiased sensitivity estimators for the SABR model. Due to the page limitation, we use the sensitivity with respect to ν as a showcase. The main methodology we adopt is *derivatives of random variables*. It is worth pointing out that unbiased estimators of other sensitivities are achievable in a similar manner. The authors leave a more comprehensive investigation in the future work.

3.1 Derivatives of Random Variables

We would like to use the PD method to derive the unbiased estimator. Generically speaking, it runs as follows. Suppose that we have collected a class of random variables $\{D(\theta,\omega) : \theta \in \Theta\}$ on a common probability space $(\Omega, \mathscr{F}, P)$, where $\omega \in \Omega$ and θ is the parameter whose sensitivity we are interested in. The method suggests that we can interchange the order of differentiation and expectation, under some appropriate conditions, to get an unbiased estimator to $dE[H(D(\omega,\theta))]/\theta$. That is,

$$\frac{d}{d\theta}E[H(D(\omega,\theta))] = E\left[H'(D(\omega,\theta)) \cdot \frac{d}{d\theta}D(\omega,\theta)\right]. \tag{7}$$

Consider a special case of the above general method. Assume that the cumulative distribution function of $D(\omega,\theta)$ is known as $G(x;\theta)$ for any given $\theta \in \Theta$. Then, in distribution, we have

$$D(\theta) \stackrel{d}{=} G^{-1}(U;\theta),$$

where $U \sim U(0,1)$ and $\theta \in \Theta$. Under this representation, Eq. (7) and the implicit function theorem implies that

$$\begin{aligned}
\frac{d}{d\theta}E[H(D(\theta))] &= \frac{d}{d\theta}E[H(G^{-1}(U;\theta))] = E\left[H'(D(\theta)) \cdot \frac{d}{d\theta}G^{-1}(U,\theta)\right] \\
&= E\left[H'(D(\theta)) \cdot \left(-\frac{\partial G(x;\theta)/\partial\theta}{\partial G(x;\theta)/\partial x}\right)\Big|_{x=D(\theta)}\right].
\end{aligned}$$

Then, we obtain an unbiased estimator for $dE[H(D(\theta))]/\theta$ as

$$H'(D(\theta)) \cdot \left(-\frac{\partial G(x;\theta)/\partial\theta}{\partial G(x;\theta)/\partial x}\right)\Big|_{x=D(\theta)}$$

Such methodology leading to a PD estimator is referred to as derivative of random variables in the simulation literature (see, e.g., Fu (2006)). This method grants us much convenience in deriving the PD estimator for the SABR model since we use extensively the inverse transform method to simulate random variables in the aforementioned exact simulation scheme.

3.2 Unbiased Sensitivity to ν

Consider the call option $c(f,\alpha)$ (cf. (3)). Conditioning on α_T and $\int_0^T \alpha_s^2 ds$, we have the following representation:

$$c(f,\alpha) = \mathbb{E}\left[\psi\left(\nu;\alpha_T,\int_0^T \alpha_s^2 ds\right)\Big|F_0 = f,\alpha_0 = \alpha\right],$$

where

$$\psi(\nu;\alpha_T,\int_0^T \alpha_s^2 ds) = \mathbb{E}\left[(F_T - K)^+\Big|\alpha_T,\int_0^T \alpha_s^2 ds\right].$$

Note that F_T, α_T and $\int_0^T \alpha_s^2$ are all influenced by the vol of vol ν. We include it into the arguments of the function ψ to emphasize this dependency. We shall study the PD estimator to the sensitivity of c with respect to ν in this section. Taking derivatives under the expectation, we have

$$\frac{\partial c}{\partial \nu} = \mathbb{E}\left[\frac{\partial \psi}{\partial \nu} + \frac{\partial \psi}{\partial[\alpha_T]}\cdot\frac{\partial \alpha_T}{\partial \nu} + \frac{\partial \psi}{\partial[\int_0^T \alpha_s^2 ds]}\cdot\frac{\partial[\int_0^T \alpha_s^2 ds]}{\partial \nu}\Big|F_0 = f,\alpha_0 = \alpha\right]. \tag{8}$$

For $\partial\psi/\partial\nu$, $\partial\psi/\partial\alpha_T$ and $\partial\psi/\partial\int_0^T \alpha_s^2 ds$ inside the expectation of (8), the following theorem states unbiased estimators for them. We can easily establish it through the technique of derivative of random variables.

Theorem 3 Introduce a dummy variable Θ. The derivatives $\partial\psi/\partial\nu$, $\partial\psi/\partial\alpha_T$ and $\partial\psi/\partial\int_0^T \alpha_s^2 ds$ admit a uniform representation such as

$$\frac{\partial \psi}{\partial \Theta} = -\mathbb{E}\left[1_{\{F_T\geq K;\ Q(A;2-\delta,C(F_T))<q(A;2-\delta)\}}\cdot\left(\frac{\partial Q(A;2-\delta,C(u))/\partial\Theta}{\partial Q(A;2-\delta,C(u))/\partial u}\right)\Big|_{u=F_T}|\nu;\alpha_T,\int_0^T \alpha_s^2 ds\right].$$

In particular, when $\Theta = \nu$,

$$\begin{aligned}&\left(\frac{\partial Q(A;2-\delta,C(u))/\partial\nu}{\partial Q(A;2-\delta,C(u))/\partial u}\right)\Big|_{u=F_T}\\ =\ &\frac{2(\beta-1)\rho(\alpha_T-\alpha_0)f(A;2-\delta,C(F_T))}{(Q(A;4-\delta,C(F_T))-Q(A;2-\delta,C(F_T)))F_T^{1-2\beta}\nu^2}\cdot\left(\frac{F_0^{1-\beta}}{1-\beta}+\frac{\rho}{\nu}(\alpha_T-\alpha_0)\right);\end{aligned}$$

when $\Theta = \alpha_T$,

$$\begin{aligned}&\left(\frac{\partial Q(A;2-\delta,C(u))/\partial[\alpha_T]}{\partial Q(A;2-\delta,C(u))/\partial u}\right)\Big|_{u=F_T}\\ =\ &\frac{2(1-\beta)\rho f(A;2-\delta,C(F_T))}{(Q(A;4-\delta,C(F_T))-Q(A;2-\delta,C(F_T)))F_T^{1-2\beta}\nu}\cdot\left(\frac{F_0^{1-\beta}}{1-\beta}+\frac{\rho}{\nu}(\alpha_T-\alpha_0)\right);\end{aligned}$$

and when $\Theta = \int_0^T \alpha_s^2 ds$,

$$\begin{aligned}&\left(\frac{\partial Q(A;2-\delta,C(u))/\partial[\int_0^T \alpha_s^2 ds]}{\partial Q(A;2-\delta,C(u))/\partial u}\right)\Big|_{u=F_T}\\ =\ &\frac{(\beta-1)(1-\rho^2)}{F_T^{1-2\beta}}\cdot\left(\frac{f(A;2-\delta,C(F_T))\cdot A}{(Q(A;4-\delta,C(F_T))-Q(A;2-\delta,C(F_T)))}+\frac{C(F_T)}{2}\right).\end{aligned}$$

Next, we consider $\partial\alpha_T/\partial\nu$ and $\partial[\int_0^T \alpha_s^2 ds]/\partial\nu$. The derivation of the first one is straightforward. Thanks to Eq. (4),

$$\frac{\partial\alpha_T}{\partial\nu} = \alpha_T \cdot (-\nu T + \nu\sqrt{T}Z). \tag{9}$$

As for the latter, recall that we simulate it through the distribution of $h(\int_0^T \alpha_s^2 ds)$, with h being a reciprocal function, and observe the relationship

$$\frac{\partial[\int_0^T \alpha_s^2 ds]}{\partial\nu} = -\left(\int_0^T \alpha_s^2 ds\right)^2 \cdot \frac{\partial[h(\int_0^T \alpha_s^2 ds)]}{\partial\nu}.$$

Therefore it is sufficient that we know how to obtain $\partial[h(\int_0^T \alpha_s^2 ds)]/\partial\nu$. Applying the method of derivatives of random variables yields

$$\frac{\partial[h(\int_0^T \alpha_s^2 ds)]}{\partial\nu} = -\frac{\dot{F}_h(y)}{f_h(y)}\Big|_{y=h(\int_0^T \alpha_s^2 ds)}, \tag{10}$$

where $f_h(y)$ is the probability density function of $h(\int_0^T \alpha_s^2 ds)$ and

$$\dot{F}_h(y) = \frac{\partial F_h(y)}{\partial\nu}.$$

From Section 2, the Laplace transform of F_h is explicitly available. Hence, the Laplace transform of f_h should be given by

$$\hat{f}_h(\theta) = \int_0^\infty e^{-\theta u} f_h(u) du = \int_0^\infty e^{-\theta u} dF_h(u) = \theta \cdot \hat{F}_h(\theta),$$

where the third equality uses integration-by-parts. Meanwhile, Lemma 1 in Glasserman and Liu (2010) implies that the Laplace transform of $\dot{F}_h(y)$ can be expressed in the form

$$\hat{\dot{F}}_h(\theta) := \int_0^\infty e^{-\theta u} \dot{F}_h(u) du = \frac{\partial}{\partial\nu}\hat{F}_h(\theta).$$

In summary, to evaluate the right hand side of (10), we apply the numerical inversion method of Abate and Whitt (1992) on $\theta\hat{F}_h(\theta)$ and $\partial\hat{F}_h(\theta)/\partial\nu$ to produce f_h and $\dot{F}_h$.

At the end of this section, we present a complete description about the exact simulation of the SABR model and its accompanying PD estimation of ν:

1. Simulate α_T from α_0 and at the same time use (9) to obtain $\partial\alpha_T/\partial\nu$.
2. Given α_T, generate a sample of $h(\int_0^T \alpha_s^2 ds)$ and take h^{-1} on it to get $\int_0^T \alpha_s^2 ds$. Substitute $h(\int_0^T \alpha_s^2 ds)$ to the right hand side of (10) to evaluate $\partial[h(\int_0^T \alpha_s^2 ds)]/\partial\nu$.
3. Simulate F_T. Evaluate $(F_T - K)^+$ and

$$\frac{\partial\psi}{\partial\nu} + \frac{\partial\psi}{\partial[\alpha_T]} \cdot \frac{\partial\alpha_T}{\partial\nu} + \frac{\partial\psi}{\partial[\int_0^T \alpha_s^2 ds]} \cdot \frac{\partial[\int_0^T \alpha_s^2 ds]}{\partial\nu}$$

   by Theorem 3.

Averaging across a large number of samples, we can obtain unbiased estimators for both price and sensitivity.

4 NUMERICAL EXAMPLES

In this section, we conduct some numerical experiments to illustrate the accuracy of our method. The following three tables show that the estimator developed in the last section performs well under a wide range of parameters. Especially, Table 1, 2, and 3 test the performance of our estimators under different values of ν, β, and α_0, respectively. The codes for all experiments were written in MATLAB 7.9.0 (R2009b), and were implemented on a PC desktop with an Intel core2 Q9400 2.66GHZ processor.

For the purpose of comparison, we use the closed-form formula in Hagan et al. (2002) as our benchmark to price a European call option. In the meanwhile, differentiating the formula with respect to ν produces an approximation to the sensitivity. We document the outcomes in the columns under the "Analytical" category in the tables. We also compare our results with those given by the Euler discretization method, which are shown in the "Euler" columns.

One interesting common feature in these experiments is that, although the Euler discretization scheme can generate comparable results to price the European option, it performs very poorly in the sensitivity estimation. We need to investigate this more deeply in our future work.

Table 1: Price and Sensitivity with respect to ν for a European Call Option with strike price $K = 100$ under different values of ν. The other parameters are $F_0 = 100$, $\rho = -0.2$, $\beta = 0.8$, $\alpha_0 = 0.3$, and $T = 0.75$. We simulate 100,000 sample paths for both Euler and our exact simulation methods. The numbers in the parentheses are the standard errors of Monte Carlo.

	Analytical		Euler		Exact	
ν	Price	Greek	Price	Greek	Price	Greek
0.2	4.1313	0.0821	4.1134(0.0198)	0.0007(0.0006)	4.1337(0.0197)	0.0827(0.0123)
0.5	4.1777	0.2273	4.1984(0.0203)	0.0004(0.0007)	4.1821(0.0203)	0.2178(0.0157)
0.8	4.2677	0.3725	4.2697(0.022)	0.0013(0.0007)	4.2659(0.0204)	0.3621(0.0202)

Table 2: Price and Sensitivity with respect to ν for a European Call Option with strike price $K = 100$ under different values of β. The other parameters are identical to Table 1.

	Analytical		Euler		Exact	
β	Price	Greek	Price	Greek	Price	Greek
0.2	0.261	0.0061	0.2625(0.0012)	0.0001(0.00004)	0.262(0.0012)	0.0062(0.0007)
0.5	1.0388	0.0238	1.0338(0.0048)	-0.0001(0.0002)	1.0373(0.0048)	0.0251(0.0029)
0.8	4.1313	0.0821	4.1134(0.0198)	0.0007(0.0006)	4.1337(0.0197)	0.0827(0.0123)

Table 3: Price and Sensitivity with respect to ν for a European Call Option with strike price $K = 100$ under different values of α_0. The other parameters are identical to Table 1.

	Analytical		Euler		Exact	
α_0	Price	Greek	Price	Greek	Price	Greek
0.3	4.1313	0.0821	4.1134(0.0198)	0.0007(0.0006)	4.1337(0.0197)	0.0827(0.0123)
0.6	8.246	0.1341	8.1424(0.0415)	0.0018(0.0014)	8.2038(0.0414)	0.1454(0.0266)
0.8	10.9749	0.152	10.6697(0.057)	0.0008(0.0019)	10.9841(0.0568)	0.1392(0.0347)

A PROOF OF THEOREM 3

Proof. From the generation algorithm for F_T in Section 2.2.3, we can see:

$$F_T = 0 \cdot 1_{\{U \le 1-q(A;2-\delta)\}} + \hat{U} \cdot 1_{\{U > 1-q(A;2-\delta)\}},$$

where $\hat{U}$ satisfies equation (6). Moreover, the fact that

$$1 - Q(A; 2-\delta, C(0)) = 1 - q(A; 2-\delta)$$

implies the continuity of F_T, as a function of U, at $U = 1 - q(A; 2-\delta)$. Therefore,

$$\frac{\partial F_T}{\partial \Theta} = \frac{\partial \hat{U}}{\partial \Theta} \cdot 1_{\{U > 1-q(A;2-\delta)\}}. \tag{11}$$

Applying the implicit function theorem to equation (6), we have

$$\frac{\partial \hat{U}}{\partial \Theta} = -\frac{\partial Q(A; 2-\delta, C(u))/\partial \Theta}{\partial Q(A; 2-\delta, C(u))/\partial u}\Big|_{u=\hat{U}}. \tag{12}$$

For the dummy variable Θ, if we differentiate under the expectation sign,

$$\frac{\partial \psi}{\partial \Theta} = \frac{\partial}{\partial \Theta}\mathbb{E}\left[(F_T - K)^+ \Big| \alpha_T, \int_0^T \alpha_s^2 ds\right] = \mathbb{E}\left[1_{\{F_T \ge K\}} \cdot \frac{\partial F_T}{\partial \Theta}\right].$$

Combining this with (11) and (12), we can easily show the first half of the theorem statements. Some straightforward computation can lead to the results in the second half. We omit them in the interest of space.

B NONCENTRAL CHI-SQUARE DISTRIBUTIONS

We use the concept of noncentral chi-square distributions in several points of the paper. Here we introduce it briefly. A noncentral chi-square random variable $\chi^2(\mu;\lambda)$ with μ degrees of freedom and noncentrality parameter λ has probability density function

$$f(x; \mu, \lambda) = \frac{1}{2}\exp\left(-\frac{x+\lambda}{2}\right)\left(\frac{x}{\lambda}\right)^{\frac{\mu-2}{4}} I_{\frac{\mu}{2}-1}\left(\sqrt{\lambda x}\right)$$

for $x > 0$, where I_a is the *modified Bessel function of the first kind* given by

$$I_a(x) = \sum_{k=0}^{+\infty} \frac{(x/2)^{a+2k}}{k!\Gamma(a+k+1)}$$

and Γ denotes the gamma function. Its cumulative distribution function is given by

$$Q(x; \mu, \lambda) = P[\chi^2(\mu;\lambda) \le x] = \int_0^x f(y; \mu, \lambda) dy.$$

The central chi-square random variable $\chi^2(\mu)$ is just a special example of the noncentral chi-square random variable when $\lambda = 0$. Its distribution density is given by

$$g(x; \mu) = \frac{e^{-x/2} x^{(\mu/2)-1}}{2^{\mu}\Gamma(\mu/2)}.$$

and its cumulative distribution functions of χ_μ^2 is

$$q(x; \mu) = P[\chi^2(\mu) \le x] = \int_0^x g(y; \mu) dy.$$

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