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IIMPORTANCE SAMPLING FOR RISK CONTRIBUTIONS OF CREDIT PORTFOLIOS

Guangwu Liu

Department of Management Sciences
City University of Hong Kong
Tat Chee Avenue, Kowloon Tong, Hong Kong, CHINA

ABSTRACT

Value-at-Risk is often used as a risk measure of credit portfolios, and it can be decomposed into a sum of risk contributions associated with individual obligors. These risk contributions play an important role in risk management of credit portfolios. They can be used to measure risk-adjusted performances of subportfolios and to allocate risk capital. Mathematically, risk contributions can be represented as conditional expectations, which are conditioned on rare events. In this paper, we develop a restricted importance sampling (IS) method for simulating risk contributions, and devise estimators whose mean square errors converge in a rate of n^{-1}. Furthermore, we combine our method with the IS method in the literature to improve the efficiency of the estimators. Numerical examples show that the proposed method works quite well.

1 INTRODUCTION

Risk management of credit portfolios requires not only the measurement of portfolio credit risk, e.g., Value-at-Risk (VaR), but also the measurement of the contribution of each subportfolio to the overall risk. These risk contributions play an important role in risk capital allocation and the measurement of risk-adjusted performances of subportfolios. For more details of risk contributions, interested readers are referred to Martin, Thompson, and Brown (2001), Kalkbrener, Lotter, and Overbeck (2004), Mausser and Rosen (2008) and Glasserman (2008).

Several risk measures have been proposed for portfolio credit risk, including VaR, expected shortfall and many others. In this paper we focus exclusively on risk contributions associated with VaR, which is widely used in the financial industry, for instance, VaR is the risk measure adopted by the 2003 Basel Capital Accord (Basel Committee on Bank Supervision 2003).

While measuring portfolio credit risk is already a difficult task, see, e.g., Glassermand and Li (2005) and Bassamboo, Juneja, and Zeevi (2008), measuring risk contributions of credit portfolios is even more computationally demanding. The major difficulty is that risk contributions are conditional expectations and they are conditioned on rare events. When the positive portfolio losses have continuous distributions, these rare events can even be probability-zero events. This feature makes it inefficient to estimate the risk contributions by ordinary Monte Carlo simulation.

To address the difficulty of simulating risk contributions, some work has been done in the literature. When the portfolio loss distributions are discrete, Glasserman (2005) developed importance sampling estimators by applying the importance sampling method proposed in Glasserman and Li (2005) with the purpose of variance reduction. When the positive losses have continuous distributions, kernel methods can be applied to estimate risk contributions, see, e.g., Gouriéroux, Laurent, and Scaillet (2000). In the continuous case, Tasche (2009) combined the kernel method with the importance sampling method of Glasserman and Li (2005) to derive more efficient estimators. In spite of the progress in the above work, however, estimating a large number of risk contributions remains a computationally demanding problem (Glasserman 2008).

In this paper we develop a new method for simulating risk contributions when the positive portfolio losses have continuous distributions. To circumvent the difficulty of simulating a conditional expectation, we develop an importance sampling method which represents the risk contribution as a ratio of two ordinary expectations. We refer to this method as *restricted importance sampling*, as the main idea is to restrict all generated observations to a specific region with probability 1. Furthermore, we combine the restricted importance sampling method with the importance sampling method of Glasserman and Li (2005) to further improve the efficiency of the estimator.

The major advantage of the restricted importance sampling method is that its mean square error converges at a rate of n^{-1}, which is the best that can be achieved. By applying the restricted importance sampling method, in principle estimating risk contributions is no more difficult than estimating ordinary expectations.

The rest of the paper is organized as follows. Section 2 reviews the background of risk contributions of credit portfolios and methods of estimating conditional expectations. In Section 3 we develop a restricted importance sampling method for portfolios with independent obligors, and then extend it to portfolios with dependent obligors in Section 4. In Section 5 we apply the importance sampling method of Glasserman and Li (2005) to improve the efficiency of the estimator. Numerical examples are provided in Section 6, followed by conclusions in Section 7.

2 BACKGROUND

One of the key issues of credit portfolio modeling is the dependence structure among obligors. For ease of exposition, all through the paper we focus mainly on the widely used normal copula model (see, e.g., Gupton, Finger, and Bhatia (1997) and Li 2000 for details). However, the method we propose can be extended to other models, e.g., the model with extremal dependence of Bassamboo, Juneja, and Zeevi (2008). We will return to a brief discussion of such extensions in Section 7.

In particular, for a credit portfolio with m obligors, we let Y_k denote the default indicator for the kth obligor, i.e., Y_k being 1 if the obligor defaults over a time horizon $[0,T]$, and being 0 otherwise. Then the dependence among Y_k's is captured by the covariance structure of a multivariate normal latent variables $X = (X_1,\ldots,X_m)$ with $Y_k = 1_{\{X_k > x_k\}}$, where x_k is chosen to match the pre-specified marginal default probability p_k of the kth obligor, and the covariance structure of X is specified by the following factor model:

$$X_k = a_{k1}Z_1 + \ldots + a_{kd}Z_d + b_k\varepsilon_k,$$

where $Z = (Z_1,\ldots,Z_d)$ denotes the systematic risk factors, $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_m)$ denotes the idiosyncratic risk factors of individual obligors, $A = (a_{ij})$ is the factor loading matrix, and $b_i = \sqrt{1-\sum_{j=1}^d a_{ij}^2}$ (we implicitly assume that $\sum_{j=1}^d a_{ij}^2 \le 1$ for all i). The normal copula model assumes that Z and ε are independent and both of them follow standard multivariate normal distributions.

Note that in the normal copula model, Y_k's are independent conditional on the systematic risk factors Z. Specifically, conditional on Z, Y_k is a Bernoulli random variable with probability of success being $p_k(Z)$, where

$$p_k(Z) = \Pr(Y_k = 1|Z) = \Phi\left(\frac{a_kZ + \Phi^{-1}(p_k)}{b_k}\right), \tag{1}$$

with $a_k = (a_{k1},\ldots,a_{kd})$, and Φ denoting the standard normal cumulative distribution function.

2.1 Risk Contributions of Credit Portfolio

In the paper, we focus on cases where the losses given default (LGDs) of obligors are continuous random variables, the same setting as in Tasche (2009). Specifically, we let a constant e_k denote the exposure at default of the kth obligor, and B_k denote the LGD where B_k is a continuous random variable with a support $(0,1)$. Then the portfolio credit loss over the time horizon is

$$L = \sum_{k=1}^m e_kB_kY_k = \sum_{k=1}^m L_k,$$

where L_k denotes $e_kB_kY_k$ for notational ease, and we assume that B_k's are independent of each other and are independent of Y_k's for simplicity.

The risk of the credit portfolio can be measured by VaR of the loss L. Essentially, the VaR at a confidence level of $100\times\alpha\%$ of the credit portfolio is the α-quantile of L. In practice, α can be 0.95, 0.99 or even 99.9%. We let $v_\alpha(L)$ denote the α-VaR of the credit portfolio.

While the estimation $v_\alpha(L)$ has been studied extensively in the literature, see, e.g., Glasserman and Li (2005) and Bassamboo, Juneja, and Zeevi (2008), in this paper we are interested in the risk contributions associated with $v_\alpha(L)$. Specifically, if we let c_k denote the marginal risk contribution of the kth obligor, then under appropriate conditions (see, e.g., Kalkbrener, Lotter, and Overbeck (2004)),

$$c_k = \mathrm{E}\left[L_k|L = v_\alpha(L)\right],$$

and hence

$$v_\alpha(L) = \sum_{k=1}^m c_k$$

is indeed a decomposition of the overall portfolio risk.

Due to the complexity of the credit portfolio model, one usually has to resort to Monte Carlo simulation to estimate c_k's. To do so, a typical two-phrase procedure (see Glasserman 2005) is to first estimate $v_\alpha(L)$, and then estimate

$\tilde{c}_k = \mathrm{E}[L_k | L = \hat{v}_\alpha]$ where $\hat{v}_\alpha$ denotes an estimate of $v_\alpha(L)$. In this paper we focus on the latter phrase. Mathematically, we are interested in estimating

$$\gamma(y) = \mathrm{E}[L_m | L = y],$$

for a given threshold value y, where without loss of generality we assume that $y < \sum_{k=1}^m e_k$. Note that we consider the risk contribution of the last obligor just for expositional ease. Risk contributions of other obligors can be handled in a similar manner.

2.2 Estimating Conditional Expectations

Recall that the quantity $\gamma(y)$ we want to estimate is a conditional expectation, whose definition is somewhat ambiguous when $L = y$ is a probability-zero event. This is indeed the case in our context where LGDs are continuous random variables, and hence L may have a point mass at $L = 0$ and a density function for $L > 0$.

Since $y > 0$, all through the paper we assume that L has a continuous and positive density in a neighborhood of y. Then it is mathematically rigorous to define $\gamma(y)$ as (Durrett 2005)

$$\gamma(y) = \mathrm{E}[L_m | L = y] := \lim_{\delta \to 0} \frac{\mathrm{E}\left(L_m \cdot 1_{\{y-\delta \le L \le y+\delta\}}\right)}{\mathrm{E}\left(1_{\{y-\delta \le L \le y+\delta\}}\right)}. \quad (2)$$

By Equation (2), one may choose a small δ and then estimate $\mathrm{E}\left(L_m \cdot 1_{\{y-\delta \le L \le y+\delta\}}\right) / \mathrm{E}\left(1_{\{y-\delta \le L \le y+\delta\}}\right)$ as an approximation of $\gamma(y)$. This is the essential idea of the kernel method (see, e.g., Gouriéroux, Laurent, and Scaillet (2000) and Tasche (2009)). Given n identically and independently distributed (i.i.d.) observations of (L_m, L), an estimator of $\gamma(y)$ is

$$\bar{M}_n = \frac{\sum_{i=1}^n L_m^i \cdot 1_{\{y-\delta \le L^i \le y+\delta\}}}{\sum_{i=1}^n 1_{\{y-\delta \le L^i \le y+\delta\}}}, \quad (3)$$

for some small $\delta > 0$, where (L_m^i, L^i) denotes the ith observations of (L_m, L).

Under appropriate conditions, it has been shown that (see, e.g., Bosq 1998 and Pagan and Ullah 1999) the estimator $\bar{M}_n$ has a bias of order $O(\delta^2)$, and a variance of order $O(1/(n\delta))$[1]. Then it can be easily seen that the optimal δ is of order $O(n^{-1/5})$ and the optimal convergence rate of mean square error is of order $O(n^{-4/5})$, which is slower than the n^{-1} convergence rate of a typical Monte Carlo estimator.

The major disadvantages of $\bar{M}_n$ include its reliance on an appropriate selection of δ, and its slower convergence rate. As a remedy, in the rest of this paper we propose an importance sampling method based on which we can devise more efficient estimators that do not suffer from the above disadvantages.

3 RESTRICTED IMPORTANCE SAMPLING FOR INDEPENDENT OBLIGORS

To illustrate the main idea, we first consider a simple case where the default indicators Y_i's are independent Bernoulli random variables, i.e., corresponding to the factor loading matrix being $A = 0$ in the normal copula model.

By the representation in Equation (2), we can see that the major difficulty of estimating $\gamma(y)$ is that for a small δ, very few observations of L fall into the important region $\{y - \delta \le L \le y + \delta\}$. This is the case especially when the default probabilities p_k's are very small, say, e.g., 2% or even 0.1%, while the threshold value y is relatively large. In principle, number of observations that fall into the important region is of order $O(n\delta)$. This is an intuitive explanation why the variance of $\bar{M}_n$ is of order $O(1/(n\delta))$.

To have more observations in the important region, Tasche (2009) applied the importance sampling method proposed in Glasserman and Li (2005) so that under the new probability measure L has a mean of y. It has been shown in Tasche (2009) that such a method may reduce the variance of the estimator significantly. However, in general this method has a slower convergence rate than n^{-1}.

To accelerate the convergence rate, we propose to first twist the probability measure such that under the new measure all observations generated fall into the important region $\{y - \delta \le L \le y + \delta\}$, and then use the corresponding likelihood ratio to correct the estimate. By using importance sampling in such a manner we restrict all the generated observations of L to the important region. That is why we call it *restricted importance sampling*.

[1] In the paper we use the notation $o(\cdot)$ and $O(\cdot)$ with the meaning that $a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$, and $a_n = O(b_n)$ if $\limsup_n a_n/b_n < C$ for some constant C.

3.1 Restricted Importance Sampling

To be practically meaningful, we assume that $y-\delta>0$. Then, to restrict all the generated observations of L to the important region $\{y-\delta \le L \le y+\delta\}$, we propose the following importance sampling scheme.

Importance Sampling Scheme for Independent Obligors

1. Generate U_k uniformly for $k=1,\ldots,m$.
2. For $k=1,\ldots,m$, let

$$\tilde{p}_k = 1_{\{\sum_{i=1}^{k-1} e_i Y_i + \sum_{j=k+1}^{m} e_j < y-\delta\}} + p_k 1_{\{\sum_{i=1}^{k-1} e_i Y_i + \sum_{j=k+1}^{m} e_j \ge y-\delta\}},$$

   and $Y_k = 1_{\{U_k \le \tilde{p}_k\}}$.
3. Let $Y_{(1)},\ldots,Y_{(\tau)}$ denote the nonzero Y_k's, and $e_{(1)},\ldots,e_{(\tau)}$ and $B_{(1)},\ldots,B_{(\tau)}$ denote the corresponding exposures at default and LGDs respectively. For $k=1,\ldots,\tau$, let

$$l_k = \max\left(\frac{y-\delta-\sum_{i=1}^{k-1} e_{(i)}B_{(i)} - \sum_{j=k+1}^{\tau} e_{(j)}}{e_{(k)}}, 0\right), \quad u_k = \min\left(\frac{y+\delta-\sum_{i=1}^{k-1} e_{(i)}B_{(i)}}{e_{(k)}}, 1\right).$$

   Generate $B_{(k)}$ according to the density function

$$\tilde{g}_{(k)}(x) = \frac{g_{(k)}(x)}{G_{(k)}(u_k) - G_{(k)}(l_k)} 1_{\{l_k \le x \le u_k\}},$$

   where g_k and G_k denote respectively the probability density function and cumulative distribution function of B_k.
4. Calculate the likelihood ratio H:

$$H = \prod_{k=1}^{m} \left(p_k 1_{\{\tilde{p}_k=1\}} + 1_{\{\tilde{p}_k \ne 1\}}\right) \prod_{j=1}^{\tau} \left(G_{(j)}(u_j) - G_{(j)}(l_j)\right).$$

Step 2 in the above importance sampling scheme guarantees that

$$L = \sum_{k=1}^{m} e_k Y_k \ge y-\delta \quad \text{with probability 1.}$$

Specifically, this result is summarized in the following lemma, whose proof is omitted.

Lemma 1. *If $\tilde{p}_k = 1$ whenever $\sum_{i=1}^{k-1} e_i Y_i + \sum_{j=k+1}^{m} e_j < y+\delta$ for $k=1,\ldots,m$, then*

$$\widetilde{\Pr}\{L \ge y-\delta\} = 1$$

where $\widetilde{\Pr}$ denotes the probability under the IS measure.

Note that under the IS measure τ is always greater or equal to 1, since $y-\delta>0$. Then, Step 3 guarantees that under the IS measure,

$$L = \sum_{k=1}^{m} e_k B_k Y_k \in [y-\delta, y+\delta] \quad \text{with probability 1.}$$

Step 4 calculates the likelihood ratio of the IS measure. For details of how to calculate likelihood ratios for an importance sampling scheme, interested readers are referred to Asmussen and Glynn (2007).

Let E_Q denote the expectation under the IS scheme. Based on the construction of the IS scheme, an immediate result is summarized in the following theorem.

Theorem 1. *Let $r(L_1,\ldots,L_m)$ be a function of $(L_1,\ldots,L_m)$. For a given $\delta>0$, if $r(L_1,\ldots,L_m)=0$ whenever $L \notin [y-\delta, y+\delta]$, then*

$$\mathrm{E}\left[r(L_1,\ldots,L_m)\right] = \mathrm{E}_Q\left[r(L_1,\ldots,L_m)\cdot H\right].$$

Moreover, under the IS measure, $L \in [y-\delta, y+\delta]$ with probability 1.

By Theorem 1, we immediately obtain that

$$\mathrm{E}\left[L_m \cdot 1_{\{y-\delta \le L \le y+\delta\}}\right] = \mathrm{E}_Q\left(L_m \cdot H\right), \quad \text{and} \quad \mathrm{E}\left[1_{\{y-\delta \le L \le y+\delta\}}\right] = \mathrm{E}_Q\left(H\right).$$

Then by Equation (2), we may rewrite the conditional expectation $\gamma(y)$ as

$$\gamma(y) = \lim_{\delta \to 0} \frac{\mathrm{E}_Q(L_m \cdot H)}{\mathrm{E}_Q\left(H\right)}. \tag{4}$$

The right-hand-side of Equation (4) still involves a limit operator. But fortunately, we will show that this limit operator can actually be removed.

First, it can be easily seen that letting $\delta \to 0$ in the IS scheme is equivalent to replacing $\tilde{p}_k$, Y_k, l_k and u_k respectively by

$$\bar{p}_k = 1_{\{\sum_{i=1}^{k-1} e_i \bar{Y}_i + \sum_{j=k+1}^{m} e_j < y\}} + p_k 1_{\{\sum_{i=1}^{k-1} e_i \bar{Y}_i + \sum_{j=k+1}^{m} e_j \ge y\}},$$

$\bar{Y}_k = 1_{\{U_k \le \bar{p}_k\}}$, and

$$\bar{l}_k = \max\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)} \bar{B}_{(i)} - \sum_{j=k+1}^{\tau} e_{(j)}}{e_{(k)}}, 0\right), \quad \bar{u}_k = \min\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)} \bar{B}_{(i)}}{e_{(k)}}, 1\right),$$

where $\bar{B}_{(k)}$ is generated according to the density function $\bar{g}_{(k)}(x) = \frac{g_{(k)}(x)}{G_{(k)}(\bar{u}_k) - G_{(k)}(\bar{l}_k)} 1_{\{\bar{l}_k \le x \le \bar{u}_k\}}$.

Second, we can show that $H/(2\delta)$ has a non-degenerate limit as $\delta \to 0$, which is summarized in the following lemma.

Lemma 2. *Suppose that for $k = 1, \ldots, m$, $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then with probability 1,*

$$\lim_{\delta \to 0} \frac{H}{2\delta} = H_1 := \prod_{k=1}^{m} \left(p_k 1_{\{\bar{p}_k = 1\}} + 1_{\{\bar{p}_k \ne 1\}}\right) \prod_{j=1}^{\tau-1} \left(G_{(j)}(\bar{u}_j) - G_{(j)}(\bar{l}_j)\right) g_{(\tau)} \left(\frac{y - \sum_{i=1}^{\tau-1} e_{(i)} \bar{B}_{(i)}}{e_{(\tau)}}\right) \frac{1}{e_{(\tau)}}.$$

By observing that

$$\lim_{\delta \to 0} \frac{\mathrm{E}_Q(L_m \cdot H)}{\mathrm{E}_Q(H)} = \lim_{\delta \to 0} \frac{\frac{1}{2\delta} \mathrm{E}_Q(L_m \cdot H)}{\frac{1}{2\delta} \mathrm{E}_Q\left(H\right)},$$

we immediately establish the following limit based on Lemma 2.

Theorem 2. *Suppose that for $k = 1, \ldots, m$, $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then*

$$\gamma(y) = \lim_{\delta \to 0} \frac{\mathrm{E}_Q(L_m \cdot H)}{\mathrm{E}_Q(H)} = \frac{\mathrm{E}_Q(L_m \cdot H_1)}{\mathrm{E}_Q(H_1)},$$

where H_1 is defined in Lemma 2.

The proof of Theorem 2 is straightforward by applying the dominated convergence theorem (Durrett 2005) and is hence omitted. Theorem 2 represents $\gamma(y)$ as a ratio of two expectations, which do not involve δ. Therefore, during the implementation, we simply simulate the portfolio loss by setting $\delta = 0$ in the importance sampling scheme. Then a straightforward estimator of $\gamma(y)$ can be easily derived:

$$\bar{M}_n^{IS} = \frac{\sum_{i=1}^{n} L_m^i H_1^i}{\sum_{i=1}^{n} H_1^i},$$

where (L_m^i, L^i, H_1^i) denotes the ith observation of (L_m, L, H_1).

Remark 1. *In the IS scheme, we may replace p_k by any $\hat{p}_k \in (0,1)$, and correspondingly replace the term $p_k 1_{\{\bar{p}_k = 1\}} + 1_{\{\bar{p}_k \ne 1\}}$ in the likelihood ratio by*

$$p_k 1_{\{\bar{p}_k = 1\}} + \left(\frac{p_k}{\hat{p}_k}\right)^{Y_k} \left(\frac{1 - p_k}{1 - \hat{p}_k}\right)^{1 - Y_k} 1_{\{\bar{p}_k \ne 1\}}.$$

However, one needs to select $\hat{p}_k$ carefully, since an arbitrary choice of $\hat{p}_k$ may lead to a large variance of the estimator. We will return to the question of how to select an appropriate $\hat{p}_k$ in Section 5.

3.2 Asymptotic Properties

In this subsection we analyze the asymptotic properties of the proposed estimator $\bar{M}_n^{IS}$, and show that its mean square error has a convergence rate of n^{-1}.

We let $f(\cdot)$ denote the density function of L for $L > 0$, and define

$$S_n = \frac{1}{\gamma(y)f(y)}\left(\frac{1}{n}\sum_{i=1}^{n} L_m^i H_1^i - \gamma(y)f(y)\right), \quad \text{and} \quad V_n = \frac{1}{f(y)}\left(\frac{1}{n}\sum_{i=1}^{n} H_1^i - f(y)\right).$$

Then it can be easily verified that

$$\bar{M}_n^{IS} = \gamma(y)\frac{1+S_n}{1+V_n} = \gamma(y)\left(1 + S_n - V_n + V_n^2 - S_nV_n + R_n\right), \tag{5}$$

where $R_n = \frac{V_n^2}{\gamma(y)}(\bar{M}_n - \gamma(y))$.

Note that $\bar{M}_n^{IS}$ is a ratio of two sample means, and asymptotic properties of such ratio estimators have been studied extensively in the literature of statistics, see, e.g., David and Sukhatme (1974) and the references therein. In particular, the asymptotic properties of $\bar{M}_n^{IS}$ are summarized in the following proposition.

Proposition 1. *Suppose that for $k = 1,\ldots,m$, $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then*

$$\sqrt{n}(\bar{M}_n^{IS} - \gamma(y)) \Rightarrow \sigma N(0,1),$$

where "$\Rightarrow$" denotes the convergence in distribution, $N(0,1)$ denotes a standard normal distribution, and $\sigma^2 = \frac{1}{f^2(y)}\left\{\mathrm{E}_Q(L_m^2 H_1^2) + \gamma^2(y)\mathrm{E}_Q(H_1^2) - 2\gamma(y)\mathrm{E}_Q(L_m H_1^2)\right\}$.

Furthermore, if the sequences $\{n(\bar{M}_n^{IS} - \gamma(y))^2, n \geq 1\}$ and $\{nR_n, n \geq 1\}$ are uniformly integrable, then the mean square error of $\bar{M}_n^{IS}$ is

$$\mathrm{MSE}(\bar{M}_n^{IS}) = \frac{\sigma^2 + o(1)}{n}.$$

Proposition 1 shows that the convergence rate of the mean square error of $\bar{M}_n^{IS}$ is n^{-1}, which is the best that can be achieved in a Monte Carlo simulation.

4 RESTRICTED IMPORTANCE SAMPLING FOR DEPENDENT OBLIGORS

In Section 3, we describe a restricted importance sampling method for credit portfolios with independent obligors. The extension to dependent obligors is straightforward by observing that Y_k's are independent conditional on the systematic risk factors Z. We may first generate Z, and then for any given Z, we apply the restricted importance sampling method of Section 3. In particular, the IS scheme is listed as follows:

Importance Sampling Scheme for Dependent Obligors

1. Generate the systematic risk factors Z.
2. Generate U_k uniformly for $k = 1,\ldots,m$.
3. For $k = 1,\ldots,m$, let

$$\bar{p}_k(Z) = 1_{\{\sum_{i=1}^{k-1} e_i\bar{Y}_i + \sum_{j=k+1}^{m} e_j < y\}} + p_k(Z)1_{\{\sum_{i=1}^{k-1} e_i\bar{Y}_i + \sum_{j=k+1}^{m} e_j \geq y\}},$$

where $p_k(Z)$ is defined in Equation (1). Let $\bar{Y}_k = 1_{\{U_k \leq \bar{p}_k(Z)\}}$.

4. Let $\bar{Y}_{(1)},\ldots,\bar{Y}_{(\tau)}$ denote the nonzero $\bar{Y}_k$'s, and $e_{(1)},\ldots,e_{(\tau)}$ and $\bar{B}_{(1)},\ldots,\bar{B}_{(\tau)}$ denote the corresponding credit exposures and LGDs respectively. For $k = 1,\ldots,\tau-1$, let

$$\bar{l}_k = \max\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)}\bar{B}_{(i)} - \sum_{j=k+1}^{\tau} e_{(j)}}{e_{(k)}}, 0\right), \quad \bar{u}_k = \min\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)}\bar{B}_{(i)}}{e_{(k)}}, 1\right).$$

Generate $\bar{B}_{(k)}$ according to the density function

$$\bar{g}_{(k)}(x) = \frac{g_{(k)}(x)}{G_{(k)}(\bar{u}_k) - G_{(k)}(\bar{l}_k)} 1_{\{l_k \le x \le u_k\}}.$$

5. Calculate the likelihood ratio $H_1(Z)$:

$$H_1(Z) = \prod_{k=1}^{m} \left(p_k(Z) 1_{\{\bar{p}_k(Z)=1\}} + 1_{\{\bar{p}_k(Z)\neq 1\}} \right) \prod_{j=1}^{\tau-1} \left(G_{(j)}(\bar{u}_j) - G_{(j)}(\bar{l}_j) \right) g_{(\tau)} \left(\frac{y - \sum_{i=1}^{\tau-1} e_{(i)} \bar{B}_{(i)}}{e_{(\tau)}} \right) \frac{1}{e_{(\tau)}}.$$

The validation of the importance sampling scheme is a straightforward extension of that in Section 3. In particular, the theoretical support of the importance sampling scheme for dependent obligors is summarized in the following theorem whose proof is omitted.

Theorem 3. *Suppose that for $k = 1, \ldots, m$, $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then*

$$\gamma(y) = \frac{\mathrm{E}_Q(L_m \cdot H_1(Z))}{\mathrm{E}_Q(H_1(Z))}.$$

Then if n observations based on the IS scheme is generated, an estimator of $\gamma(y)$ is

$$\bar{M}_n^{ISD} = \frac{\sum_{i=1}^{n} L_m^i H_1^i(Z^i)}{\sum_{i=1}^{n} H_1^i(Z^i)},$$

where $(Z^i, H_1^i(Z^i))$ denotes the ith observation of $(Z, H_1(Z))$.

Similar to the analysis of Section 3, asymptotic properties of $\bar{M}_n^{ISD}$ are summarized in the following proposition, whose proof is similar to that of Proposition 1 and is hence omitted.

Proposition 2. *Suppose that for $k = 1, \ldots, m$, $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then*

$$\sqrt{n}(\bar{M}_n^{ISD} - \gamma(y)) \Rightarrow \sigma_1 N(0,1),$$

where $\sigma_1^2 = \frac{1}{f^2(y)} \left\{ \mathrm{E}_Q(L_m^2 H_1^2(Z)) + \gamma^2(y) \mathrm{E}_Q(H_1^2(Z)) - 2\gamma(y) \mathrm{E}_Q(L_m H_1^2(Z)) \right\}$.

Furthermore, if the sequences $\{n(\bar{M}_n^{ISD} - \gamma(y))^2, n \ge 1\}$ and $\{nR_n, n \ge 1\}$ are uniformly integrable, then the mean square error is

$$\mathrm{MSE}(\bar{M}_n^{ISD}) = \frac{\sigma_1^2 + o(1)}{n}.$$

5 IMPROVEMENT OF EFFICIENCY

In Sections 3 and 4 we propose a restricted importance sampling method for simulating risk contributions of credit portfolios with independent and dependent obligors respectively. The proposed method restricts the simulated portfolio loss to the important region such that all observations generated are useful in the estimation. By doing so the method enjoys a convergence rate of n^{-1}, which is the major advantage of the method.

Intuitively, under the original measure without importance sampling, an observation of L may lie somewhere far from y, while under the IS measure we force it to lie in a small neighborhood of y, and then use the likelihood ratio to correct the estimate. Then one may expect that if under the original measure most observations of L lie far from y then the variance of the likelihood ratio could be large, and hence leads to a large variance of the estimator. To reduce the variance, we propose a two-step importance sampling procedure, where in the first step we twist the probability measure such that the portfolio loss L has a mean of y under the new measure, and then in the second step we apply the restricted importance sampling method of Sections 3 and 4. The former step is not new and has been used in Glasserman (2005) and Tasche (2009). In the rest of this section we illustrate how to apply the importance sampling method of Glasserman and Li (2005) (see, also, Glasserman, Kang, and Shahabuddin 2008) in the former step. Henceforth we refer to the IS measure of Glasserman and Li (2005) as GL measure.

Without loss of generality, we work exclusively on the cases with dependent obligors. Then following the notation in Glasserman et al. (2008), we define the cumulant generating function of $e_k B_k$ as

$$\Lambda_k(\lambda) = \log \mathrm{E}\left[e^{\lambda e_k B_k} \right].$$

Note that conditional on Z, the default indicators Y_k's are independent Bernoulli random variables. Then under GL measure, the conditional default probability $p_k(Z)$ is exponentially twisted to

$$p_{k,\theta}(Z) = \frac{p_k(Z)e^{\Lambda_k(\theta)}}{1 + p_k(Z)\left(e^{\Lambda_k(\theta)} - 1\right)}.$$

The conditional likelihood ratio associated with this change of measure is given by

$$\prod_{k=1}^{m}\left(\frac{p_k(Z)}{p_{k,\theta}(Z)}\right)^{Y_k}\left(\frac{1-p_k(Z)}{1-p_{k,\theta}(Z)}\right)^{1-Y_k} = e^{-\sum_{k=1}^{m} Y_k\Lambda_k(\theta)+m\psi(\theta,Z)},$$

where $\psi(\theta,z)$ is the conditional cumulant generating function of L divided by m, i.e.,

$$\psi(\theta,z) := \frac{1}{m}\log \mathrm{E}\left[e^{\theta L}\middle| Z = z\right] = \frac{1}{m}\sum_{k=1}^{m}\log\left(1 + p_k(Z)\left(e^{\Lambda(\theta)} - 1\right)\right).$$

For given Z and Y_k's, another exponential-twisting change of measure can be applied to B_k's. In particular, the density function of B_k under the new probability measure is given by

$$g_{k,\theta}(t) = g_k(t)e^{\theta e_k Y_k t - \Lambda_k(\theta Y_k)}. \tag{6}$$

Then given Z and Y_k's, the conditional likelihood ratio is

$$\prod_{k=1}^{m}\frac{g_k(B_k)}{g_{k,\theta}(B_k)} = \prod_{k=1}^{m} e^{-\theta e_k Y_k B_k + \Lambda_k(\theta Y_k)} = e^{-\theta\sum_{k=1}^{m} e_k Y_k B_k + \sum_{k=1}^{m}\Lambda_k(\theta Y_k)}.$$

Note that $\Lambda_k(0) = 0$. Then $Y_k\Lambda_k(\theta) = \Lambda_k(\theta Y_k)$, and hence the likelihood ratio of GL measure is

$$\begin{aligned} L_r &= e^{-\sum_{k=1}^{m} Y_k\Lambda_k(\theta)+\psi(\theta,Z)} \times e^{-\theta\sum_{k=1}^{m} e_k Y_k B_k + \sum_{k=1}^{m}\Lambda_k(\theta Y_k)} \\ &= e^{-\theta\sum_{k=1}^{m} e_k Y_k B_k + m\psi(\theta,Z)} = e^{-\theta L + m\psi(\theta,Z)}, \end{aligned}$$

where the last equality follows from the fact that $L = \sum_{k=1}^{m} e_k Y_k B_k$.

Therefore, under the GL measure, we may rewrite $\mathrm{E}\left(1_{\{y-\delta\le L\le y+\delta\}}\right)$ and $\mathrm{E}\left(L_m \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right)$ as:

$$\begin{aligned} \mathrm{E}\left(1_{\{y-\delta\le L\le y+\delta\}}\right) &= \widetilde{\mathrm{E}}_\theta\left(e^{-\theta L+m\psi(\theta,Z)} \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right), \\ \mathrm{E}\left(L_m \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right) &= \widetilde{\mathrm{E}}_\theta\left(L_m e^{-\theta L+m\psi(\theta,Z)} \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right), \end{aligned}$$

where $\widetilde{\mathrm{E}}_\theta$ denotes the expectation under the GL IS measure.

Then by Equation (2) we have the following representation of $\gamma(y)$:

$$\gamma(y) = \lim_{\delta\to 0}\frac{\widetilde{\mathrm{E}}_\theta\left(L_m e^{-\theta L+m\psi(\theta,Z)} \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right)}{\widetilde{\mathrm{E}}_\theta\left(e^{-\theta L+m\psi(\theta,Z)} \cdot 1_{\{y-\delta\le L\le y+\delta\}}\right)}. \tag{7}$$

In the GL IS measure, θ is a parameter used to control the exponential twisting. Intuitively, a good choice would be setting θ such that more observations drop into a small neighborhood of $L = y$, which is the important region in our context. Note that θ depends on Z. Then similar to Glasserman (2005), an appropriate choice of $\theta(z)$ for any given $Z = z$ is defined as

$$\theta^*(z) = \arg\min\{-\theta y + m\psi(\theta,z)\}.$$

Equivalently, $\theta^*(z)$ is set to be the unique solution of $m\partial_\theta\psi(\theta,z) = y$, which implies that $\widetilde{\mathrm{E}}_{\theta^*}[L|Z = z] = y$ (see, e.g., Glasserman 2008) .

With this choice of θ^*, we can express $\gamma(y)$ as a conditional expectation under the GL measure. This result is summarized in the following lemma.

Lemma 3. *Suppose that $\Lambda_k(\theta)$ is continuous in θ for all k. Then*

$$\gamma(y) = \frac{\widetilde{\mathrm{E}}_{\theta^*}[L_m e^{-\theta^*(Z)y + m\psi(\theta^*(Z),Z)}|L=y]}{\widetilde{\mathrm{E}}_{\theta^*}[e^{-\theta^*(Z)y + m\psi(\theta^*(Z),Z)}|L=y]}.$$

Combining the GL method with the restricted importance sampling method, the simulation procedure for estimating $\gamma(y)$ is described as follows:

Two-Step Importance Sampling Scheme

1. Generate the systematic risk factors Z. Set $\theta^*(Z)$ equal to the unique solution of $m\partial_\theta \psi(\theta, Z) = y$. We suppress the dependence of θ^* on Z when there is no confusion. Calculate

$$p_{k,\theta^*}(Z) = \frac{p_k(Z)e^{\Lambda_k(\theta^*)}}{1 + p_k(Z)\left(e^{\Lambda_k(\theta^*)} - 1\right)}.$$

2. Generate U_k uniformly for $k = 1, \ldots, m$.
3. For $k = 1, \ldots, m$, let

$$\bar{p}_{k,\theta^*}(Z) = 1_{\{\sum_{i=1}^{k-1} e_i \bar{Y}_i + \sum_{j=k+1}^{m} e_j < y\}} + p_{k,\theta^*}(Z) 1_{\{\sum_{i=1}^{k-1} e_i \bar{Y}_i + \sum_{j=k+1}^{m} e_j \geq y\}},$$

and $\bar{Y}_k = 1_{\{U_k \leq \bar{p}_{k,\theta^*}(Z)\}}$.
4. Let $\bar{Y}_{(1)}, \ldots, \bar{Y}_{(\tau)}$ denote the nonzero $\bar{Y}_k$'s, and $e_{(1)}, \ldots, e_{(\tau)}$ and $\bar{B}_{(1)}, \ldots, \bar{B}_{(\tau)}$ denote the corresponding credit exposures and LGDs respectively. For $k = 1, \ldots, \tau - 1$, let

$$\bar{l}_k = \max\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)}\bar{B}_{(i)} - \sum_{j=k+1}^{\tau} e_{(j)}}{e_{(k)}}, 0\right), \quad \bar{u}_k = \min\left(\frac{y - \sum_{i=1}^{k-1} e_{(i)}\bar{B}_{(i)}}{e_{(k)}}, 1\right).$$

Generate $\bar{B}_{(k)}$ according to the density function

$$\bar{g}_{(k),\theta^*}(x) = \frac{g_{(k),\theta^*}(x)}{G_{(k),\theta^*}(\bar{u}_k) - G_{(k),\theta^*}(\bar{l}_k)} 1_{\{l_k \leq x \leq u_k\}},$$

where $g_{k,\theta}$ is defined in Equation (6), and $G_{k,\theta}$ is the corresponding cumulative distribution function.
5. Calculate the likelihood ratio $H_2(Z)$:

$$\begin{aligned} H_2(Z) \quad = \quad & e^{-\theta^*(Z)y + m\psi(\theta^*(Z),Z)} \prod_{k=1}^{m} \left(p_{k,\theta^*}(Z) 1_{\{\bar{p}_{k,\theta^*}(Z)=1\}} + 1_{\{\bar{p}_{k,\theta^*}(Z)\neq 1\}} \right) \\ & \times \prod_{j=1}^{\tau-1} \left(G_{(j),\theta^*}(\bar{u}_j) - G_{(j),\theta^*}(\bar{l}_j) \right) g_{(\tau),\theta^*}\left(\frac{y - \sum_{i=1}^{\tau-1} e_{(i)}\bar{B}_{(i)}}{e_{(\tau)}} \right) \frac{1}{e_{(\tau)}}. \end{aligned}$$

If we generate n i.i.d. observations of $(L_m, L, H_2(Z))$ with $(L_m^i, L^i, H_2^i(Z^i))$ denoting the ith observation, then an estimator of $\gamma(y)$ is

$$\bar{M}_n^{Com} = \frac{\sum_{i=1}^{n} L_m^i H_2^i(Z^i)}{\sum_{i=1}^{m} H_2^i(Z^i)}.$$

To theoretically validate the two-step IS scheme, we need the following theorem.

Theorem 4. *Suppose that for $k = 1, \ldots, m$, $\Lambda_k(\theta)$ is continuous in θ and $g_k(x)$ is bounded by a constant C for all $x \in (0,1)$. Then*

$$\gamma(y) = \frac{\widetilde{\mathrm{E}}_Q(L_m \cdot H_2(Z))}{\widetilde{\mathrm{E}}_Q(H_2(Z))},$$

where $\widetilde{\mathrm{E}}_Q$ denotes the expectation operator under the two-step IS measure.

6 NUMERICAL EXAMPLES

In this section we illustrate the performance of the proposed method using two examples. In both example we consider a credit portfolio with 100 obligors, where the marginal default probability of each obligor is 1%, the exposure at default for each obligor is 1, and the LGD of each obligor follows a normal distribution with a mean 50% and a standard deviation 20%, truncated at $(0,1)$.

For the first example we assume that defaults of obligors are independent, while for the second example we assume that defaults follow a three-factor normal copula model, with the factor loading matrix A being randomly generated. We select the threshold value y by setting it approximately equal to the 90%, 95%, 99%, and 99.9% Value-at-Risk respectively. We let α denote the confidence level of the value-at-risk associated with which we estimate the risk contributions.

For both examples, we use a very large number (10^9) of observation to obtain an accurate estimate of $\gamma(y)$, and then use it as the true value to examine the performances of the estimators. We use the estimated relative root mean square error (RRMSE) as a benchmark, which measures the percentage of the root mean square error to the absolute value of the quantity being estimated. The RRMSE reported are based on 1,000 replications.

The numerical results for these two examples are summarized in Tables 1 and 2 respectively. From the tables we can see that the restricted importance-sampling estimator $\bar{M}_n^{IS}$ performs quite well for both examples when the sample size n is reasonably large and the threshold value is not too extreme. We can also see that the estimator ($\bar{M}_n^{Com}$) combined with the importance sampling method of Glasserman and Li (2005) indeed performs better than $\bar{M}_n^{IS}$. When we estimate the risk contributions for the α-VaR for α very close to 1, e.g., 99.9%, it could be necessary to apply $\bar{M}_n^{Com}$ rather than $\bar{M}_n^{IS}$.

Table 1: Performances of the estimators for the first example

	RRMSE (%)											
	$n=10^3$				$n=10^4$				$n=10^5$			
α (%)	90	95	99	99.9	90	95	99	99.9	90	95	99	99.9
$\bar{M}_n^{IS}$	21.1	47.1	104.2	265.2	8.5	17.1	25.4	72.5	2.6	5.3	8.9	25.0
$\bar{M}_n^{Com}$	16.4	28.1	25.3	32.2	5.9	8.7	8.6	9.6	1.8	3.0	2.6	2.9

Table 2: Performances of the estimators for the second example

	RRMSE (%)											
	$n=10^3$				$n=10^4$				$n=10^5$			
α (%)	90	95	99	99.9	90	95	99	99.9	90	95	99	99.9
$\bar{M}_n^{IS}$	45.2	69.9	115.8	238.6	15.1	20.9	35.7	73.9	4.5	7.3	10.5	26.0
$\bar{M}_n^{Com}$	41.6	55.4	96.3	135.9	12.4	18.2	26.3	50.6	4.1	5.7	8.1	16.4

7 CONCLUSIONS

In this paper we propose a restricted importance sampling method for simulating risk contributions of credit portfolios. The proposed method achieves the fastest convergence rate of n^{-1}. To improve the efficiency of the method, we combine the restricted importance sampling method with the IS method of Glasserman and Li (2005). Some preliminary examples show that the proposed method works quite well.

Although in the paper we focus on the normal copula model, the restricted importance sampling method can also be applied to other models, e.g., models with extremal dependence of Bassamboo et al. (2008). However, for these models, one may need to correspondingly change the importance sampling method in Section 5 in order to improve the efficiency. Detailed analysis of the extensions to these models is an interesting topic for future research.

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AUTHOR BIOGRAPHIES

GUANGWU LIU is an assistant professor in the Department of Management Sciences at City University of Hong Kong. His research interests include financial risk management and stochastic simulation. His e-mail address is <msgw.liu@cityu.edu.hk>.