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DEMAND CURVE PREDICTION VIA BAYESIAN PROBABILITY ASSIGNMENT OVER A FUNCTIONAL SPACE

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ABSTRACT

One of the important aspects of energy modeling is the process of demand curve prediction. Existing demand curve prediction methods generally rely on statistical curve fittings which assume a certain functional form such as constant price elasticity. There are a number of disadvantages to this approach. For one, this method makes certain assumptions about the functional form of the price-demand curve that may not be exhibited in practice. In addition, since curve fits rely on only a single function, and not a distribution of functions, they do not capture the uncertainty about price-demand curves. In this work, demand curve prediction is instead treated by assigning a probability measure to the space of all functions that meet the global regularity (non-decreasing conditions). Using this method, a numerical example of Bayesian demand curve prediction is presented.

1 INTRODUCTION

Currently, the majority of demand curve predictions are based on curve fits. While these may be adequate when making simple decisions, there are a number of issues that make them somewhat impractical in real world decision making. For one, they only output a single curve prediction. As a result, they are not compatible with rigorous decision making in the presence of uncertainty. In addition, with curve fits it is impossible to construct a predictive model for demand curves that is both globally regular (meets the monotonicity constraints of demand curves) and locally flexible (given enough observations of an arbitrary "true" demand curve meeting the global constraints, the prediction will converge to the true curve) (Barnett 1998). In this work, we present an alternative method of demand curve prediction which is fundamentally different from curve fitting. In particular, a probability distribution is assigned over the space of all functions that satisfy the regularity conditions. First, a hyperbolic coordinate transformation is used to impose the regularity conditions is introduced. Then, the problem of assigning a probability distribution over the space of functions and its update with information is discussed.

2 THE HYPERBOLIC COORDINATE TRANSFORMATION

For a price P>0 and demand Q>0, consider the hyperbolic coordinate transformation given by:

$$u = \frac{1}{2}(\log P - \log Q).$$

$$v = \frac{1}{2}(\log P + \log Q).$$

The inverse transformation is given by:

$$P = e^{v+u}.$$

$$Q = e^{v-u}.$$

This transformation maps the first quadrant to the entire R^2 plane. In particular, the point $\{P = 0, Q = \infty\}$ is mapped to the line $u = -\infty$, the point $\{P = \infty, Q = 0\}$ is mapped to the line $u = \infty$, the P axis and Q axis are mapped to the line $v = -\infty$, and the lines $P = \infty$ and $Q = \infty$ are mapped to the line $v = \infty$ (Figure 1).

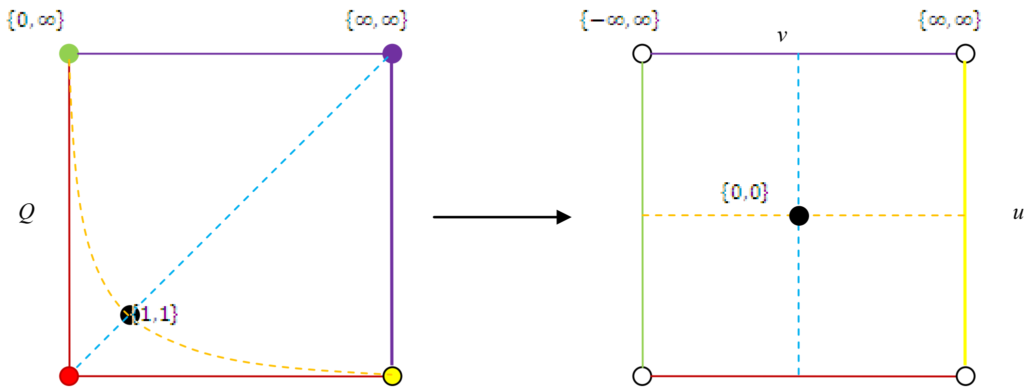

Figure 1: Mapping of the hyperbolic coordinate transformation

The advantage of using hyperbolic coordinates is that the regularity conditions imposed on the demand system can be expressed nicely in u-v coordinates (Dunkel and Hänggi 2009). Any function $Q(P)$ that is monotonically decreasing, and has the positive real values as both a domain and range can be expressed as a function $v(u)$ which satisfies

$$\operatorname{dom} v(u) = \mathbb{R}.$$

$$\left|\frac{dv}{du}\right| < 1. \tag{1}$$

In addition, u-v coordinates have nice interpretations in terms of economic variables. For one, the total revenue R generated (the product $P * Q$ in P-Q coordinates) is given by

$$R = e^{2v}.$$

As a result, maximizing the revenue for a given demand curve is equivalent to finding the maximum v of that curve. Furthermore, the elasticity E, given in P-Q space by

$$E = -\frac{P}{Q}\frac{dQ}{dP} = -\frac{d\ln Q}{d\ln P},$$

is given in u-v space by

$$E = \frac{\left(1 - \frac{dv}{du}\right)}{\left(1 + \frac{dv}{du}\right)}. \tag{2}$$

Therefore, lines of constant slope in the u-v plane are equivalent to lines of constant elasticity and horizontal lines are equivalent to lines of unit elasticity.

3 CONSTRUCTING A PROBABILITY DISTRIBUTION ON THE HYPERBOLIC PLANE

In order to construct a probability distribution over the space of functions, a discrete random walk model can be used. In particular, the following two assumptions are made:

Assumption 1 *When predicting the behavior of the demand curve at any price P_0, only information regarding the curve in an infinitesimal neighborhood of P_o needs to be considered.*

Assumption 2 *Knowledge of the second or higher derivatives is irrelevant when predicting the demand curve.*

Now, suppose that it is known that the demand curve goes through the point $\{P_1, Q_1\}$ and one is interested in calculating the probability that the demand curves travels through the point $\{P_1 + \Delta P, Q_1 - \Delta Q\}$ for some $0 < \Delta P$ and $0 < \Delta Q < Q_1$. Because of assumptions 1 and 2, this conditional probability can be calculated using a lattice walk model. In P-Q space, these conditional probabilities are difficult to evaluate since the transition probabilities MUST depend explicitly on P and Q, and these dependencies are necessarily singular near the axes (Figure 2).

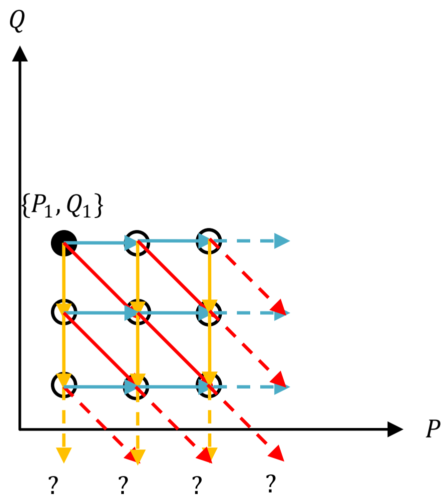

Figure 2: Evaluating a random lattice walk in ***P-Q*** space is difficult due to the boundaries on the axes. In ***u-v*** space, this is not an issue since the ***P-Q*** axes are mapped out of the plane (to the line v=-∞).

However, in u-v space these dependencies are already accounted for. Because of this and assumptions 1 and 2, the transition probabilities are independent and identically distributed. If equation (1) (resulting from the monotonicity in P-Q space) was not imposed, these transition probabilities would converge to a wiener process and the resultant conditional probabilities would be normally distributed (Nagasawa 1993). With the requirement of equation (1), however, the conditional probabilities must be of compact support and therefore cannot be normally distributed. Instead, the conditional probabilities are given by a Lorentz invariant distribution (Ikeda and Matsumoto 1999). The lattice walk and the resultant conditional probability distribution are shown in Figure 3. After transforming back into P-Q space, by using these conditional probabilities and assumptions 1 and 2, the probability that the demand curve goes through any point $\{P, Q\}$ can be calculated.

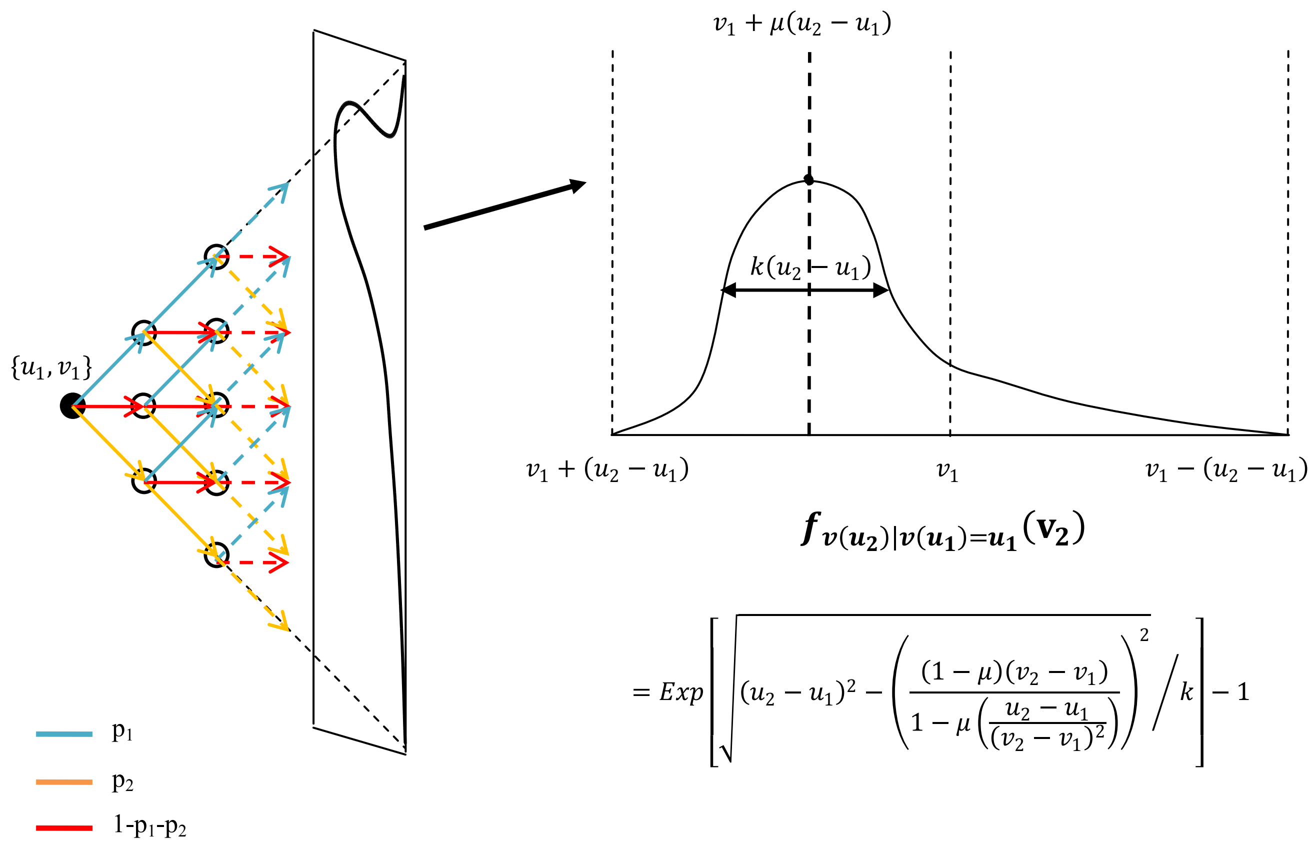

Figure 3: On the left, the random lattice walk is shown. As the number of steps increases to infinity, the distribution converges to a two parameter Lorentz-invariant distribution (shown on the right) where the two parameters (mean $\boldsymbol{\mu}$ and scale / spread parameter $\boldsymbol{k}$) are determined from the transition probabilities p_1 and p_2.

4 USING THE PROBABILITY MODEL

To make an initial prediction of the demand curve, one needs to define which points of the demand curve are already known, as well as assigning values to the μ and k parameters. In the general case, they can take different values at each $\{u, v\}$ coordinate, so initial parameter *fields*, $\mu_0(u, v)$and $k_0(u, v)$ need to be chosen. In terms of the P-Q coordinate system, assigning these parameter fields is equivalent to answering the following question for every $\{\mathrm{P}_0, \mathrm{Q}_0\}$:

Given that the demand curve travels through $\{\mathrm{P}_0, \mathrm{Q}_0\}$*, what is the expectation and variance of the slope of the demand curve at this point?*

Or using equation (2)

Given that the demand curve travels through $\{\mathrm{P}_0, \mathrm{Q}_0\}$*, what is the expectation and variance of the elasticity of the demand curve at this point?*

For example, suppose someone is interested in predicting the demand curve for some type of product X. Using historical sales data, they believe that the demand curve travels through the points $\{P, Q\}$={\$29, 3960 units}, {\$60, 2040 units} , {\$200, 810 units} and {\$400, 240 units}. In addition, they define the μ_0 and k_0 fields as constant between any two known points (Table 1). In particular, they choose the μ_0 field as the average elasticity between two known points, and a k_0 field that decreases near the axes (less uncertainty/spread near axes). Using these assumptions, the resultant prediction for the demand curve of product X is as shown in Figure 4. The price which maximizes expected revenue generated by selling product X can then be calculated from this prediction, which in this case is \$200. In addition, these predictions can be used to place a value on information gathering activities. For example, suppose that the demand Q_{test} for product X at price P_{test} can be estimated in a small test market. Since the expected revenue generated prior to the test, the probability of each possible outcome of the test (the demand curve prediction at P_{test}) and the expected revenue generated with each outcome can all be calculated prior to performing the test, the expected value gained from a market test at any P_{test} can be calculated. P_{test} can then be chosen to maximize this value.

Table 1: Parameter fields for predicting the demand curve of product X.

Region:	**Drift Field $\mu_0(u,v)$:**	**Diffusion Field $k_0(u,v)$:**
\$0< P <\$29	0.310*	0.03
\$29< P <\$60	0.0460	0.05
\$60< P <\$200	0.132	0.1
\$200< P <\$400	-0.274	0.05
P >\$400	-0.263*	0.03

*-Values chosen arbitrarily such that $P * Q$ goes to 0 as P or Q goes to 0.

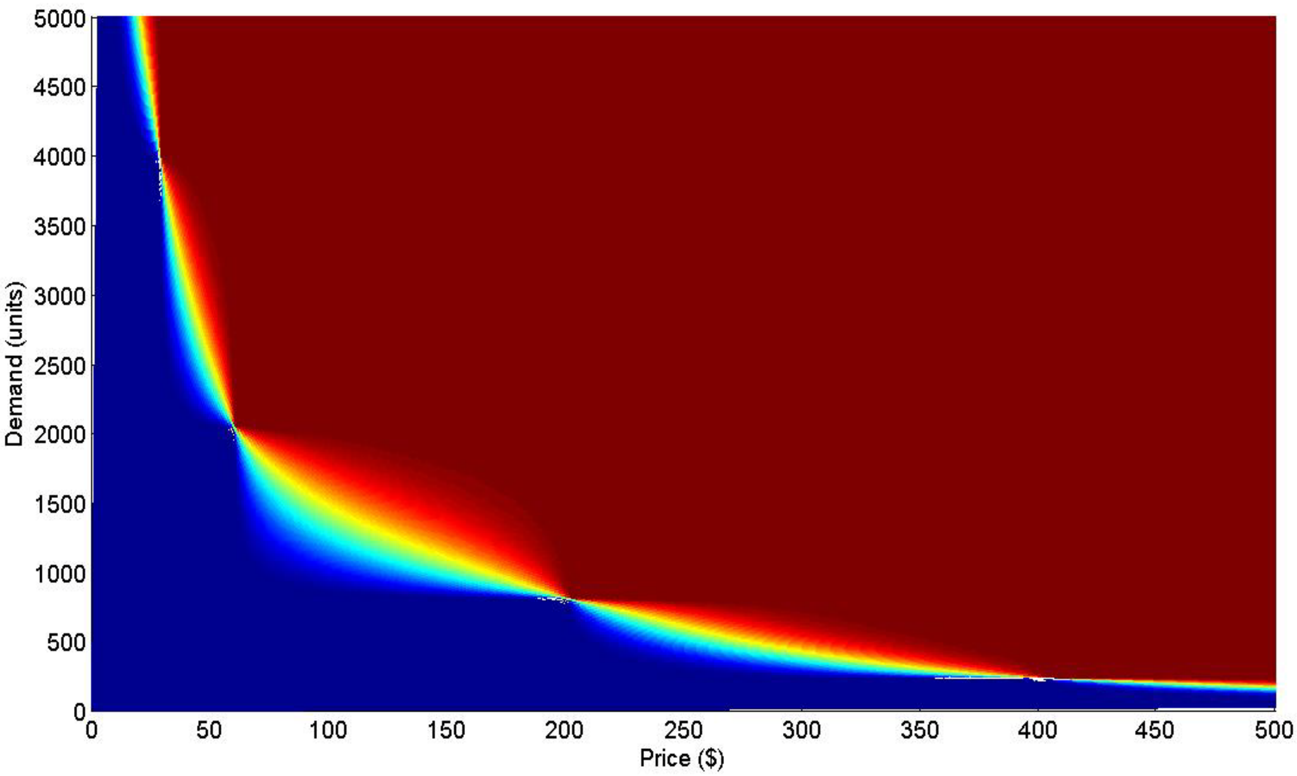

Figure 4: Demand curve prediction for product X.

5 CONCLUSION

In conclusion, the predictive model presented here has a number of advantages over traditional demand curve predictions which utilize statistical curve fits. Fundamentally, it differs from curve fitting since a probability is assigned over a full space of functions instead of assuming a strict functional form for the demand curve. As a result, these methods result in predictions that are both globally regular and locally flexible, which is impossible with traditional curve fitting. In addition, since this predictive model is compatible with rational decision making, it can be used to determine not only optimal pricing strategies, but also optimal information gathering strategies. Overall, although these methods are somewhat more complex mathematically than curve fits, they also provide stronger functionality which is often preferred when making real world decisions involving real dollars.

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