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# FAST SIMULATION OF EQUITY-LINKED LIFE INSURANCE CONTRACTS WITH A SURRENDER OPTION

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## ABSTRACT

In this paper, we consider equity-linked life insurance contracts that give their holder the possibility to surrender their policy before maturity. Such contracts can be valued using simulation methods proposed for the pricing of American options, but the mortality risk must also be taken into account when pricing such contracts. Here, we use the least-squares Monte Carlo approach of Longstaff and Schwartz coupled with quasi-Monte Carlo sampling and a control variate in order to construct efficient estimators for the value of such contracts. We also show how to incorporate the mortality risk into these pricing algorithms without explicitly simulating it.

## 1 INTRODUCTION

Life insurance companies used to sell life insurance contracts that provide a fixed capital at time of death of the insured. In the past ten years, individuals are more and more willing to combine investment and life insurance. Nowadays popular contracts often involve a return linked to the market. Equity index annuities are products that offer investors downside protection because of the presence of a minimum maturity benefit and death benefit with a potential upside rate in case the index performs well (see for instance Palmer (2006) or Hardy (2003) for more details on contracts currently offered).

In this paper we investigate some of the options embedded in an equity-linked life insurance contract. We include standard options such as a guaranteed minimum maturity benefit and a guaranteed minimum death benefit. But many more complex options exist (see for instance Ballotta and Haberman (2003) or Boyle and Hardy (2003) to cite only a few). The possibility of early withdrawals is investigated by Milevsky and Salisbury (2006). In this study we focus on the surrender option, which gives policyholders the right to terminate their policies before the maturity indicated in their contract. It is the possibility of withdrawing the total value of the contract. Literature on surrender options in equity-linked life insurance products is recent since these options have been neglected for a long time. In the 1990s, they were indeed considered as very low-risk options since interest rates in the market were much higher than the minimum guaranteed rate and the stock market performed very well. After several bankruptcies in the insurance sector (for instance around 2001, the British company, *Equitable Life* went bankrupt due to options embedded in pension annuities where mortality risk and financial risk were underestimated and not hedged), embedded options are now taken more seriously into account in the valuation of the contracts.

Each option offers investors some additional rights that might cause some profit reduction for insurer if optimally exercised. It is thus of utmost importance for companies to understand all options embedded in contracts they sell to better price and hedge them. Concerning the surrender option, there are several consequences. First, the insurer might incur financial losses from early payments caused by surrendered policies due to liquidity of their assets (they might have to liquidate depreciated assets to face surrender demand). Indeed companies face a trade-off between long term investment (and high returns) but with few liquidity or investing in liquid assets (with a lower return). Second, in the long run insurer might suffer from adverse selection since policyholders that have health problems and thus insurability difficulties will not surrender. To avoid adverse selection, some contracts (for example contracts that have only a benefit in case of survival) do not allow surrender. To minimize risks associated with the surrender option, insurers can give high penalties. However insured are keen on fair surrender conditions since these contracts are often long-term investments and their needs might change and force them to surrender. Regulators, lawyers and market competition between companies protect investors against unfair conditions in case of early termination.

To value this surrender option, no unified framework exists in previous literature. Surrender decision can be of two types: an *exogenous* surrender (for example, depending on personal reasons: family problem, unemployment, sudden need of liquidity) or an *economic* or *endogenous* surrender, linked to interest rates fluctuations, financial environment changes, etc. Actuaries estimate exogenous surrender from historical data on the lapse rate. Exogenous surrender risk can be diversified. Thus insurers should be more threatened by endogenous surrender risk that is difficult to estimate (in particular because of lack of data) and leads to systematic risks. It seems that there were only weak historical evidence of situations when surrender was optimal. Any approach based on historical lapses might underestimate the real cost of surrender options. Kuo, Tsai, and Chen (2003) study how the lapse rate is influenced by unemployment rate and interest rate fluctuations.

To deal with the second type of surrender decisions (financial, endogenous surrender risk), the alternative is to use a pure financial approach based on the option theory. Surrender option is seen as an American option. Indeed equity-linked contracts with a fixed term can be expressed as a portfolio of European options (Brennan and Schwartz (1976), and Boyle and Schwartz (1977) are the first ones proposing this approach). The cost of the surrender option is then the American premium that is added if these options can be exercised before maturity. In the Black and Scholes framework, Grosen and Jørgensen (1997) give the optimal exercise barrier by using results from Myneni (1992). First studies of the surrender option market value use numerical schemes to solve partial differential equations (Jensen, Jørgensen, and Grosen 2001)) or binomial trees (Bacinello 2003, Bacinello 2005). Bacinello takes into account mortality risk, periodical premiums and annual bonus, but both of these approaches are extremely slow. Recently, Shen and Xu (2005) have also investigated surrender options by way of partial differential equations.

These works use the no-arbitrage principle and the market value of the surrender option is the market value of an optimal surrender decision. New accounting rules force insurers to evaluate their liabilities (consisting mainly of the sold contracts) at their market value even though the contracts are not really traded on a market. There is thus an important need for financial modeling of insurance contracts. The no arbitrage approach to value the surrender option has been criticized since this option is not traded. Moreover, it implies that if it is optimal for one insured to surrender, it is thus optimal for all of them to surrender, which is not realistic even if it is the greatest risk faced by the issuing company. Albizzati and Geman (1994) propose an interesting model to address this issue. The idea is that the value of the surrender option at a given date is known and then it can be split over different dates using an exogenous lapse rate. In this paper, we are also interested in giving a new way to tackle this problem by introducing a parameter that will depends on the individual, and will nuance his "optimal" decision. All individuals will thus not have the same optimal decision.

Our approach is a financial approach and we want to obtain a "market value" of the surrender option, in a similar way to what has been done in Andreatta and Corradin (2003), Bacinello, Biffis, and Millossovich (2008a), and Bacinello, Biffis, and Millossovich (2008b). We use the least-squares Monte Carlo approach of Longstaff and Schwartz (2001) to perform the valuation, but we include the mortality risk using a different model and a different approach than Bacinello, Biffis, and Millossovich (2008a), Bacinello, Biffis, and Millossovich (2008b). In particular, by making the usual assumption that the mortality risk is independent of the financial risk, we show that survival times do not need to be simulated. This reduces the variability of the obtained estimators for the value of the surrender option. In addition, we use the value of the European contract as a control variate and perform simulations using quasi-random sampling.

The rest of this paper is organized as follows. In Section 2, we describe the life insurance contract, a point to point Equity Indexed Annuity with a guaranteed minimum death benefit and a guaranteed minimum surrender benefit. The surrender decision is first based exclusively on financial criteria, in other words a policyholder surrenders his policy if the surrender amount gives him at least the opportunity to buy the same contract at a lower price on the market. But this approach overestimates the surrender option by looking at the worst case that practically never happens: all policyholders surrender at the same time (the optimal surrender time). In Section 2.3, we address this issue and introduce a parameter that represents the propensity of the individual to act optimally. In Section 3, we show how to include mortality risk in the least-squares Monte Carlo approach used to price the contract. Then we discuss in Section 4 the two methods we used to improve the efficiency of this Monte Carlo estimator: a control variate based on the European contract, and quasi-Monte Carlo sampling to simulate the financial paths. Numerical results are given in Section 5, and a brief conclusion is provided in Section 6.

## 2 FRAMEWORK

We first describe the contract, a point-to-point Equity Indexed Annuity (EIA) with a minimum guarantee at maturity or at time of death of the insured. Then, we give our assumptions for the underlying index dynamics, and we introduce the guaranteed minimum surrender benefit studied in this paper.

### 2.1 The contract

We consider a point-to-point EIA with a Guaranteed Minimum Maturity Benefit. The policy has a fixed maturity date of $T$ years. The insured's initial investment is $P$. We assume that no additional payments are done after the inception of the contract. Most EIAs in the U.S. are purchased with a single premium (see Palmer (2006)).

The contract is linked to an index, for instance the S&P 500 index. We denote by $S_t$ the value of the index at time $t$. We suppose a percentage $\alpha$ of the initial premium $P$ is guaranteed at a minimum annual rate equal to $g$. These parameters $\alpha$ and $g$ are often constrained by law. For instance, the United States have adopted recently revised nonforfeiture regulations, stating that the minimum guarantee under newly issued contracts must be at least 87.5% of all premiums paid, $P$ accumulated at a minimum interest rate, here $g$, between 1% and 3%. About one third of EIAs apply their stated interest rate guarantee to only 87.5% percent of the premium, one third to 90%, and only one fifth to 100%, see Palmer (2006).

The final payoff of the contract at maturity time $T$ writes as:

$$V_T = \alpha P \ \max\left((1+g)^T, \left(\frac{S_T}{S_0}\right)^k\right). \tag{1}$$

This contract is only for the purpose of illustration and this study can easily be extended to more complicated contracts. Insurance companies propose unit-linked policies or equity-indexed annuities with additional death guarantees. Assuming that we neglect longevity risk, mortality risk can be diversified. Indeed deaths observed in a portfolio of insureds might happen at independent dates. The large law of numbers and the central limit theorem are suitable to estimate risks. Note that this is not the case when mortality is stochastic (as studied, for instance, by Bacinello, Biffis, and Millossovich (2008a) and Bacinello, Biffis, and Millossovich (2008b).

Here we assume the equity-linked contract described by (1) is combined with a *Guaranteed Minimum Death Benefit*. We assume the guaranteed minimum death benefit is paid at the end of the year when the death of the insured occurs and that it can be written as:

$$D_t = \ \alpha P \ \max\left((1+g_d)^t, \left(\frac{S_t}{S_0}\right)^{k_d}\right), \qquad t = 1,\ldots,T, \tag{2}$$

if death occurs between $t-1$ and $t$, where we assume there is no penalty but the guaranteed rate $g_d$ and the participation $k_d$ in the index performance might be different than the ones involved in the contract's payoff described by (1). Palmer (2006) note that few EIAs assess a surrender charge if the EIA is cashed in due to the contract owner's death.

Unlike the mortality risk, the financial risk cannot be hedged by pooling arguments. All policies depend on the same index and if it goes up, the insurer will have to pay a high participation rate for all policies at the same time. This cannot be hedged by accepting more policies. Theoretically, those risks can be hedged by using the following decomposition of the contract's final value as

$$V_T = \alpha P \ (1+g)^T + \max\left(\left(\frac{S_T}{S_0}\right)^k - \alpha P \ (1+g)^T, 0\right), \tag{3}$$

which is a portfolio made up of a guaranteed amount $\alpha P$—usually yielding a lower interest rate $g$ than the risk-free rate $r$ of the market— plus a long position in a call option written on the underlying $S_T^k$. Using Black&Scholes-type results, companies can adjust their portfolio continuously and replicate the above payoff. This decomposition and its pricing in a Black and Scholes framework were first done in Brennan and Schwartz (1976) and in Boyle and Schwartz (1977).

## 2.2 Model

For the ease of exposition, here we neglect all types of costs, and assume the market is complete and perfectly liquid. We set ourselves in Black and Scholes framework, although our methodology could easily be applied to more complex models. The risk-free interest rate is constant and denoted by $r$. Since the market is complete, there is a unique risk-neutral measure, $Q$. Dynamics of the index $S$ under the $Q$-measure write as:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

where $\sigma$ is the volatility of the index and $W_t$ is a $Q$-standard Brownian motion. The fair value of the contract is obtained by taking the expectation under the risk-neutral probability $Q$ of the discounted cashflows:

$$\xi(S_0, g, k, T) \quad = \quad e^{-rT} E\left[V_T\right] \tag{4}$$

where $V_T$ is given by (1). After straightforward computations in the Black and Scholes framework, we obtain a closed-form formula for the European contract:

$$\begin{aligned}\xi(S_0, g, k, T) \quad &= \quad e^{(g-r)T} S_0 \Phi\left(\frac{g - k\left(r - \frac{\sigma^2}{2}\right)}{k\sigma}\sqrt{T}\right) \\ &\quad + \quad S_0 e^{(k-1)rT + k(k-1)\frac{\sigma^2 T}{2}} \Phi\left(-\alpha + k\sigma\sqrt{T}\right),\end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal variable.

This is the "risk-neutral price" the initial amount of money needed to perfectly hedge the final cash-flows and provide the payoff (1) to the insured for the European version of the contract where surrender is not allowed and there is no death benefit. This formula does not include the cost of hedging, transaction costs and the discrete hedging error that will obviously occurred in practice.

We now include the minimum guaranteed death benefit defined earlier in (2). Let us assume the probability that a policyholder of age $x+t$ will die before the end of the year is $q_{x+t}$, for $t=0,\dots,T-1$, and we let $p_{x+t}=1-q_{x+t}$. More precisely, if we include the mortality risk in the formula (4), then we have that the European contract's value at time 0, denoted $V_{0,e}$, is given by

$$V_{0,e} = {}_Tp_x \times \xi(S_0,g,k,T) + \sum_{t=0}^{T-1} {}_tp_x q_{x+t} \times \xi(S_0,g_d,k_d,t+1), \quad (5)$$

where $\xi$ is defined by the formula (4) and ${}_tp_x = \prod_{j=0}^{t-1} p_{x+j}$ is the probability that an individual of age $x$ survives at least $t$ years.

### 2.3 The surrender option

We are interested in a contract where there is a *Guaranteed Minimum Surrender Benefit*. That is, we assume that the above policy also offers a minimum surrender guarantee, and that the right to surrender can only be exercised at the end of each year until the maturity of the contract. Hence this is a Bermudan-type option rather than a truly American one.

In practice, this minimum guarantee is often independent of the return of the index, and thus we make the assumption that the policyholder receives the following amount $L_t$ if he surrenders the policy at time $t$:

$$L_t = (1-\beta_t)\ \alpha P\ (1+h)^t, \quad (6)$$

where $0 \le h < r$ is the guaranteed rate, and $\beta_t$ is the penalty charged for surrendering at time $t$, for $t=1,\dots,T-1$. A standard penalty would be for instance a decreasing rate over years. Examples of penalty functions are given in Palmer (2006). Sometimes a market value adjustment is applied when market performs poorly. By law, the policy's cash surrender value cannot fall below the guaranteed minimum value ($h \ge g$).

The individual will surrender because of financial reasons if

$$L_t > C_t,$$

where $C_t$ is the market value at time $t$ of the contract. In other words, if the insured has an opportunity to make a profit if he surrenders and that at the same time he can buy exactly the same policy.

However, and as mentioned in the introduction, it is plausible to assume that a policyholder may not make a decision that is optimal from the purely financial point of view when choosing to surrender or not. For this reason, we assume there is an extra decision parameter $\lambda \ge 1$ such that the contract is surrendered only if

$$L_t > \lambda C_t.$$

In some sense, it means that some agents will react faster to the surrender criteria than others. Informed agents will have $\lambda = 1$. For uninformed agents that do not worry about their investments, they will surrender if the surrender condition is really interesting and thus if $L_t$ is significantly higher than the market value of the contract $C_t$. It should be clear that as $\lambda$ increases, the value of the (financial) surrender option goes to 0, a fact that we verify in our experiments of Section 5.

## 3 LEAST-SQUARES MONTE CARLO WITH MORTALITY

Let us first review the least-squares Monte Carlo approach as proposed in Longstaff and Schwartz (2001) applied to the problem of pricing an equity-linked contract when there is no mortality risk.

The method uses $n$ realization paths $\{S_t^i, t = 0,1,\dots,T; i=1,\dots,n\}$ of the index, and then estimates for each path $i$ when is the optimal exercise time $t_i^*$. This is done by proceeding backward from $T$ as follows: set $t_i^* = T$, and the contract's value on path $i$ to $V_T^i$, which is the value of $V_T$ as in (1), but with $S_T$ replaced by the final value $S_T^i$ of the index on path $i$. Then at time $t = T-1, T-2, \dots, 1$, set $t_i^* = t$ and $V_t^i = L_t^i$ if $L_t^i > \hat{C}_t^i$, where $\hat{C}_t^i$ is an estimate of the continuation value of the contract at time $t$ given $S_t^i$, given by $e^{-r}\mathrm{E}(V_{t+1}|S_t^i)$, and $L_t^i$ is the surrender value at time $t$ on path $i$. This estimate $\hat{C}_t^i$ is obtained by regression of the discounted contract's value at the next time step against the current value of the index. More precisely, a finite set of multivariate basis functions $\{\psi_l(\cdot), l=0,1,\dots,M\}$ is chosen, and the regression coefficients are estimated as

$$(\hat{\beta}_0,\dots,\hat{\beta}_M)^T = (\Psi^T\Psi)^{-1}\Psi^T(y_1,\dots,y_n)^T,$$

where $y_i = e^{-r}V_{t+1}^i$, and $\Psi_{i,l} = \psi_l(S_t^i)$ for $i=1,\dots,n, l = 0,\dots,M$. Then $\hat{C}_t^i = \sum_{l=0}^M \hat{\beta}_l \psi_l(S_t^i)$. When $L_t^i \le \hat{C}(t,S_t^i)$, then we simply update the contract's value on path $i$ by discounting it for one more step, i.e., we set $V_t^i = e^{-r}V_{t+1}^i$.

Once the optimal exercise times $t_i^*$ are estimated for each path, the contract's value is approximated by the low-biased estimator

$$\hat{V}_{0,fin} = \frac{1}{n}\sum_{i=1}^n e^{-rt_i^*} L_{t_i^*}^i, \quad (7)$$

where we used the convention that $L_T^i = V_T^i$.

We now want to include the mortality risk in the above pricing methodology, as well as the surrender parameter $\lambda$ described in the previous section. When deciding whether to surrender or not, the policyholder must take into account the fact that he/she might die in the coming year, thus receiving $D_{t+1}$ at time $t+1$ rather than holding on to the contract. That is, the continuation value can be written as

$$\begin{aligned} \mathrm{E}(V_{t+1}|S_t^i) &= e^{-r}(q_{x+t}\mathrm{E}(V_{t+1}|S_t^i, death) \\ &\quad + p_{x+t}\mathrm{E}(V_{t+1}|S_t^i, survival)) \\ &= q_{x+t}e^{-r}\mathrm{E}(D_{t+1}|S_t^i) + p_{x+t}C_t \\ &= q_{x+t}\xi(S_t^i, g_d, k_d, 1) + p_{x+t}C_t. \end{aligned}$$

where $\xi$ is defined by (4). Hence at time $t$, the contract is surrendered if

$$L_t > \tilde{C}_t := q_{x+t}e^{-r}D_{t+1} + p_{x+t}\hat{C}_t,$$

where $\hat{C}_t$ is the continuation value *conditioned on survival until* $t+1$, and is calculated as before. In this case, we set $V_t = L_t$. However the contract's value $V_t$ must be updated differently when the surrender value $L_t$ does not exceed the weighted continuation value $\tilde{C}_t$. More precisely, we have

$$V_t = e^{-r}(q_{x+t}\xi(S_t, g_d, k_d, 1) + p_{x+t}V_{t+1}).$$

Once we have an estimate of the optimal exercise time $t_i^*$ on each path $i$, then we must take into account the mortality risk by replacing the estimator (7) by

$$\hat{V}_0 = \frac{1}{n}\sum_{i=1}^{n} A_i, \tag{8}$$

where

$$A_i = \left( {}_{t_i^*}p_x e^{-rt_i^*} L_{t_i^*}^i + \sum_{t=0}^{t_i^*-1} {}_t p_x q_{x+t} e^{-r(t+1)} D_{t+1}^i \right),$$

and ${}_t p_x = \prod_{j=0}^{t-1} p_{x+j}$ is the probability that an individual of age $x$ survives at least $t$ years. The idea is that if the policyholder dies before the estimated optimal surrender time, then a death benefit will instead be paid at the end of the year of death.

Summing up, we proceed as in the pseudocode given in Figure 1.

```
LsMCEqLinkMort()
for i ← 1 to n
    t*(i) ← T
    generate S^i_1, ..., S^i_T
    V[i] ← V^i_T
for j ← T − 1 downto 1
    q ← q_{x+t}
    p = 1 − q
    compute β̂_0, ..., β̂_M
    for i ← 1 to n
        C^i_t ← Σ_{l=0}^{M−1} β̂_{l,j} ψ_l(S^i_t)
        C̃^i_t ← e^{−r} q ξ(S^i_t, g_d, k_d, 1) + p C^i_t
        if L^i_t > λ × C^i_t then
            t*[i] ← t
            V[i] ← L^i_t
        else
            V[i] ← e^{−r}(q ξ(S^i_t, g_d, k_d, 1) + p V[i])
for i ← 1 to n
    A[i] ← _{t*[i]}p_x e^{−rt*[i]} L^i_{t*[i]}
    for t = 0 to t*[i]
        A[i] ← A[i] + e^{−rt} _t p_x q_{x+t} D^i_{t+1}
return Σ_{i=1}^{n} A[i]/n
```

Figure 1: Pseudocode describing regression-based Monte Carlo for an equity-linked contract with mortality

## 4 EFFICIENCY IMPROVEMENT

A first way to improve the quality of the estimator (8) is to use the value of the corresponding European contract as a control variate. We know the European contract value when the mortality is included. It is given by $V_{0,e}$ defined earlier by (5). Hence the control variate estimator is given by

$$\hat{V}_{0,cv} = \hat{V}_0 + \hat{\gamma}(V_{0,e} - \hat{V}_{0,e}), \tag{9}$$

where $\hat{V}_{0,e}$ is given by

$$\hat{V}_{0,e} = \frac{1}{n}\sum_{i=1}^{n} E^i,$$

where

$$E^i = e^{-rT}{}_T p_x V_T^i + \sum_{t=0}^{T-1} e^{-r(t+1)}{}_t p_x q_{x+t} D_{t+1}^i.$$

The coefficient $\hat{\gamma}$ in (9) is given by the estimated value of $\mathrm{Cov}(E_i, A_i)/\mathrm{Var}(E_i)$. The surrender option can then be estimated as

$$\hat{X}_{0,cv} = \hat{V}_{0,cv} - V_{0,e}. \tag{10}$$

Secondly, we can use randomized quasi-Monte Carlo methods to simulate the $n$ paths of the index. This was done in the context of the least-squares Monte Carlo algorithm in Lemieux (2004), among others. Here, we do this using a randomly digitally shifted Sobol' point set, and apply the Brownian bridge technique (Caflisch and Moskowitz 1995) in order to reduce the effective dimension of the problem.

More precisely, what we need here is a $T$-dimensional point set $P_n$ in $[0,1)^T$, obtained using the first $n$ points of the Sobol' sequence (Sobol' 1967) with the direction numbers given in Bratley and Fox (1988). Then we randomize this point set using a random digital shift (L'Ecuyer and Lemieux 2002). By repeating the randomization process $m$ times independently, we thus obtain $m$ estimators $\hat{V}_{0,l}$ of the form (8) for the contract (with the surrender option), but where the path $S_1^i,\ldots,S_T^i$ is based on the randomized point $\tilde{\mathbf{u}}_{i,l}$, obtained by adding the $l$th digital shift to the $i$th point of $P_n$, for $i=,1\ldots,n$ and $l=1,\ldots,m$. The application of the Brownian bridge technique in this case amounts to use the first coordinate of $\tilde{\mathbf{u}}_{i,l}$ to generate $S_T^i$, when constructing the $l$th estimator $\hat{V}_{0,l}$, then the second coordinate is used to generate $S^i_{\lfloor T/2\rfloor}$, the third one for $S^i_{\lfloor T/4\rfloor}$, and so on.

Once we have these $m$ estimators $\hat{V}_{0,l}$, we can construct a 95% confidence interval for $V_0$ with half-width

$$1.96\sqrt{\frac{1}{m(m-1)}\sum_{l=1}^{m}(\hat{V}_{0,l}-\bar{V}_0)^2},$$

where $\bar{V}_0=\sum_{l=1}^m \hat{V}_{0,l}/m$. A similar approach can be used for the control variate estimator.

## 5 RESULTS

In what follows, we give results for different equity-linked contracts. In each case, we report the value of the contract, $\hat{V}_0$, the simulated value of the contract without a surrender option, given by $\hat{V}_{0,e}$, and the value $\hat{X}_{0,cv}$ of the surrender option. Note that we also report the exact value of the contract in the line starting with "exact". For each estimate, we report results using either Monte Carlo (MC) or randomized quasi-Monte Carlo (RQMC) sampling, and give below each estimate the half-width of a 95% confidence interval for the value that we are trying to estimate. To price the contract, we run simulations either with a control variate—as given in lines "MC-cv" and "RQMC-cv"—or without—as given in lines "MC-nocv" and "RQMC-nocv". Note that since the estimate $\hat{X}_{0,cv}$ for the surrender option differs only by a constant from the one $V_{0,cv}$ for the contract itself, the half-width of the confidence interval is the same for both estimators.

The default (benchmark) values for the parameters are: $T=10$ years, $\sigma=20\%$, $P=100$, $\alpha=0.85$, $r=4\%$, the minimum guaranteed rates $g$, $h$ and $g_d$ are all set to 2%, and the participating coefficient $k$ and $k_d$ to 90%. The penalties are as follows: $\beta_1=0.05$, $\beta_2=0.04$, $\beta_3 0.02$, $\beta_4=0.01$, $\beta_t=0$ for $t\geq 5$. We also assume that the policyholder is rational and perfectly informed: $\lambda=1$.

For mortality, we use a simple parametric model: the Makeham's model. The survival function $s(x)$, which gives the probability that an individual of age 0 will survive at least $x$ years, is given by

$$s(x)=\exp\left(-\int_0^x \mu(s)ds\right)=\exp\left(-Ax-B\left(c^x-1\right)\right)$$

where $\mu(x)=A+Bc^x$ with $B>0, A\geq -B, c>1, x\geq 0$. We can then calculate the probability to live and to die as follows:

$$_tp_x=\frac{s(x+t)}{s(x)},\quad _tq_x=1-{}_tp_x.$$

Melnikov and Romaniuk (2006) calibrated Makeham's model for the mortality in the USA and give the following estimates of the parameters $A$, $B$ and $c$:

$$A=9.5666\times 10^{-4},\quad B=5.162\times 10^{-5},\quad c=1.09369.$$

All experiments have been performed by generating 25 i.i.d. groups of 8192 simulations, for a total of 204,800 simulation runs. This is about 10 times less than the number of simulations—19,000 simulations with 140 seeds, for a total of $2,66\times 10^6$ runs—used in Bacinello, Biffis, and Millossovich (2008a), Bacinello, Biffis, and Millossovich (2008b), who get results with about the same precision as our RQMC estimators with the control variate (but use stochastic mortality, as explained before). For the basis functions of the regression, we use the first four Legendre polynomials.

Table 1: Result with the default parameters

| | $\hat{V}_0$ | $\hat{V}_{0,e}$ | $\hat{X}_0$ |
|---|---|---|---|
| MC-nocv | 94.479 | 92.689 | – |
| | 0.208 | 0.216 | – |
| MC-cv | 94.565 | – | 1.785 |
| | 0.024 | – | 0.024 |
| RQMC-nocv | 94.593 | 92.781 | – |
| | 0.009 | 0.009 | – |
| RQMC-cv | 94.593 | – | 1.813 |
| | 0.009 | – | 0.009 |
| exact | – | 92.7795 | – |

In Table 1, we report our results with the benchmark parameters. The European contract has the value 92.7795. Note that the initial premium of the contract is $P=100$ and thus the market value of the contract is lower than the premium charged at time 0. This is consistent with real contracts that often include commissions and fees for different services and options. Including the surrender option increases the value of the contract by approximately 1.8.

Thus, the surrender option represents about 1.9% of the total value of the (European) contract.

We now experiment with different values of $\lambda$, participating parameters, and model parameters.

Table 2: Varying $\lambda$

| | $\hat{V}_0$ | $\hat{V}_{0,e}$ | $\hat{X}_0$ |
|---|---|---|---|
| $\lambda = 1.05$ | | | |
| MC-nocv | 93.716 | 92.689 | – |
| | 0.206 | 0.216 | – |
| MC-cv | 93.801 | – | 1.022 |
| | 0.030 | – | 0.030 |
| RQMC-nocv | 93.750 | 92.781 | – |
| | 0.024 | 0.009 | – |
| RQMC-cv | 93.750 | – | 0.970 |
| | 0.024 | – | 0.024 |
| exact | – | 92.7795 | – |
| $\lambda = 1.15$ | | | |
| MC-nocv | 92.718 | 92.689 | – |
| | 0.219 | 0.216 | – |
| MC-cv | 92.810 | – | 0.031 |
| | 0.015 | – | 0.015 |
| RQMC-nocv | 92.786 | 92.781 | – |
| | 0.011 | 0.009 | – |
| RQMC-cv | 92.784 | – | 0.005 |
| | 0.004 | – | 0.004 |
| exact | – | 92.7795 | – |

We first increase the value of $\lambda$ to 1.05 or 1.15 and we report the results in Table 2. We observe that when $\lambda$ is increasing, the policyholder becomes less and less interested in the surrender option. Its value becomes almost zero. In fact, insurers rely a lot on this fact. Policyholders will not have a financially optimal behaviour and will be reluctant to surrender their policies. A situation where $\lambda > 1$ as it is the case in Table 2 is thus more realistic. Note that it would be interesting to look at the risk of a portfolio of policies with a pool of policyholders with different parameter $\lambda$.

We can also look at the impact of changes in the participating coefficient $k$ and $k_d$, or in the minimum guaranteed rates $g$, $g_d$, and $h$. Our results suggest that the surrender option is less valuable when the minimum guarantees at maturity, for death, and for surrender are higher (that is, $g = h = g_d$ is higher) or if the participating coefficient $k$ at maturity or $k_d$ at time of death increases. Indeed, the surrender option represents $0.82/96.73 = 0.85\%$ of the value of the European contract when the guaranteed rates increase to $g = h = g_d = 3\%$ instead of the benchmark case of $g = h = g_d = 2\%$, and $1.68/95.27 \approx 1.8\%$ when the participating coefficients $k$ and $k_d$ are 95% instead of 90%, instead of a ratio of 1.95% with the default parameters. In the second case, we expect that an increase in the participation at maturity and in case of death are incentive

Table 3: Varying participation parameters

| | $\hat{V}_0$ | $\hat{V}_{0,e}$ | $\hat{X}_0$ |
|---|---|---|---|
| $g = h = 0.03$ | | | |
| MC-nocv | 97.437 | 96.640 | – |
| | 0.206 | 0.205 | – |
| MC-cv | 97.530 | – | 0.798 |
| | 0.023 | – | 0.023 |
| RQMC-nocv | 97.558 | 96.734 | – |
| | 0.014 | 0.009 | – |
| RQMC-cv | 97.556 | – | 0.823 |
| | 0.010 | – | 0.010 |
| exact | – | 96.7327 | – |
| $k = k_d = 0.95$ | | | |
| MC-nocv | 96.827 | 95.170 | – |
| | 0.233 | 0.242 | – |
| MC-cv | 96.924 | – | 1.652 |
| | 0.024 | – | 0.024 |
| RQMC-nocv | 96.956 | 95.273 | – |
| | 0.013 | 0.011 | – |
| RQMC-cv | 96.955 | – | 1.683 |
| | 0.011 | – | 0.011 |
| exact | – | 95.2718 | – |
| $h = 0.03$ | | | |
| MC-nocv | 96.907 | 92.689 | – |
| | 0.205 | 0.216 | – |
| MC-cv | 96.992 | – | 4.213 |
| | 0.026 | – | 0.026 |
| RQMC-nocv | 97.023 | 92.781 | – |
| | 0.011 | 0.009 | – |
| RQMC-cv | 97.022 | – | 4.243 |
| | 0.008 | – | 0.008 |
| exact | – | 92.7795 | – |

to not surrender. But in the first case, there is a tradeoff between the increased guarantee at maturity and death—which are incentive for not exercising—and the increase in the guarantee in case of surrender. Our results suggest that the increase at maturity and death offset the increase in the surrender payoff. However, if only the guaranteed rate $h$ for the surrender option increases to 3%, then, as expected, the option's value increases from about 1.8 to about 4.24.

Finally, we can analyze the impact of the financial assumptions and in particular the volatility of the underlying index. We can compare the value of the surrender option equal to 1.8 when volatility is 20% (benchmark parameter in Table 1), with the values in Table 4, given by 1.47 if $\sigma = 10\%$ or 1.91 if $\sigma = 30\%$. The value of the surrender option thus appears to be increasing with the volatility $\sigma$. This is consistent with the well-known fact that the value of an option increases with the volatility of the underlying index.

Table 4: Varying financial model

| | $\hat{V}_0$ | $\hat{V}_{0,e}$ | $\hat{X}_0$ |
|---|---|---|---|
| $\sigma = 0.1$ | | | |
| MC-nocv | 87.270 | 85.817 | – |
| | 0.095 | 0.100 | – |
| MC-cv | 87.294 | – | 1.451 |
| | 0.017 | – | 0.017 |
| RQMC-nocv | 87.310 | 85.844 | – |
| | 0.006 | 0.002 | – |
| RQMC-cv | 87.310 | – | 1.466 |
| | 0.006 | – | 0.006 |
| exact | – | 85.8434 | – |
| exact | – | 85.8434 | – |
| $\sigma = 0.3$ | | | |
| MC-nocv | 100.995 | 99.072 | – |
| | 0.357 | 0.374 | – |
| MC-cv | 101.176 | – | 1.914 |
| | 0.035 | – | 0.035 |
| RQMC-nocv | 101.176 | 99.264 | – |
| | 0.023 | 0.026 | – |
| RQMC-cv | 101.174 | – | 1.913 |
| | 0.014 | – | 0.014 |
| exact | – | 99.2618 | – |

## 6 CONCLUSIONS

This paper extends the Least Squares Monte Carlo method proposed by Longstaff and Schwartz (2001) to the valuation of surrender benefits embedded in life insurance index linked contracts. With a relatively small number of simulations, we get a quite precise approximation of the surrender benefit using quasi-Monte Carlo simulations and incorporating a powerful control variate. We show how to include mortality risk when this risk is modeled using fixed probabilities that are the same for everybody. In practice, policyholders who decide to surrender a life insurance contract are healthier than the average. The surrender option will introduce adverse selection (worse risks stay with the insurer and healthy people leave the portfolio) and thus will increase a lot the potential value of the surrender benefits. We believe that the proposed method could easily be extended, for instance, to the case where the optimal decision of the policyholder is taken conditionally to his health state. We plan to study this model in the near future.

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