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PATHWISE DERIVATIVE METHODS ON SINGLE-ASSET AMERICAN OPTION SENSITIVITY ESTIMATION

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ABSTRACT

In this paper, we investigate efficient Monte Carlo estimators to American option sensitivities on single asset. Using two features of the exercising boundary of the optimal stopping problem, the "continuous-fit" and "smooth-pasting" conditions, we derive unbiased pathwise estimators for first and second-order derivatives. Our method can be easily embedded into some popular algorithms for pricing one-dimensional American options. Numerical examples on vanilla puts illustrate accuracy and efficiency of the method.

1 INTRODUCTION

Options are financial assets which entitle the holder the right to buy/sell a specific underlying at a contracted price and time in the future. The vast majority of options are either European or American. An American option may be exercised at any time before its expiry date while a European-style option can only be exercised at the expiry date. Most of the exchange-traded options are American.

The evaluation of American-style options poses a challenge to the community of computational finance. Finding the option price entails solving an optimal stopping problem. This embedded optimization problem makes the pricing task a difficult problem for simulation. Several treads of research lines are suggested in the literature to tackle the problem. Fu and Hu (1995), Andersen (2000), Garcia (2003), and so on, parameterize exercise regions or stopping rules and reduce the optimal stopping problem to a much more tractable finite-dimensional optimization problem. Broadie and Glasserman (1997) and Broadie and Glasserman (2004) propose random tree and stochastic mesh methods to obtain valid confidence intervals for the American option prices. Tsitsiklis and Roy (1999) and Longstaff and Schwartz (2001) use regression to estimate continuation values from the simulated paths and then to price American options. Haugh and Kogan (2004), Rogers (2002) and Andersen and Broadie (2004) establish dual formulation of the pricing problem through which a useful approximation on upper bounds on price is obtainable. A more comprehensive review on the related literature can be found in Glasserman (2004).

This paper investigates how to develop efficient Monte Carlo estimators to the price sensitivities of American options. Price sensitivities, or "Greeks" in the jargon of option markets, reflect the derivatives of option prices with respect to the changes of parameters which affect the value of an option. They play a vital role in risk management of options. For instance, the risk in a short position in an option can be offset significantly by a delta-hedging strategy, holding delta units of underlying assets (see, e.g. Hull (2009)). Here the delta is simply the partial derivative of the option price with respect to the current price of the underlying asset. Implementation of this strategy requires inputs of the related price sensitivities. Whereas the option prices themselves can often be observed in the market, their sensitivities cannot. Therefore, accurate calculation of sensitivities is arguably even more important than calculation of prices.

Despite intensive attempts to estimating American options prices, the simulation methods on their sensitivity estimation are underdeveloped. We use a *pathwise derivative* method in this paper to derive estimators for the first and second order sensitivities. The pathwise derivative method differentiates each simulated outcome with the respect to the parameter of interest to produce unbiased sensitivity estimates. There is a large literature on this method in the discrete-event simulation literature, where it is usually referred as to *infinitesimal perturbation analysis*. Broadie and Glasserman (1996) apply it to the field of option pricing and develop unbiased estimators for European-style

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options mainly. In this sense, our work can be viewed as a continuation of their paper in the direction of American options.

Like the difficulty we encounter in the pricing problem, the embedded optimal stopping problem also complicates the sensitivity analysis of American options. Perturbation in a parameter has a two-folded impact on the option price: on one hand it will change the dynamic of the underlying asset and then affect the payoff to the holder directly; on the other hand it will also lead to changes in the holder's exercising rule and in turn translate into a change in the option value. Two nice properties of optimal exercise boundaries, the *continuous-fit* and *smooth-pasting* conditions (see, e.g. Shiryayev (2000)), turn out to be very helpful in obtaining unbiased estimators. They imply that the latter impact is negligible when the perturbation is small. In light of this key observation, we manage to achieve the desired estimators for price sensitivities. For the sake of presentation simplicity, we focus on vanilla American puts in this paper, where the underlying asset price is driven by the Black-Scholes model. This setting simplifies the mathematical treatments significantly. However, we should stress that our method does not confine itself within a one-dimensional environment. A more general discussion on how to employ both of the pathwise derivative method and the *likelihood ratio method*, another generator of unbiased sensitivity estimators, are left to a working paper of the authors (Chen and Liu 2010).

Our estimators are unbiased as long as the optimal exercising rules are known. Most of the algorithms mentioned before yield very accurate approximation to the rules. Therefore the estimators developed in this paper generate quite accurate outcomes. In addition, we can easily embed the estimators into any of these algorithms to produce sensitivities as a by-product of the price estimation without considerably extra computational efforts.

In the literature, two papers are closely related with the main theme of this paper. Piterbarg (2004a) and Piterbarg (2004b) present sensitivity estimators for American-style swaptions, which are options based on interest rate swaps. However, the approach in these two papers is very informal and not rigorous. The author exploits many approximating heuristics to develop estimators. The current paper establishes a theoretically rigorous platform to derive sensitivity estimators for American-style derivatives. With the help of the continuous-fit and smooth-pasting properties, the essence of our method does not depend much on the derivative and underlying asset's structure. We can easily generalize it to consider the optimal stopping problem in a more broader setting.

The remainder of the paper is organized as follows. Section 2 introduces the formulation of the American option pricing problem and its Monte Carlo recipe. We present the main results of this paper — unbiased estimators of the first and second order price sensitivities — in Section 3. The aforementioned two conditions on the optimal exercise boundaries are established in this section as well. The numerical experiments conducted in Section 4 show accuracy and efficiency of our estimators. All of the technical issues arising in the body text are deferred to the Appendix.

2 AMERICAN OPTIONS AND SIMULATION

Let S_t be the price of an underlying asset at time t. It follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \quad S_0 = s \tag{1}$$

in the risk neutral probability measure, where r is the risk-free interest rate, σ is the volatility of asset price and W is a standard Brownian motion. Consider a vanilla American put option contract maturing at time $T > 0$. The option holder is allowed to exercise it anytime up to T. When she exercise at time τ, she can sell the underlying asset at a pre-specified price K. Thus, the option payoff is given by the function $(K - S_\tau)^+$. Assume that the holder has no access to the future information and all of her decisions are made based on the information available currently. Mathematically, pricing the American option can be formulated as to find a solution to the following optimal stopping problem

$$\sup_{\tau \in \mathfrak{T}} E[e^{-r\tau} \cdot (K - S_\tau)^+ | S_0 = s], \tag{2}$$

where $\mathfrak{T}$ is a class of stopping time valued in $[0,T]$ with respect to the filtration $\mathfrak{F} = \{\mathscr{F}_t, 0 \le t \le T\}$, which is generated by $\{S_t, 0 \le t \le T\}$.

Consider Monte Carlo schemes to obtain the solution to (2) numerically. In simulation, a discrete version of the problem is more relevant because any computer in principle can only generate finite random numbers in any given period. Hence, from now on we restrict ourselves to discretely monitored options which can be exercised only at a fixed set of dates: $0 = t_0 < t_1 < ... < t_N = T$, where $t_i - t_{i-1} = \Delta t$. Denote $\mathfrak{T}_N$ to be the set of all stopping time which are valued in $\{t_0, \cdots, t_N\}$. The problem (2) then becomes finding an optimal stopping time to maximize

$$Q_0(s) := \sup_{\tau \in \mathfrak{T}_N} E[e^{-r\tau} \cdot (K - S_\tau)^+ | S_0 = s].$$

As N tends to infinity, the discretely-monitored option value will converge to the original option value.

The theoretical foundation for a variety of simulation methods on American option pricing is the following dynamic-programming characterization of the option value. Let $Q_i(S_{t_i})$ be the option value at t_i given S_{t_i}, assuming that the option has not been exercised previously. Then, Q_i should satisfy the following recursion:

$$Q_N(S_T) = (K - S_T)^+ \quad \text{and} \quad Q_i(S_{t_i}) = \max\{(K - S_{t_i})^+, E[e^{-r(t_{i+1}-t_i)} Q_{i+1}(S_{t_{i+1}})|S_{t_i}]\}, \text{ for } 0 \le i \le N-1. \tag{3}$$

Once all $Q_i's$ are known, we can specify an optimal stopping rule by letting:

$$\tau^* = \min\{i \in \{0,1,\ldots,N\} : (K - S_{t_i})^+ \ge E[e^{-r(t_{i+1}-t_i)} Q_{i+1}(S_{t_{i+1}})|S_{t_i}]\}. \tag{4}$$

For the convenience of later reference, introduce a new notation $C_i(s) := E[e^{-r(t_{i+1}-t_i)} Q_{i+1}(S_{t_{i+1}})|S_{t_i} = s]$. It reflects the value of holding the American option rather than exercising it at time t_i. In this sense we call it by the continuation value of the option. Then, Eq. (4) tells us that we should stop to exercise the option the first time the option payoff exceeds its continuation value. As illustrated in Figure 1, the option holder has a strong incentive to exercise the option if the underlying asset price goes deeply low. In other words, there exists critical values B_i^* for all $0 \le i \le N$ such that

$$(K - S_{t_i})^+ \ge E[Q_{i+1}(S_{t_{i+1}})|S_{t_i}]\} \Leftrightarrow S_{t_i} \le B_i^*.$$

Therefore, the optimal stopping time given by (4) has the form $\tau^* = \min\{i \in \{0,1,\ldots,N\} : S_{t_i} \le B_i^*\}$.

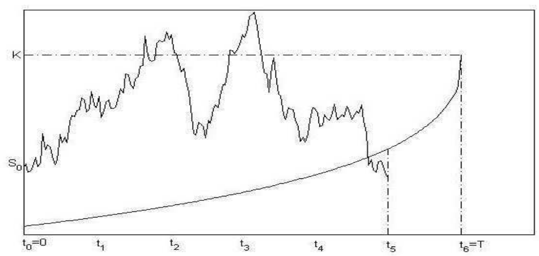

Figure 1: Exercise boundary for American put with payoff $(K - S_\tau)^+$. The holder exercises the option at t_5, the first time the underlying asset crosses the boundary.

One property of the optimal boundary turns out to be very useful later: at B_i^*, the function Q_i satisfies $Q_i(B_i^*) = C_i(B_i^*) = K - B_i^*$. This condition is usually referred as to the *continuous-fit condition* in the literature of the optimal stopping problem (see, e.g. Shiryayev (2000)). It indicates that the holder will be indifferent between the choice of exercising the option and the choice of continuing to hold it when the underlying price $S_{t_i} = B_i^*$.

To solve Q_0 through the recursions (3), one key step is to evaluate the conditional expectation C_i repeatedly. With the help of Monte Carlo, a naive approach is as follows:

1. Let $\hat{Q}_N(s) = h(s)$ for all s;
2. For any $0 \le i \le N-1$, given that $\hat{Q}_{i+1}$ is obtained, simulate a set of samples of $S_{t_{i+1}}$, $\{S^1_{t_{i+1}}, \cdots, S^M_{t_{i+1}}\}$, from $S_{t_i} = s$ and use $\hat{C}_i(s) = \frac{1}{M}\sum_{j=1}^M \hat{Q}_{i+1}(S^j_{t_{i+1}})$ to estimate $C_i(s)$;
3. Let $\hat{Q}_i(s) = \max\{h(s), \hat{C}_i(s)\}$.

Repeat steps (1)-(3) from $i = N$ back to $i = 0$ and we can find an estimation to the option price Q_0. Of course, the above Monte Carlo method is very time-consuming. A host of algorithms are proposed in the literature to improve its efficiency. One may refer to Chapter 8 in Glasserman (2004) and the references therein for a comprehensive overview.

3 FIRST AND SECOND ORDER SENSITIVITIES ON AMERICAN PUTS

We derive unbiased estimators for American-put option price sensitivities in this section.

3.1 The Smooth-Pasting Condition

In addition to the continuous property of Q_i, we can further prove that it should be smooth at B_i^* too. The following theorem summarizes the related result, which plays an important role in deriving the second-order price sensitivities. Note that the derivative of function $(K - s)^+$ equals -1 when $s < K$. Theorem 1 simply states that the two pieces of the option value function should be joined together smoothly across the exercising boundary B_i^*.

Theorem 1 (The smooth-pasting condition). *For $0 \le i \le N-1$, the derivative of Q_i exists at the optimal exercise boundary B_i^* and furthermore, $Q_i'(B_i^*) = -1$.*

The preceding smooth pasting condition can also be regarded as a variation of the first-order minimization condition in the setting of optimal stopping problems. Recall that $Q_i(s) \ge K - s$ for all s. Hence B_i^* must be a minimizer of the function $Q_i(s) - (K - s)$. Thus, we have

$$0 = \frac{d}{ds}[Q_i(s) - (K-s)](B_i^*) = \frac{dQ_i}{ds}(B_i^*) - (-1).$$

A more rigorous argument for Theorem 1 is provided in the Appendix.

3.2 First-Order Sensitivities

Now turn to develop unbiased estimators to the first-order sensitivities of American puts. The American option price under (1) is affected by the current price of the underlying S_0, the volatility σ and the risk-free interest rate r. Correspondingly, we consider three sensitivities: delta, vega and rho. They are the first-order price sensitivities with respect to S_0, σ and r, respectively. We have the following theorem to encapsulate the main results.

Theorem 2. *Suppose that τ^* is the optimal stopping time defined as in Section 2. Then,*

$$Delta = \frac{\partial Q_0(S_0)}{\partial S_0} = -E[e^{-r\tau^*}\frac{S_{\tau^*}}{S_0}], \quad Vega = \frac{\partial Q_0(S_0)}{\partial \sigma} = -E[e^{-r\tau^*}S_{\tau^*}(-\sigma\tau^* + W_{\tau^*})]$$

and

$$Rho = \frac{\partial Q_0(S_0)}{\partial r} = (-K)\cdot E[\tau^* e^{-r\tau^*}].$$

We can justify the statements of Theorem 2 heuristically as follows. Take the delta as an illustration. Given a realization of the Brownian motion $\{W_t, 0 \le t \le T\}$, the sample path of the underlying price S_t is determined by the initial position S_0 through the following relationship:

$$S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right),\ 0 \le t \le T.$$

And the optimal stopping time τ^*, defined by (4), is also dependent on S_0. Applying the pathwise derivative method, we have

$$\frac{\partial Q_0(S_0)}{\partial S_0} = \frac{\partial}{\partial S_0}E[e^{-r\tau^*}(K - S_{\tau^*})^+] = E\left[\frac{\partial}{\partial S_0}e^{-r\tau^*}(K - S_{\tau^*})^+\right],$$

where the derivative in the expectation on the right hand side can be interpreted as the differentiation of the random payoff with respect to S_0 with $\{W_t, 0 \le t \le T\}$ held fixed.

Consider a small positive ∂S_0 and two processes indexed by S_0 and $S_0 - \partial S_0$:

$$S_t^1 = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \text{ and } S_t^2 = (S_0 - \partial S_0)\exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right),\ 0 \le t \le T. \quad (5)$$

Denote $\tau^*(S_0)$ and $\tau^*(S_0 - \partial S_0)$ to be the optimal stopping times in these two processes respectively. The effect of this perturbation is two-folded. For some sample paths of W, it does not change the value of τ^*, i.e., $\tau^*(S_0) = \tau^*(S_0 - \partial S_0)$ (see the the left plot of Figure 2). Therefore,

$$\begin{aligned} \frac{\partial}{\partial S_0}e^{-r\tau^*}(K - S_{\tau^*})^+ &= \frac{e^{-r\tau^*(S_0 - \partial S_0)}(K - S^2_{\tau^*(S_0 - \partial S_0)})^+ - e^{-r\tau^*(S_0)}(K - S^1_{\tau^*(S_0)})^+}{\partial S_0} \\ &= e^{-r\tau^*(S_0)} \cdot \frac{(K - S^2_{\tau^*(S_0)}) - (K - S^1_{\tau^*(S_0)})}{\partial S_0} \end{aligned}$$

in these sample paths. By (5), we know that the right hand side of the above equality should be $e^{-r\tau^*}S_{\tau^*}/S_0$.

As shown by the right plot in Figure 2, the small perturbation on S_0 can also change the value of the optimal stopping time τ^* under some other sample paths of W. Starting from S_0, the trajectory of S_t driven by such W approaches

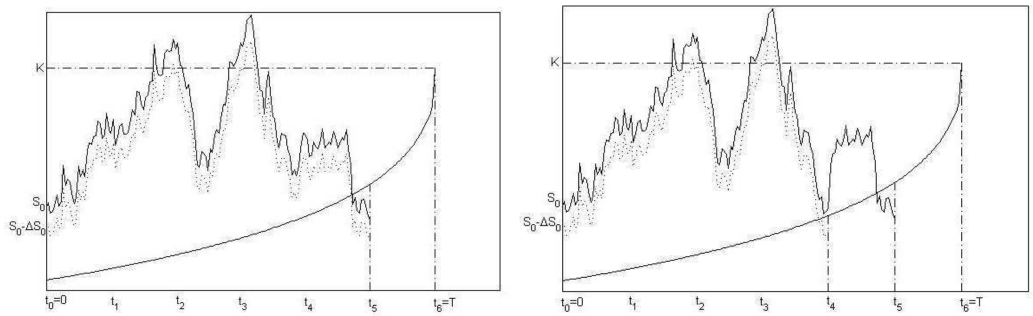

Figure 2: The effects of perturbation on S_0. In the left plot, it just changes the payoff value. Small perturbation ΔS_0 (shown by the dot line) has no impact on the optimal exercising time. But the right plot shows that the perturbation can change the optimal stopping time in some sample paths. For the path in the plot, the holder exercises at t_5. With a small perturbation ΔS_0, he should exercise the option at t_4.

the exercise boundary B^* at least once (say, t_4 in the plot) before it is stopped at τ^* (t_5 in the plot). When we apply the perturbation ∂S_0 to the initial price S_0, it results in a dramatic value change in τ^*. More precisely, such sample path of W must satisfy that

$$B_t^* < S_t^1 < B_t^* + O(\partial S_0)$$

for some $t = \tau^*(S_0 - \partial S_0) < \tau^*(S_0)$. Changing the initial point from S_0 to $S_0 - \partial S_0$ leads to $S_t^2 < B_t^*$ and makes the option holder stop at $t = \tau^*(S_0 - \partial S_0)$.

In light of the observations in the last paragraph, the expectation of the option value change under such realization of W is given by

$$E\left[\frac{e^{-r\tau^*(S_0-\partial S_0)}(K - S^2_{\tau^*(S_0-\partial S_0)})^+ - e^{-r\tau^*(S_0)}(K - S^1_{\tau^*(S_0)})^+}{\partial S_0} \cdot \mathbf{1}_{\{B^*_{\tau^*(S_0-\partial S_0)} < S^1_{\tau^*(S_0-\partial S_0)} < B^*_{\tau^*(S_0-\partial S_0)} + O(\partial S_0)\}}\right]. \tag{6}$$

Given $W_{\tau^*(S_0-\partial S_0)}$, the values of S^1 and S^2 at $\tau^*(S_0 - \partial S_0)$ are determined through (5). Meanwhile, the conditional expectation

$$\begin{aligned} E[e^{-r\tau^*(S_0)}(K - S^1_{\tau^*(S_0)})^+ | S^1_{\tau^*(S_0-\partial S_0)}] &= e^{-r\tau^*(S_0-\partial S_0)} E[e^{-r(\tau^*(S_0)-\tau^*(S_0-\partial S_0))}(K - S^1_{\tau^*(S_0)})^+ | S^1_{\tau^*(S_0-\partial S_0)}] \\ &= e^{-r\tau^*(S_0-\partial S_0)} C(S^1_{\tau^*(S_0-\partial S_0)}). \end{aligned}$$

according to the definition of the continuation value. Therefore,

$$\begin{aligned} & E\left[\frac{e^{-r\tau^*(S_0-\partial S_0)}(K - S^2_{\tau^*(S_0-\partial S_0)})^+ - e^{-r\tau^*(S_0)}(K - S^1_{\tau^*}(S_0))^+}{\partial S_0} \cdot \mathbf{1}_{\{B^*_{\tau^*(S_0-\partial S_0)} < S^1_{\tau^*(S_0-\partial S_0)} < B^*_{\tau^*(S_0-\partial S_0)} + O(\partial S_0)\}} \Big| W_{\tau^*(S_0-\partial S_0)}\right] \\ = & \frac{e^{-r\tau^*(S_0-\partial S_0)}(K - S^2_{\tau^*(S_0-\partial S_0)})^+ - e^{-r\tau^*(S_0-\partial S_0)} C(S^1_{\tau^*(S_0-\partial S_0)})}{\partial S_0} \cdot \mathbf{1}_{\{B^*_{\tau^*(S_0-\partial S_0)} < S^1_{\tau^*(S_0-\partial S_0)} < B^*_{\tau^*(S_0-\partial S_0)} + O(\partial S_0)\}}. \end{aligned} \tag{7}$$

Substituting (7) to (6) and using the tower rule of conditional expectation, the expectation (6) equals to

$$E\left[e^{-r\tau^*(S_0-\partial S_0)}\left((K - S^2_{\tau^*(S_0-\partial S_0)})^+ - C(S^1_{\tau^*(S_0-\partial S_0)})\right) \cdot \frac{1}{\partial S_0} \cdot \mathbf{1}_{\{B^*_{\tau^*(S_0-\partial S_0)} < S^1_{\tau^*(S_0-\partial S_0)} < B^*_{\tau^*(S_0-\partial S_0)} + O(\partial S_0)\}}\right].$$

When ∂S_0 tends to zero, both $S^2_{\tau^*(S_0-\partial S_0)}$ and $S^1_{\tau^*(S_0-\partial S_0)}$ converge to the exercising boundary B^*. By the continuous-fit property, we have

$$\lim_{\partial S_0 \to 0} (K - S^2_{\tau^*(S_0-\partial S_0)})^+ - C(S^1_{\tau^*(S_0-\partial S_0)}) = (K - B^*) - C(B^*) = 0.$$

In addition,

$$\lim_{\partial S_0 \to 0} E\left[\frac{1}{\partial S_0} \cdot \mathbf{1}_{\{B^*_{\tau^*(S_0-\partial S_0)} < S^1_{\tau^*(S_0-\partial S_0)} < B^*_{\tau^*(S_0-\partial S_0)} + O(\partial S_0)\}}\right]$$

exists, which converges to the density function of $S^1_{\tau^*(S_0-\partial S_0)}$ at $B^*_{\tau^*(S_0-\partial S_0)}$.

In summary, we know that the expectation (6) is actually zero. The unbiased estimator of delta should be given by the statement of the theorem. The other estimates can be understood in a similar way.

3.3 Second-order Sensitivities

Gamma is the most important one among the second-order sensitivities. It is defined as the second-order derivative of the option price with respect to the current underlying price S_0.

Using the smooth-pasting property of the exercising boundary, we can establish the following theorem. Its proof appears in the Appendix part.

Theorem 3. *Suppose that τ^* is the optimal exercising time. Then,*

$$Gamma = \frac{\partial^2 Q_0(S_0)}{\partial S_0^2} = e^{-rT}\left(\frac{K}{S_0}\right)^2 \cdot E\left[1_{\{\tau^* > t_{N-1}\}} \cdot g(S_{t_{N-1}}, K)\right],$$

where

$$g(S_{t_{N-1}}, K) = \frac{1}{K\sigma\sqrt{\Delta t}}\phi\left(\frac{\log(K/S_{t_{N-1}}) - (r - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right).$$

4 NUMERICAL EXPERIMENTS

In this section we conduct numerical experiments to test the accuracy and efficiency of the estimators in the last two sections. Our estimators involve τ^*, the optimal stopping times and it would be very difficult to obtain the exact value of τ^* in Monte Carlo simulation. But a host of algorithms, such as those we mentioned in the introduction section, do provide very accurate approximations of the optimal stopping rule. Therefore, we can easily embed the estimators in those algorithms to generate the sensitivities as a by-products of American option pricing.

The results are summarized in Table 4. We use the numerical example in Huang, Subrahmanyam, and Yu (1996) as a benchmark for comparison. That paper relies on the PDE approach to yield the price and sensitivities for American put options. In the table, we make use of a regression-based method (Longstaff and Schwartz 2001) to obtain the prices and sensitivities, which are shown in the columns with the superscript PW. The table illustrates that our method produces quite accurate estimation.

Table 1: Sensitivities for American Put Options. The sensitivities with superscript $*$ are cited from Huang et al. (1996). The defaulting parameters are S_0=40, $r = 0.0488$ and $\sigma = 0.2$. The step number is $N = 400$ between $t = 0$ and T. The total sample number is 1 million for simulation. The numbers in the parentheses are the standard errors of Monte Carlo.

K	T	$Price^*$	$Price^{MC}$	$Delta^*$	$Delta^{PW}$
35	0.3333	0.2010	0.2004(0.0007)	-0.0904	-0.0906(0.0003)
35	0.5833	0.4345	0.4344(0.0011)	-0.1343	-0.1345(0.0003)
40	0.3333	1.5859	1.5831(0.0019)	-0.4454	-0.4451(0.0005)
40	0.5833	1.9985	1.9982(0.0024)	-0.4304	-0.4309(0.0004)
45	0.3333	5.1098	5.1081(0.002)	-0.8849	-0.8854(0.0003)
45	0.5833	5.2859	5.2854(0.0028)	-0.7966	-0.7963(0.0004)
$Gamma^*$	$Gamma^{PW}$	$Vega^*$	$Vega^{PW}$	Rho^*	Rho^{PW}
0.0358	0.0355(0.0002)	3.7473	3.7553(0.0107)	-1.0995	-1.0977(0.0033)
0.0365	0.0368(0.0002)	6.5595	6.5864(0.0147)	-2.7835	-2.791(0.0065)
0.0928	0.0924(0.0003)	8.946	8.9625(0.01)	-4.2627	-4.2729(0.0052)
0.0721	0.0724(0.0002)	11.6561	11.6616(0.0127)	-7.0988	-7.1098(0.0088)
0.0828	0.0825(0.0002)	3.7578	3.7491(0.0048)	-1.9717	-1.9635(0.004)
0.0783	0.0784(0.0002)	7.5223	7.5164(0.0064)	-5.7571	-5.7614(0.0076)

A PROOF OF THE SMOOTH PASTING CONDITION

Proof of Theorem 1. For any small $\varepsilon > 0$ and $1 \le i \le n$, we have

$$Q_i(B_i^* + \varepsilon) = \max\{K - (B_i^* + \varepsilon), E[Q_{i+1}(S_{t_{i+1}})|S_{t_i} = B_i^* + \varepsilon]\} \ge K - (B_i^* + \varepsilon)$$

according to (3). Therefore,

$$\liminf_{\varepsilon \downarrow 0} \frac{Q_i(B_i^* + \varepsilon) - Q_i(B_i^*)}{\varepsilon} = \liminf_{\varepsilon \downarrow 0} \frac{Q_i(B_i^* + \varepsilon) - (K - B_i^*)}{\varepsilon} \ge \liminf_{\varepsilon \downarrow 0} \frac{K - (B_i^* + \varepsilon) - (K - B_i^*)}{\varepsilon} = -1. \quad (8)$$

On the other hand, we claim that

$$\limsup_{\varepsilon \downarrow 0} \frac{Q_i(B_i^* + \varepsilon) - Q_i(B_i^*)}{\varepsilon} \le -1. \quad (9)$$

Combining (8) with (9) will yield the theorem.

To show (9), consider a new geometric Brownian motion starting from $B_i^* + \varepsilon$, i.e., letting $S_{t_i}^\varepsilon = B_i^* + \varepsilon$ and

$$S_{t_j}^\varepsilon = S_{t_{j-1}}^\varepsilon \exp\left((r - \frac{1}{2}\sigma^2) + \sigma(W_{t_j} - W_{t_{j-1}}) \right)$$

for all $i \le j \le N$. Define a stopping time such that

$$\tau_i^\varepsilon := \min\{j \in \{i, \cdots, N\} : K - S_{t_j}^\varepsilon \ge E[Q_{j+1}(S_{t_{j+1}}^\varepsilon)|S_{t_j}^\varepsilon]\}.$$

This is an optimal exercising rule for an option issued at t_i as the underlying price is given by S^ε. Therefore, we have

$$Q_i(B_i^* + \varepsilon) = E[(K - S_{\tau_i^\varepsilon}^\varepsilon)^+ | S_{t_i}^\varepsilon = B_i^* + \varepsilon].$$

In the meantime, note that this τ_i^ε is suboptimal if the underlying state process follows the original dynamic S_t, which implies

$$Q_i(B_i^*) \ge E[(K - S_{\tau_i^\varepsilon})^+ | S_{t_i} = B_i^*].$$

Consequently,

$$\limsup_{\varepsilon \downarrow 0} \frac{Q_i(B_i^* + \varepsilon) - Q_i(B_i^*)}{\varepsilon} \le \frac{d}{d\varepsilon} E\left[(K - S_{\tau_i^\varepsilon}^\varepsilon)^+ - (K - S_{\tau_i^\varepsilon})^+ \middle| S_{t_i} = B_i^*,\ S_{t_i}^\varepsilon = B_i^* + \varepsilon \right]. \quad (10)$$

It is easy to verify that the function $g(x) = (K - x)^+$ is Lipschitz and there exists an integrable random variable κ such that

$$|S_{\tau_i^\varepsilon}^\varepsilon - S_{\tau_i^\varepsilon}| \le \kappa\varepsilon$$

for all $\varepsilon > 0$. The dominated convergence theorem then implies that we can interchange the order of expectation and differentiation on the right hand side to obtain

$$\limsup_{\varepsilon \downarrow 0} \frac{Q_i(s^* + \varepsilon) - Q_i(s^*)}{\varepsilon} \le E\left[\lim_{\varepsilon \downarrow 0} \frac{(K - S_{\tau_i^\varepsilon}^\varepsilon)^+ - (K - S_{\tau_i^\varepsilon})^+}{\varepsilon} \right].$$

As $\varepsilon \to 0$, $S_{t_j}^\varepsilon \to S_{t_j}$ for all $i \le j \le N$. Then, the stopping time τ_i^ε converges to

$$\tau_i := \min\{j \in \{i, \cdots, N\} : K - S_{t_j} \ge E[Q_{j+1}(S_{t_{j+1}})|S_{t_j}]\}.$$

In addition, $\tau_i = t_i$ when we start S_t from $S_{t_i} = B_i^*$. Therefore,

$$\lim_{\varepsilon \downarrow 0} \frac{(K - S_{\tau_i^\varepsilon}^\varepsilon)^+ - (K - S_{\tau_i^\varepsilon})^+}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{(K - S_{t_i}^\varepsilon)^+ - (K - S_{t_i})^+}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{(K - (B_{t_i}^* + \varepsilon))^+ - (K - B_{t_i}^*)^+}{\varepsilon} = -1. \quad (11)$$

The smooth pasting condition is proved if we plug (11) into (10). □

B A USEFUL LEMMA

Lemma 1. *Suppose that $\Psi(\theta)$ and $\nu(\theta)$ are random variables that depend on θ. Denote $f_\theta(u)$ to be the probability density function of $\nu(\theta)$. Assume*
(1) $\Psi(\theta)$ and $\nu(\theta)$ are differentiable with respect to θ with probability 1,
(2) there exists an integrable random variable K_Ψ^θ such that $|\Psi(\theta+\Delta\theta)-\Psi(\theta)| \leq K_\Psi^\theta \cdot |\Delta\theta|$ when $|\Delta\theta|$ is sufficiently small.
Then, we have:

$$\frac{\partial}{\partial\theta}E[\Psi(\theta)\cdot 1_{\{\nu(\theta)\geq 0\}}] = E[\Psi'(\theta)\cdot 1_{\{\nu(\theta)\geq 0\}}] + f_\theta(0)\cdot E[\Psi(\theta)\cdot\nu'(\theta)|\nu(\theta)=0]. \tag{12}$$

Proof. It is Theorem 1 in Wang, Fu, and Marcus (2009). A similar result appears in Liu and Hong (2009). □

C PROOFS OF THE THEOREMS

Proof of Theorem 2. We prove the estimator of delta first. Note that there exists a boundary $B_0^* < K$ such that

$$Q_0(S_0) = (K-S_0)1_{\{S_0\leq B_0^*\}} + C_0(S_0)1_{\{S_0>B_0^*\}}.$$

Therefore,

$$\frac{\partial Q_0(S_0)}{\partial S_0} = \frac{\partial(K-S_0)}{\partial S_0}\cdot 1_{\{S_0\leq B_0^*\}} + \frac{\partial C_0(S_0)}{\partial S_0}\cdot 1_{\{S_0>B_0^*\}}. \tag{13}$$

Focus on the second term of the above equality. Denote

$$I_i = E[(K-S_{t_i})1_{\{S_{t_0}>B_0^*,\ S_{t_1}>B_1^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\leq B_i^*\}}].$$

Then the continuation value function has the following decomposition: $C_0(S_0) = \sum_{i=1}^N e^{-rt_i} I_i$.

Taking the partial derivative of I_i with respect to S_0 and invoking Lemma 1, we can show that

$$\begin{aligned}\frac{\partial I_i}{\partial S_0} &= E\left[(-1)\cdot\frac{\partial S_{t_i}}{\partial S_0}\cdot 1_{\{S_{t_0}>B_0^*,\ S_{t_1}>B_1^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\leq B_i^*\}}\right]\\ &\quad+\sum_{j=1}^{i-1} E\left[(K-S_{t_i})\cdot 1_{\{S_{t_0}>B_0^*,\ \ldots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\leq B_i^*\}}\cdot\frac{\partial S_{t_j}}{\partial S_0}\middle|S_{t_j}=B_j^*\right]\cdot f_{S_{t_j}}(B_j^*)\\ &\quad-E\left[(K-S_{t_i})1_{\{S_{t_0}>B_0^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*\}}\cdot\frac{\partial S_{t_i}}{\partial S_0}\middle|S_{t_i}=B_i^*\right]\cdot f_{S_{t_i}}(B_i^*),\end{aligned} \tag{14}$$

where $f_{S_{t_i}}(x)$ is the probability density function of S_{t_i} at x, i.e.,

$$f_{S_{t_i}}(x) = \frac{1}{x\sigma\sqrt{t_i}}\phi\left(\frac{\log(x/S_0)-(r-\frac{1}{2}\sigma^2)t_i}{\sigma\sqrt{t_i}}\right)$$

and ϕ is the pdf of a standard normal.

Denote $\tilde{I}_i^j$ to be the jth-summand in the second line of the right hand side of (14). By the Markov property of S_t, the expectation inside each $\tilde{I}_i^j$ can be represented as follows:

$$\begin{aligned}&E\left[(K-S_{t_i})\cdot 1_{\{S_{t_0}>B_0^*,\ \ldots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\leq B_i^*\}}\cdot\frac{\partial S_{t_j}}{\partial S_0}\middle|S_{t_j}=B_j^*\right]\\ =\;&E\left[1_{\{S_{t_0}>B_0^*,\ \ldots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot\frac{\partial S_{t_j}}{\partial S_0}\middle|S_{t_j}=B_j^*\right]\cdot E\left[(K-S_{t_i})\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \ldots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\leq B_i^*\}}\middle|S_{t_j}=B_j^*\right].\end{aligned}$$

Therefore,

$$\sum_{i=1}^{N}\sum_{j=1}^{i-1}e^{-rt_i}\tilde{I}_i^j=\sum_{j=1}^{N-1}\sum_{i=j+1}^{N}e^{-rt_i}\tilde{I}_i^j$$

$$=\sum_{j=1}^{N-1}E\left[1_{\{S_{t_0}>B_0^*,\ \dots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot\frac{\partial S_{t_j}}{\partial S_0}\Big|S_{t_j}=B_j^*\right]\cdot f_{S_{t_j}}(B_j^*)\cdot E\left[\sum_{i=j+1}^{N}e^{-rt_i}(K-S_{t_i})\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\le B_i^*\}}\Big|S_{t_j}=B_j^*\right].$$

Note that the final term in the above equality satisfies that

$$\begin{aligned}E\left[\sum_{i=j+1}^{N}e^{-rt_i}(K-S_{t_i})\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\le B_i^*\}}\Big|S_{t_j}=B_j^*\right] &= E\left[e^{-rt_{j+1}}Q_{j+1}(S_{t_{j+1}})\Big|S_{t_j}=B_j^*\right]\\ &= e^{-rt_j}C_j(B_j^*)=e^{-rt_j}(K-B_j^*),\end{aligned}$$

where the first equality is due to the definition of Q_{j+1} and we have the second equality because

$$C_j(B_j^*)=E\left[e^{-r(t_{j+1}-t_j)}Q_{j+1}(S_{t_{j+1}})\Big|S_{t_j}=B_j^*\right].$$

In summary, we have

$$\sum_{i=1}^{N}\sum_{j=1}^{i-1}e^{-rt_i}\tilde{I}_i^j=\sum_{j=1}^{N-1}e^{-rt_j}E\left[(K-S_{t_j})1_{\{S_{t_0}>B_0^*,\ \dots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot\frac{\partial S_{t_j}}{\partial S_0}\Big|S_{t_j}=B_j^*\right]\cdot f_{S_{t_j}}(B_j^*).\tag{15}$$

To obtain unbiased estimator to the delta, we sum (14) across all $1\le i\le N$. It leads to

$$\begin{aligned}\frac{\partial C_0(S_0)}{\partial S_0} &= -E[\sum_{i=1}^{N}e^{-rt_i}\frac{\partial S_{t_i}}{\partial S_0}\cdot 1_{\{\tau^*=t_i\}}]+\sum_{i=1}^{N}\sum_{j=1}^{i-1}e^{-rt_i}\tilde{I}_i^j-\sum_{i=1}^{N}e^{-rt_i}E\left[(K-S_{t_i})1_{\{S_{t_0}>B_0^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*\}}\cdot\frac{\partial S_{t_i}}{\partial S_0}\Big|S_{t_i}=B_i^*\right]\cdot f_{S_{t_i}}(B_i^*)\\ &= -E[\sum_{i=1}^{N}e^{-rt_i}\frac{\partial S_{t_i}}{\partial S_0}\cdot 1_{\{\tau^*=t_i\}}]=-E[e^{-r\tau^*}\frac{\partial S_{\tau^*}}{\partial S_0}],\end{aligned}\tag{16}$$

where the second equality holds because of (15) and $B_N^*=K$. Combing the above equality with the fact that $\partial S_t/\partial S_0=S_t/S_0$, we prove the case of delta.

The proofs of vega and rho are very similar. Here we only illustrate how to obtain the unbiased estimator for vega briefly. Emulating (14), we can establish that

$$\begin{aligned}\frac{\partial I_i}{\partial\sigma} &= E\left[(-1)\cdot\frac{\partial S_{t_i}}{\partial\sigma}\cdot 1_{\{S_{t_0}>B_0^*,\ S_{t_1}>B_1^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\le B_i^*\}}\right]\\ &\quad+\sum_{j=1}^{i-1}E\left[(K-S_{t_i})\cdot 1_{\{S_{t_0}>B_0^*,\ \dots,\ S_{t_{j-1}}>B_{j-1}^*\}}\cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\le B_i^*\}}\cdot\frac{\partial(S_{t_j}-B_j^*)}{\partial\sigma}\Big|S_{t_j}=B_j^*\right]\cdot f_{S_{t_j}}(B_j^*)\\ &\quad-E\left[(K-S_{t_i})1_{\{S_{t_0}>B_0^*,\ \dots,\ S_{t_{i-1}}>B_{i-1}^*\}}\cdot\frac{\partial(S_{t_i}-B_i^*)}{\partial\sigma}\Big|S_{t_i}=B_i^*\right]\cdot f_{S_{t_i}}(B_i^*),\end{aligned}$$

where the optimal boundary B^* is under the affect of σ. Following similar arguments as in the case of delta will lead us to

$$\frac{\partial C_0}{\partial\sigma}=-E[\sum_{i=1}^{N}e^{-rt_i}\frac{\partial S_{t_i}}{\partial\sigma}\cdot 1_{\{\tau^*=t_i\}}]=-E[e^{-r\tau^*}S_{\tau^*}(-\sigma\tau^*+W_{\tau^*})].$$

By the decomposition of Q_0 at the beginning of the proof, we can show the case of vega. □

Proof of Theorem 3. From Theorem 2, we know that

$$\begin{aligned}\frac{\partial^2 Q_0(S_0)}{\partial S_0^2} &= \frac{\partial}{\partial S_0}\left[\frac{\partial Q_0(S_0)}{\partial S_0}\right]=-\frac{\partial}{\partial S_0}E\left[e^{-r\tau^*}\cdot\frac{S_{\tau^*}}{S_0}\right]\\ &= -\sum_{i=0}^{N}\frac{\partial}{\partial S_0}E\left[e^{-rt_i}\cdot\exp((r-\frac{1}{2}\sigma^2)t_i+\sigma W_{t_i})1_{\{\tau^*=t_i\}}\right],\end{aligned}\tag{17}$$

where the last equality uses the fact that $S_t = S_0 \exp((r-\sigma^2/2)t + \sigma W_t)$. Applying Lemma 1 again,

$$\begin{aligned}
&\frac{\partial}{\partial S_0} E\left[\exp((r-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i})1_{\{\tau^*=t_i\}}\right] \\
= \quad & \frac{\partial}{\partial S_0} E\left[\exp((r-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i})1_{\{S_0>B_0^*,\ S_{t_1}>B_1^*,\ \cdots,\ S_{t_{i-1}}>B_{i-1}^*,\ S_{t_i}\le B_i^*\}}\right] \\
= \quad & -\sum_{j=0}^{i-1} E\left[\exp((r-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i})1_{\{S_0>B_0^*,\dots,S_{t_{j-1}}>B_{j-1}^*\}} \cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\dots,S_{t_{i-1}}>B_{i-1}^*,S_{t_i}\le B_i^*\}} \frac{\partial S_{t_j}}{\partial S_0}\Big| S_{t_j} = B_j^*\right] \cdot f_{S_{t_j}}(B_j^*) \\
& + E\left[\exp((r-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i})1_{\{S_0>B_0^*,\dots,S_{t_{i-1}}>B_{i-1}^*\}} \frac{\partial S_{t_i}}{\partial S_0}\Big| S_{t_i} = B_i^*\right] \cdot f_{S_{t_i}}(B_i^*). \qquad (18)
\end{aligned}$$

On the other hand, summing up the third line in (18) across all $1 \le i \le N$ will yield

$$\begin{aligned}
& \sum_{i=1}^{N} e^{-rt_i} \sum_{j=0}^{i-1} E\left[\exp((r-\frac{1}{2}\sigma^2)t_i + \sigma W_{t_i})1_{\{S_0>B_0^*,\dots,S_{t_{j-1}}>B_{j-1}^*\}} \cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\dots,S_{t_{i-1}}>B_{i-1}^*,S_{t_i}\le B_i^*\}} \frac{\partial S_{t_j}}{\partial S_0}\Big| S_{t_j} = B_j^*\right] \cdot f_{S_{t_j}}(B_j^*) \\
= \quad & \sum_{j=0}^{N-1} e^{-rt_j} E\left[1_{\{S_0>B_0^*,\dots,S_{t_{j-1}}>B_{j-1}^*\}} \frac{\partial S_{t_j}}{\partial S_0}\Big| S_{t_j} = B_j^*\right] \cdot f_{S_{t_j}}(B_j^*) \cdot \frac{S_{t_j}}{S_0} \cdot \sum_{i=j+1}^{N} e^{-r(t_i-t_j)} E\left[\frac{S_{t_i}}{S_{t_j}} \cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\dots,S_{t_{i-1}}>B_{i-1}^*,S_{t_i}\le B_i^*\}}\Big| S_{t_j} = B_j^*\right]. \qquad (19)
\end{aligned}$$

According to the proof of Theorem 2, especially (16),

$$\sum_{i=j+1}^{N} e^{-r(t_i-t_j)} E\left[\frac{S_{t_i}}{S_{t_j}} \cdot 1_{\{S_{t_{j+1}}>B_{j+1}^*,\dots,S_{t_{i-1}}>B_{i-1}^*,S_{t_i}\le B_i^*\}}\Big| S_{t_j} = B_j^*\right] = C'_{t_j}(B_j^*) = -1,$$

where we use the smooth-pasting condition in the last equality. Consequently, the left hand side of (19) should equal to

$$\begin{aligned}
& -\sum_{j=0}^{N-1} e^{-rt_j} E\left[1_{\{S_0>B_0^*,\dots,S_{t_{j-1}}>B_{j-1}^*\}} \frac{\partial S_{t_j}}{\partial S_0}\Big| S_{t_j} = B_j^*\right] \cdot f_{S_{t_j}}(B_j^*) \cdot \frac{S_{t_j}}{S_0} \\
= \quad & -\sum_{j=0}^{N-1} e^{-rt_j} E\left[\exp((r-\frac{1}{2}\sigma^2)t_j + \sigma W_{t_j})1_{\{S_0>B_0^*,\dots,S_{t_{j-1}}>B_{j-1}^*\}} \frac{\partial S_{t_j}}{\partial S_0}\Big| S_{t_j} = B_j^*\right] \cdot f_{S_{t_j}}(B_j^*). \qquad (20)
\end{aligned}$$

If we substitute (18) into (17), then we have

$$\frac{\partial^2 Q_0(S_0)}{\partial S_0^2} = e^{-rT} E\left[\exp((r-\frac{1}{2}\sigma^2)T + \sigma W_T)1_{\{S_0>B_0^*,\dots,S_{t_{N-1}}>B_{N-1}^*\}} \frac{\partial S_T}{\partial S_0}\Big| S_T = B_N^*\right] \cdot f_{S_T}(B_N^*).$$

using (20). Notice that $\partial S_T/\partial S_0 = S_T/S_0$ and $B_N^* = K$. We can obtain the unbiased estimator in the theorem statement for gamma. □

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