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A LARGE DEVIATIONS VIEW OF ASYMPTOTIC EFFICIENCY FOR SIMULATION ESTIMATORS

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ABSTRACT

Consider a simulation estimator $\alpha(c)$ based on expending c units of computer time, to estimate a quantity α. One measure of efficiency is to attempt to minimize $P(|\alpha(c)-\alpha| > \epsilon)$ for large c. This helps identify estimators with less likelihood of witnessing large deviations. In this article we establish an exact asymptotic for this probability when the underlying samples are independent and a weaker large deviations result under more general dependencies amongst the underlying samples.

1 INTRODUCTION

Consider the problem of numerically computing the quantity α which can be expressed as the expectation of a random variable X. Assuming that independent copies of the random variable X can be generated, α can be computed via simulation.

In many problem contexts there may be more than one means of expressing α as an expectation. In particular, suppose that $\alpha = EX = EY$. There are then two obvious alternative approaches to computing α, one based on independent replication of X and the other based on on independent replication of Y. Given two such competing estimators for α, one then wishes to choose the estimator with maximum computational efficiency. Such a selection requires a concrete notion of computational efficiency for simulation estimators.

This problem has been previously addressed by Glynn and Whitt (1992), based on ideas going back at least as far as Hammersley and Handcomb (1964). The idea is to choose the estimator which maximizes the asymptotic convergence rate. The convergence rate of the estimator can be studied using the central limit theorem (CLT). Such a CLT needs to take into account the fact that efficiency is a function both of the variance of the estimator and the computer time required to generate the estimator.

Specifically, for a given computer budget c, let $\alpha(c)$ be the estimator for α based on independently replicating the random variable X. Note that the number of replications of X completed in c units of computer time is itself a random variable having a distribution that depends on the chracteristics of the random quantity τ_X, where τ_X is the time required to generate a single copy of X. Glynn and Whitt (1992) prove that if $E\tau_X < \infty$ and $var(X) < \infty$, then

$$c^{1/2}(\alpha(c) - \alpha) \to \sqrt{E\tau_X var(X)}N(0,1),$$

as $c \to \infty$, where $N(0,1)$ denotes the standard Gaussian random variable. Based on this CLT, it is natural to choose the estimator which maximizes the 'figure of merit' given by the reciprocal of the product of the mean time to generate each replication with the variance per replication. Glynn and Whitt (1992) also examine convergence rates for a number of more complex simulation estimators.

In this article, we introduce a new measure of computational efficiency for a simulation estimator. Suppose that our goal is to compute the quantity α to a given absolute precision ϵ ($\epsilon > 0$). It is then natural to select the estimator $\alpha(c)$ based on expending c units of computer time that maximizes $P(|\alpha(c) - \alpha| \le \epsilon)$. We will prove, in Section 2, that under certain regularity conditions on the estimator,

$$\frac{1}{c}\log P(|\alpha(c) - \alpha| > \epsilon) \to -\beta \qquad (1)$$

as $c \to \infty$, for some positive finite constant β. Hence, if we wish to maximize the 'probability concentration' of the estimator around α, we should seek to find an estimator that maximizes the value of β. As in our earlier discussion, the parameter β depends not just on the distribution of the random variables being replicated, but also the random quantity τ_X that describes the computer time required to generate X. For example, if τ_X is heavy tailed so that its positive exponential moments do not exist, then it is easy to see that $\beta = 0$ for $\epsilon < \alpha$, making the estimator unattractive based on (1). The use of probability concentration criteria to study estimator efficiency has a substantial history in the statistics literature; for example, see Serfling (1980).

From a mathematical standpoint, the most novel feature of our analysis is that our 'large deviations' result (1) describes the behavior of $\alpha(c)$ based on expending c units of computer time (rather than the more traditional sample mean estimator associated with averaging the first n replications of X). Thus our large deviations result takes into account the additional variability induced by the fact that the number of replications completed in c units of computer time is itself random.

Section 2 provides a complete mathematical description of our main result and includes some of the key proofs. Here we develop an exact asymptotic for the probability that $\alpha(c)$ has a large deviation from its mean as $c \to \infty$, in the settings where independent identically distributed samples of (X, τ_X) are generated. Under more general dependence structure we develop large deviations results as in (1). It should be noted that this large deviations result plays an important role in a related analysis of ordinal optimization algorithms that we have undertaken; see Glynn and Juneja (2004) for details. The more technical proofs that are not central to our analysis are relegated to the appendix in Section 3.

The exact asymptotics that we derive in the i.i.d. settings can also be inferred as a special case of the results in Chi (2007). Our analysis, focussed on a specialized problem is simpler. As mentioned earlier, we also develop large deviations results under more general dependence structure not considered in Chi (2007).

2 FRAMEWORK

2.1 Preliminaries

There exists a sequence $(X_n : n \geq 1)$ of simulatable random variables (rv) for which

$$\frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow \alpha. \tag{2}$$

Let $\tau_i \geq 0$ denote the time to generate X_i. Let $S_n = \sum_{i=1}^{n} X_i$ and $T_n = \sum_{i=1}^{n} \tau_i$ with $(S_0, T_0) = (0, 0)$. Let $N(c) = \sup\{n : T_n \leq c\}$. Thus, $N(\cdot)$ is a renewal counting process and it denotes the number of observations generated in a unit computer time.

Put

$$\begin{aligned} \alpha(c) &= S_{N(c)}/N(c) \text{ if } N(c) \geq 1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

In Section 2.2 we develop exact asymptotics for the probabilities $P(\alpha(c) > a)$. In Section 2.3 we compute the large deviations efficiency

$$\lim_{c\to\infty} \frac{1}{c} \log P(\alpha(c) > a)$$

under general dependency conditions for the process $((X_i, \tau_i) : i \geq 1)$. The logarithmic asymptotics $\lim_{c\to\infty} \frac{1}{c} \log P(|\alpha(c) - \alpha| > \epsilon)$ are easily inferred from these results.

Let $\psi_n : \Re^2 \to \Re \cup \{\infty\}$ denote the log-moment generating function of (S_n, T_n) so that

$$\psi_n(\theta, \eta) = \frac{1}{n} \log E[exp(\theta S_n + \eta T_n)].$$

The following assumption is important in our analysis:

Assumption 1 *For each $(\theta, \eta) \in \Re^2$, the logarithmic moment generating function defined as the limit*

$$\psi(\theta, \eta) \equiv \lim_{n\to\infty} \frac{1}{n} \log E[exp(\theta S_n + \eta T_n)]$$

exists as an extended real number. Further, it is twice continuously differentiable and strictly convex in the interior of

$$\mathcal{N} = \{(\theta, \eta) : \psi(\theta, \eta) < \infty\},$$

and there exist (θ^*, η^*) and $(-\theta^*, \eta^*)$ in the interior of $\mathcal{N}$ such that

$$\psi(\theta^*, \eta^*) = \theta^* a \tag{3}$$

and

$$\frac{\partial}{\partial\theta}\psi(\theta^*, \eta^*) = a. \tag{4}$$

The requirement that $(-\theta^*, \eta^*)$ belong to interior of $\mathcal{N}$, while not critical to our analysis, allows considerable simplifications in some of the proofs.

Significant notational simplification occurs by replacing $X_i - a$ by X_i. Then, without loss of generality, we focus on developing the asymptotic for $P(S_{N(c)} > 0)$ where now α in (2) is negative and (3) and (4) are appropriately modified so that

$$\psi(\theta^*, \eta^*) = 0, \tag{5}$$

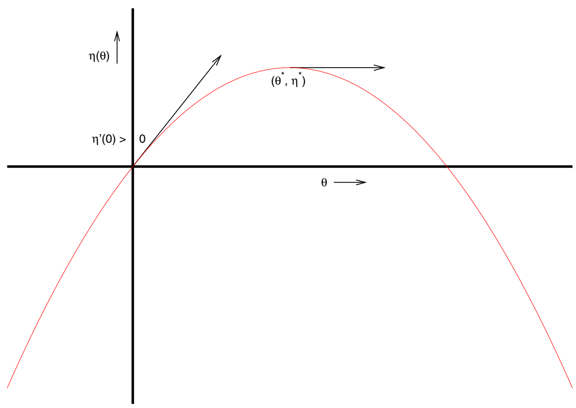

Figure 1: Typical level curve $\psi(\theta, \eta(\theta)) = 0$

and

$$\frac{\partial}{\partial \theta}\psi(\theta^*, \eta^*) = 0, \tag{6}$$

respectively. In the remaining paper, Assumption 1 denotes (5) and (6) in place of (3) and (4).

Note that $\frac{\partial}{\partial\theta}\psi(0,0) = \alpha < 0$ and since each $\tau_i \geq 0$, $\frac{\partial}{\partial\eta}\psi(\theta,\eta) > 0$.

Proofs of Lemmas 1 and 2 are peripheral to our analysis and are omitted.

Lemma 1 *Under Assumption 1, $\theta^* > 0$ and $\eta^* > 0$.*

Lemma 2 below suggests that Figure 1 depicts a typical level curve

$$\psi(\theta, \eta(\theta)) = 0 \tag{7}$$

when Assumption 1 holds.

Lemma 2 *Under Assumption 1, $\eta(\theta)$, solution to (7), is strictly concave for $\theta \in [0, \theta^*]$. In addition, $\eta'(0) > 0$ and $\eta'(\theta^*) = 0$.*

Lemma 2 also implies that the solution to (5) and (6) when it exists, is unique.

2.2 An Exact Asymptotic For Independent Identically Distributed Random Vectors

Let $((X_i, \tau_i) : i \geq 1)$ denote a sequence of i.i.d. random vectors under the probability P.

Let $F(\cdot,\cdot)$ denote the distribution function of (X_i, τ_i) under P. Then,

$$\psi(\theta, \eta) = \log \int_{(x,t)\in\Re^2} e^{\theta x + \eta t} dF(x,t).$$

Some notation is needed for further analysis.

- Let $\tilde{F}(\cdot,\cdot)$ denote the distribution obtained by exponentially twisting $F(\cdot,\cdot)$ with parameters (θ^*, η^*), i.e., for $(x,t) \in \Re^2$

$$d\tilde{F}(x,t) = e^{\theta^* x + \eta^* t} dF(x,t).$$

- Let $\tilde{P}$ denote the probability associated with the independent random vectors $((X_i, \tau_i) : i \geq 1)$ under $\tilde{F}(\cdot,\cdot)$ and let $\tilde{E}$ denote the associated expectation operator under $\tilde{P}$ (E denotes the expectation operator under P). Let $\tilde{\sigma}^2(Z)$ denote the variance of rv Z under $\tilde{P}$.
- For notational convenience let (X, τ) have the same distribution as (X_i, τ_i). Note that $\tilde{E}X = 0$. Let $\lambda = 1/\tilde{E}\tau$ and let ρ denote the correlation between X and τ under $\tilde{P}$.

The exact asymptotic result relies critically on the local limit theorems for which the following assumption is needed.

Assumption 2 *Under $\tilde{P}$, the rv (X, τ) have a joint probability density function (pdf) $\tilde{f}(\cdot,\cdot)$. Suppose that $\tilde{f}^{n*}(\cdot,\cdot)$ denotes the pdf obtained by convolution of $\tilde{f}(\cdot,\cdot)$ with itself n times. Then, there exists an $n > 0$ such that $\tilde{f}^{n*}(\cdot,\cdot)$ is bounded.*

Theorem 1 states the main result of this section:

Theorem 1 *Consider $((X_i, \tau_i) : i \geq 1)$ a sequence of i.i.d. random vectors so that $EX_i < 0$, and Assumptions 1 and 2 hold. Then,*

$$\lim_{c\to\infty} \sqrt{c}\exp(\eta^* c) P(S_{N(c)} > 0)$$

equals

$$\sqrt{\frac{\lambda}{2\pi}} \frac{1}{\tilde{\sigma}(X)\theta^*\eta^*}(\exp(\psi(0,\eta^*)) - 1). \tag{8}$$

Note that $N(c)+1$ is a stopping time w.r.t. the process $((X_i, \tau_i) : i \geq 1)$. Thus, $P(S_{N(c)} > 0) = E(I(S_{N(c)} > 0))$ may be re-expressed under the measure $\tilde{P}$ as

$$\tilde{E}(\exp(-\theta^* S_{N(c)+1} - \eta^* T_{N(c)+1}) I(S_{N(c)} > 0)).$$

This equals $\exp(-\eta^* c)$ times

$$\tilde{E}(\exp(-W_c - \theta^* S_{N(c)}) I(S_{N(c)} > 0)). \tag{9}$$

where $W_c = \theta^* X_{N(c)+1} + \eta^*(T_{N(c)+1} - c)$. Thus, proving Theorem 1 amounts to showing that in the limit, $c^{1/2}$ times (9) equals (8).

Let $\beta(c) = \lfloor \lambda c - c^{5/8} \rfloor$. Let $\tilde{T}_{\beta(c)}$ denote the centered and normalized rv

$$\frac{1}{\beta(c)^{1/2}\tilde{\sigma}(\tau)}[T_{\beta(c)} - \lambda^{-1}\beta(c)].$$

Note that $|\tilde{T}_{\beta(c)}| < \sqrt{\log(c)}$ implies that

$$\begin{aligned} T_{\beta(c)} &\leq \lambda^{-1}\beta(c) + \sqrt{\log(c)}\beta(c)^{1/2}\tilde{\sigma}(\tau) \\ &= c - \lambda^{-1}c^{5/8} + o(c^{5/8}). \end{aligned}$$

Hence, $T_{\beta(c)} < c$ and $N(c) \geq \beta(c)$ for all c sufficiently large. In our asymptotic analysis, we henceforth consider c sufficiently large so that $|\tilde{T}_{\beta(c)}| < \sqrt{\log(c)}$ implies $N(c) \geq \beta(c)$.

Also consider the centered and scaled rv $\tilde{S}_{\beta(c)} = \frac{S_{\beta(c)}}{\tilde{\sigma}(X)\beta(c)^{1/2}}$ and

$$\tilde{Y}_{\beta(c)} = \frac{S_{N(c)} - S_{\beta(c)}}{\tilde{\sigma}(X)\beta(c)^{1/2}} I(N(c) > \beta(c)).$$

Therefore, when $T_{\beta(c)} < c$, $\tilde{Y}_{\beta(c)}$ is distributed as

$$\frac{S_{N(c - T_{\beta(c)})}}{\tilde{\sigma}(X)\beta(c)^{1/2}}.$$

Then, along the set $T_{\beta(c)} < c$,

$$S_{N(c)} = \left(\tilde{S}_{\beta(c)} + \tilde{Y}_{\beta(c)}\right)\tilde{\sigma}(X)\beta(c)^{1/2}.$$

Define, B_c to equal

$$\{|\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}, W_c < \nu\log\beta(c), \\ |\tilde{Y}_{\beta(c)}| < \beta(c)^{-1/16}\}$$

where $\nu \geq 1$ is a constant sufficiently large so as to satisfy the conditions that we mention later. For any set A, let A^C denote the complement.

Lemma 3 is proved in the appendix.

Lemma 3 *Under conditions of Theorem 1,*

$$\tilde{E}(\exp(-W_c - \theta^* S_{N(c)})I(S_{N(c)} > 0), B_c^C) = o(c^{-1/2}). \tag{10}$$

In view of our discussion and Lemma 3 , Theorem 1 follows from Lemma 4 below.

Lemma 4 *Under conditions of Theorem 1,*

$$\lim_{c\to\infty} \sqrt{\beta(c)}\tilde{E}(\exp(-W_c - \theta^* S_{N(c)})I(S_{N(c)} > 0), B_c) \tag{11}$$

equals

$$\lambda\sqrt{\frac{1}{2\pi}}\frac{1}{\tilde{\sigma}(X)\theta^*\eta^*}(\exp(\psi(0,\eta^*)) - 1).$$

To see (11), note that

$$\begin{aligned} &\exp(-W_c - \theta^* S_{N(c)}) \\ &= \int_{-\infty}^{\infty} \exp(-u) I(\theta^* S_{N(c)} + W_c < u) du. \end{aligned}$$

Hence, we may re-express $\tilde{E}(\exp(-W_c - \theta^* S_{N(c)})I(S_{N(c)} > 0), B_c)$ as

$$\tilde{E}\left[\int_{W_c}^{\infty} e^{-u} I(0 < S_{N(c)} < \frac{u - W_c}{\theta^*}, B_c) du\right]. \tag{12}$$

Since, $\int_{(\nu+1)\log\beta(c)}^{\infty} \exp(-u)du = 1/\beta(c)^{\nu+1} = o(1/c^{1/2})$, $\beta(c)^{1/2}$ times

$$\tilde{E}\left[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u} I(0 < S_{N(c)} < \frac{u - W_c}{\theta^*}, B_c) du\right]. \tag{13}$$

has the same limiting value as $\beta(c)^{1/2}$ times (12).

Let $\mathcal{A}_c$ denote the σ algebra associated with random variables

$$(T_{\beta(c)} I(T_{\beta(c)} < c), \ldots, T_{N(c)+1} I(T_{\beta(c)} < c))$$

and

$$(X_{\beta(c)+1} I(T_{\beta(c)} < c), \ldots, X_{N(c)+1} I(T_{\beta(c)} < c)).$$

Then, (13) equals

$$\tilde{E}\left[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u} \tilde{P}(0 < S_{N(c)} \\ < \frac{u - W_c}{\theta^*}, B_c | \mathcal{A}_c) du\right],$$

or

$$\tilde{E}[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u} \times \\ \tilde{P}\left(-\tilde{Y}_{\beta(c)} < \tilde{S}_{\beta(c)} < -\tilde{Y}_{\beta(c)} + \\ \frac{u - W_c}{\theta^*\tilde{\sigma}(X)\beta(c)^{1/2}}, B_c | \mathcal{A}_c\right) du]. \tag{14}$$

Let

$$\gamma_n(y,w,u,t) = \tilde{P}(-y < \tilde{S}_n < -y + \frac{u-w}{\theta^*\tilde{\sigma}(X)\sqrt{n}}|\tilde{T}_n = t)$$

Then, (14) equals

$$\tilde{E}\left[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u}\gamma_{\beta(c)}(\tilde{Y}_{\beta(c)}, W_c, u, \tilde{T}_{\beta(c)})I(B_c)du\right]. \quad (15)$$

Lemma 5 relies on local central limit theorems and is useful to our analysis. Some notation is needed for its statement and proof. Due to the central limit theorem, $(\tilde{S}_n, \tilde{T}_n)$ converge to a zero mean, unit variance bivariate Gaussian random vector $(\tilde{S}, \tilde{T})$ with correlation ρ, i.e., the correlation between X and τ under $\tilde{P}$. Let $\tilde{f}_n(\cdot,\cdot)$ denote the joint pdf of $(\tilde{S}_n, \tilde{T}_n)$ under $\tilde{P}$. Let $\phi_\rho(\cdot,\cdot)$ denote the bivariate Gaussian density function of $(\tilde{S}, \tilde{T})$. Let $\tilde{f}_{\tilde{S}_n|\tilde{T}_n=t}(\cdot)$ denote the probability density of $\tilde{S}_n$ conditioned on $\tilde{T}_n = t$ and let $\phi_{\tilde{S}|\tilde{T}=t}(\cdot)$ denote the probability density of $\tilde{S}$ conditioned on $\tilde{T} = t$. Note that $\phi_{\tilde{S}|\tilde{T}=t}(\cdot)$ is simply a pdf of a Gaussian rv with mean ρt and variance $(1-\rho^2)$.

Proof of Lemma 5 is given in the appendix.

Lemma 5 *Under Assumptions 1 and 2,*

$$\sup_{u<(\nu+1)\log n, |y|<n^{-1/16}, |w|<\nu\log n, |t|<\nu\sqrt{\log n}}$$

$$\left(\sqrt{n}\gamma_n(y,w,u,t) - \frac{u-w}{\theta^*\tilde{\sigma}(X)}\phi_{\tilde{S}|\tilde{T}=t}(0)\right)$$

converges to zero as $n \to \infty$.

Proof.[**Lemma 4**]

From Lemma 5, it follows that

$$\lim_{c\to\infty} \tilde{E}[\int_{W_{\beta(c)}}^{(\nu+1)\log\beta(c)} e^{-u}\times$$

$$\left(\beta(c)^{1/2}\gamma_{\beta(c)}(\tilde{Y}_{\beta(c)}, W_c, u, \tilde{T}_{\beta(c)}) - \right.$$
$$\left. -\frac{u-W_c}{\theta^*\tilde{\sigma}(X)}\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)\right)\times$$

$$I(B_c)du] = 0 \quad (16)$$

To see (16), note that for $\epsilon > 0$ and c sufficiently large

$$\left[\beta(c)^{1/2}\gamma_{\beta(c)}(\tilde{Y}_{\beta(c)}, W_c, u, \tilde{T}_{\beta(c)}) - \right.$$
$$\left. -\frac{u-W_c}{\theta^*\tilde{\sigma}(X)}\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)\right] I(B_c)$$

is less than ϵ a.s. for all $u < (\nu+1)\log\beta(c)$. Therefore,

$$\tilde{E}[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u}\times$$

$$\left(\beta(c)^{1/2}\gamma_{\beta(c)}(\tilde{Y}_{\beta(c)}, W_c, u, \tilde{T}_{\beta(c)}) - \right.$$
$$\left. -\frac{u-W_c}{\theta^*\tilde{\sigma}(X)}\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)\right)\times$$

$$I(B_c)du] \le \epsilon\tilde{E}[e^{-W_c}].$$

We show later in the appendix (see Lemma 6) that

$$\lim_{c\to\infty}\tilde{E}[e^{-W_c}] = \tilde{E}[e^{-W_\infty}] < \infty.$$

Since, ϵ is arbitrary, (16) follows.

Therefore, (11) equals

$$\lim_{c\to\infty}\tilde{E}\left[\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u}\frac{u-W_c}{\theta^*\tilde{\sigma}(X)} - \right.$$
$$\left. \phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)I(B_c)du\right] \quad (17)$$

Since,

$$\int_{W_c}^{(\nu+1)\log\beta(c)} e^{-u}(u-W_c)du$$

equals

$$e^{-W_c}\int_0^{(\nu+1)\log\beta(c)-W_c} ye^{-y}dy,$$

the term inside the limit in (17) may be re-expressed as $\frac{1}{\theta^*\tilde{\sigma}(X)}$ times

$$\tilde{E}\left[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)e^{-W_c}I(B_c) - \right.$$
$$\left.\left(\int_0^{(\nu+1)\log\beta(c)-W_c} ye^{-y}dy\right)\right].$$

Define,

$$D_c = \{W_c < \nu \log \beta(c), |\tilde{Y}_{\beta(c)}| < \beta(c)^{-1/16}\},$$

and

$$F_c = \{|\tilde{T}_{\beta(c)}| < \nu \sqrt{\log \beta(c)}\}.$$

Then $I(B_c) = I(D_c) * I(F_c)$. Observe that

$$\lim_{c\to\infty} \tilde{E}[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) e^{-W_c} I(F_c) \times$$

$$(I(D_c) \int_0^{(\nu+1)\log\beta(c)-W_c} y e^{-y} dy - 1)] = 0. \quad (18)$$

To see this, note that $\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)$ is bounded from above. Apply the Cauchy Schwartz inequality to random variables $e^{-W_c} I(F_c)$ and

$$\left(I(D_c) \int_0^{(\nu+1)\log\beta(c)-W_c} y e^{-y} dy - 1 \right), \quad (19)$$

and observe that

$$\tilde{E}[e^{-2W_c}] \le \tilde{E}[e^{-2\theta^* X_{N(c)+1}}] \to \tilde{E}[e^{-2\theta^* X_\infty}] < \infty$$

(the last inequality is shown in the appendix). The fact that the second moment of (19) converges to zero can be seen by direct computation, noting that $\tilde{P}(D_c^C) \to 0$ (this follows from the proof of Lemma 3), and that

$$I(D_c) \int_0^{(\nu+1)\log\beta(c)-\nu \log c} y e^{-y} dy$$

is less than or equal to

$$I(D_c) \int_0^{(\nu+1)\log\beta(c)-W_c} y e^{-y} dy \le \int_0^{(\nu+1)\log\beta(c)} y e^{-y} dy.$$

Therefore, (11) equals

$$\lim_{c\to\infty} \frac{1}{\theta^* \tilde{\sigma}(X)} \tilde{E}\left[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) e^{-W_c} I(F_c)\right], \quad (20)$$

or,

$$\lim_{c\to\infty} \frac{1}{\theta^* \tilde{\sigma}(X)} \tilde{E}\left[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) I(F_c) \tilde{E}[e^{-W_c} | \tilde{T}_{\beta(c)}]\right]. \quad (21)$$

Recall that $|\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}$, implies that $c - T_{\beta(c)} = \Theta(c^{5/8})$. Therefore, (11) equals

$$\lim_{c\to\infty} \frac{1}{\theta^* \tilde{\sigma}(X)} \tilde{E}\left[\tilde{E}[e^{-W_{c-T_{\beta(c)}}} | T_{\beta(c)}] I(F_c) \phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0)\right].$$

Now,

$$\lim_{c\to\infty} \left(\tilde{E}[e^{-W_{c-T_{\beta(c)}}} | T_{\beta(c)}] - \tilde{E}[e^{-W_\infty}]\right) I(F_c) = 0.$$

Furthermore, the term above inside the limit is arbitrarily small a.s. for all c sufficiently large. Therefore, (11) equals

$$\frac{1}{\theta^* \tilde{\sigma}(X)} \tilde{E}[e^{-W_\infty}] \lim_{c\to\infty} \tilde{E}[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) I(F_c)].$$

In the appendix in Lemma 6 we show that

$$\tilde{E}[e^{-W_\infty}] = \frac{\lambda}{\eta^*}(\exp(\psi(0,\eta^*)) - 1).$$

Since, $\tilde{T}_{\beta(c)}$ converges to a standard Gaussian random variable, and

$$\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) = \frac{1}{(2\pi(1-\rho^2))^{1/2}} e^{-\frac{\rho^2 \tilde{T}^2_{\beta(c)}}{2(1-\rho^2)}},$$

it is easy to see that

$$\lim_{c\to\infty} \tilde{E}[\phi_{\tilde{S}|\tilde{T}=\tilde{T}_{\beta(c)}}(0) I(F_c)] = 1/(2\pi)^{1/2}.$$

The result then follows. ■

2.3 Large Deviations in General Settings

Again consider the sequence $((X_i, \tau_i) : i \ge 1)$. We make the following additional assumption to aid in making the analysis easier and transparent:

Assumption 3 *The rv's $(X_i : i \ge 1)$ and $(\tau_i : i \ge 1)$ are bounded in the sense that there exists an $\tilde{a}$ such that $P(|X_i| + \tau_i \le \tilde{a}) = 1$ for $i \ge 1$ and $P(\tau_i \ge 1/\tilde{a}) = 1$.*

Theorem 2 *Under Assumptions 1 and 3*

$$\frac{1}{c} \log P(S_{N(c)} > 0) \to -\eta^*$$

as $c \to \infty$.

Let $\lambda^{-1} = \frac{\partial}{\partial\eta}\psi(\theta^*, \eta^*)$.

Proof. For the upper bound, note that for $0 < \eta < \eta^*$, $P(S_{N(c)} > 0)$

$$\begin{aligned} &= \sum_{n=1}^{\infty} P(S_n > 0, T_n \le c < T_{n+1}), \\ &\le \sum_{n=1}^{\infty} P(S_n > 0, T_{n+1} > c), \\ &\le \sum_{n=1}^{\infty} P(S_n > 0, T_n > c - \tilde{a}), \\ &= \sum_{n=1}^{\infty} E[I(S_n > 0, T_n > c - \tilde{a})], \\ &\le \sum_{n=1}^{\infty} E[I(S_n > 0)\exp(\theta^* S_n) I(T_n > c - \tilde{a}) \times \\ &\quad \exp((\eta^* - \eta)(T_n - c + \tilde{a}))], \\ &\le \exp(-(\eta^* - \eta)(c - \tilde{a})) \times \\ &\quad \sum_{n=1}^{\infty} E[\exp(\theta^* S_n + (\eta^* - \eta)T_n)]. \end{aligned}$$

To complete the proof of the upper bound, it suffices to show that

$$\sum_{n=1}^{\infty} E[\exp(\theta^* S_n + (\eta^* - \eta)T_n)] < \infty. \tag{22}$$

Then,

$$\limsup_{c\to\infty} \frac{1}{c} \log P(S_{N(c)} > 0) \le -\eta^* + \eta.$$

Since the $\eta > 0$ is arbitrary, the desired upper bound follows. To see (22), note that $\frac{\partial}{\partial \eta}\psi(\theta^*, \eta^*) \ge 1/\tilde{a}$ (since $\tau_i \ge 1/\tilde{a}$) and

$$\psi(\theta^*, \eta^* - \eta) = \psi(\theta^*, \eta^*) - \eta \frac{\partial}{\partial \eta}\psi(\theta^*, \eta^*) + o(\eta).$$

Hence, for η sufficiently small and positive,

$$\psi(\theta^*, \eta^* - \eta) \le -\frac{\eta}{2}\frac{\partial}{\partial \eta}\psi(\theta^*, \eta^*). \tag{23}$$

Fixing η sufficiently small so that (23) holds, Assumption 1, part 1 implies the existence of $n_0 < \infty$ for which

$$E[\exp(\theta^* S_n + (\eta^* - \eta)T_n)]$$

is less than equal to

$$\exp[n(\psi(\theta^*, \eta^* - \eta) + \frac{\eta}{4}\frac{\partial}{\partial \eta}\psi(\theta^*, \eta^*))]$$

for $n \ge n_0$. So, using the boundedness of the $X_i's$ and the $\tau_i's$ we find that

$$E[\exp(\theta^* S_n + (\eta^* - \eta)T_n)] \le n_0 \exp(\theta^* \tilde{a} n_0$$

$$+\eta^* \tilde{a} n_0) + \sum_{j=n_0}^{\infty} \exp(-\frac{j}{4}\eta \frac{\partial}{\partial \eta}\psi(\theta^*, \eta^*)) < \infty.$$

For the lower bound we use a 'change-of-measure' argument. Two observations are useful in this: First, on the set

$$\{(1-2\epsilon)c \le T(\lfloor \lambda(1-\epsilon)c \rfloor) \le c\}, \tag{24}$$

$$\lfloor \lambda(1-\epsilon)c \rfloor \le N(c) \le \lambda(1-\epsilon)c + 2\epsilon\tilde{a}c + 1, \tag{25}$$

for $\epsilon > 0$ and small. To see this, note from (24) that $N(c) \ge \lfloor \lambda(1-\epsilon)c \rfloor$ and $N((1-2\epsilon)c) \le \lfloor \lambda(1-\epsilon)c \rfloor$. Since $\tau_i \ge 1/\tilde{a}$, $N(c) - N((1-2\epsilon)c) \le 2\epsilon\tilde{a}c + 1$, and hence (25) holds.

Second, if $S_{\lfloor \lambda(1-\epsilon)c \rfloor} \ge 3\epsilon\tilde{a}^2 c$ and (25) holds, then

$$S_{N(c)} \ge \tilde{a}^2 \epsilon c - 2\tilde{a} > 0 \tag{26}$$

for c sufficiently large.

To see this, note that $S_{N(c)} = S_{\lfloor \lambda(1-\epsilon)c \rfloor} + \sum_{\lfloor \lambda(1-\epsilon)c \rfloor}^{N(c)} X_i$ when (25) holds. Since, $|X_i| \le \tilde{a}$, $S_{N(c)}$ is greater than or equal to

$$\begin{aligned} &S_{\lfloor \lambda(1-\epsilon)c \rfloor} - (N(c) - \lfloor \lambda(1-\epsilon)c \rfloor)\tilde{a} \\ &\quad \ge S_{\lfloor \lambda(1-\epsilon)c \rfloor} - (2\epsilon\tilde{a}c + 2)\tilde{a}, \end{aligned}$$

and thus (26) holds.

Form these observations we conclude that $P(S_{N(c)} > 0)$ i s greater than or equal to

$$P(S_{\lfloor \lambda(1-\epsilon)c \rfloor} > 3\epsilon\tilde{a}^2 c, (1-2\epsilon)c \le T(\lfloor \lambda(1-\epsilon)c \rfloor) \le c)$$

which in turn is greater than or equal to

$$\begin{aligned} P(4\epsilon\tilde{a}^2 c \ge S_{\lfloor \lambda(1-\epsilon)c \rfloor} \ge 3\epsilon\tilde{a}^2 c, (1-2\epsilon)c \\ \le T(\lfloor \lambda(1-\epsilon)c \rfloor) \le c). \end{aligned}$$

Let $P(A_c)$ denote the above probability.

We are now in a position to apply the change-of-measure arguments. Let

$$\psi_n(\theta, \eta) = \frac{1}{n} \log E \exp(\theta S_n + \eta T_n),$$

$n(c) = \lfloor \lambda(1-\epsilon)c \rfloor$, $\hat{P}_c(d\omega) =$

$$\exp((\theta^* + \theta)S_{n(c)} + (\eta^* + \eta)T_{n(c)} - \\ -n(c)\psi_{n(c)}(\theta^* + \theta, \eta^* + \eta)) \times P(d\omega)$$

and let $\hat{E}_c(\cdot)$ be the corresponding expectation operator. Then, if $|\theta| < \theta^*$, $|\eta| < \eta^*$, $P(S_{N(c)} > 0) \geq P(A_c)$ and $P(A_c)$ equals

$$\hat{E}_c[\exp(-(\theta^* + \theta)S_{n(c)} - (\eta^* + \eta)T_{n(c)}$$

$$-n(c)\psi_{n(c)}(\theta^* + \theta, \eta^* + \eta))I(A_c)].$$

This is greater than or equal to

$$\exp(-(\theta^* + \theta)(4\epsilon\tilde{a}^2 c) - \\ -(\eta^* + \eta)c - n(c)\psi_{n(c)}(\theta^* + \theta, \eta^* + \eta)) \times \hat{P}_c(A_c).$$

Since, ψ is strictly convex at (θ^*, η^*) and twice-continuously differentiable there, it follows that

$$\begin{pmatrix} \frac{\partial^2}{\partial\theta^2}\psi(\theta^*, \eta^*) & \frac{\partial^2}{\partial\theta\partial\eta}\psi(\theta^*, \eta^*) \\ \frac{\partial^2}{\partial\theta\partial\eta}\psi(\theta^*, \eta^*) & \frac{\partial^2}{\partial\eta^2}\psi(\theta^*, \eta^*) \end{pmatrix}$$

is non-singular at (θ^*, η^*). Hence, for ϵ small, we may find θ and η so that

$$\frac{\partial}{\partial\theta}\psi(\theta^* + \theta, \eta^* + \eta) = \frac{7}{2}\epsilon\tilde{a}^2 \qquad (27)$$

$$\frac{\partial}{\partial\eta}\psi(\theta^* + \theta, \eta^* + \eta) = \frac{\partial}{\partial\eta}\psi(\theta^*, \eta^*)(= \lambda^{-1}) \qquad (28)$$

For this choice of (θ, η), one may replicate the proof from Bucklew (1990) to conclude that

$$\tilde{P}_c(A_c) \to 1$$

as $c \to \infty$. Consequently $\liminf_{c\to\infty} \frac{1}{c} \log P(S_{N(c)} > 0)$ is greater than or equal to

$$-(\theta^* + \theta)(4\epsilon\tilde{a}^2) - (\eta^* + \eta) + \psi(\theta^* + \theta, \eta^* + \eta).$$

The solutions $\theta = \theta(\epsilon)$ and $\eta = \eta(\epsilon)$ to (27) and (28) may be chosen so that $\theta(\epsilon) \to 0$ and $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, we conclude that

$$\liminf_{c\to\infty} \frac{1}{c} \log P(S_{N(c)} > 0) \geq -\eta^*,$$

completing the proof. ■

3 APPENDIX

3.1 Proofs for Section 2.2

Some notation is needed to aid in proving the results in this section. Let

$$\Gamma_c(\theta, \eta) = \tilde{E}[\exp(-\theta X_{N(c)+1} - \eta(T_{N(c)+1} - c))].$$

Let $F_{\theta,\eta}(\cdot,\cdot)$ denote the distribution obtained by exponentially twisting $F(\cdot,\cdot)$ with parameters (θ, η), i.e.,

$$dF_{\theta,\eta}(x,t) = e^{\theta x + \eta t - \psi(\theta,\eta)} dF(x,t).$$

Let $P_{\theta,\eta}$ denote the probability of events associated with the collection of independent random vectors $((X_i, \tau_i) : i \geq 1)$ when each (X_i, τ_i) has the distribution $F_{\theta,\eta}(\cdot,\cdot)$.

Let

$$\Gamma(\theta, \eta) = \\ \frac{1}{\eta\tilde{E}(\tau)}(\exp(\psi(\theta^* - \theta, \eta^*)) - \exp(\psi(\theta^* - \theta, \eta^* - \eta))).$$

Let $\mathcal{D}$ denote the domain of finiteness of $\Gamma(\cdot,\cdot)$. Note that under Assumption 1 it includes a neigbhorhood around $(0, 0)$.

Lemma 6 *Under Assumption 1 for $(\theta, \eta) \in \mathcal{D}$,*

$$\lim_{c\to\infty} \Gamma_c(\theta, \eta) = \Gamma(\theta, \eta).$$

Proof.

The following renewal equation holds:

$$\Gamma_c(\theta, \eta) = h_c(\theta, \eta) + \int_{s \leq c} \Gamma_{c-s}(\theta, \eta) d\tilde{F}_\tau(s),$$

where

$$h_c(\theta, \eta) = \tilde{E}[\exp(-\theta X_1 - \eta(\tau_1 - c))I(\tau_1 \geq c)]. \qquad (29)$$

This in turn equals

$$\exp(\eta c)\times$$

$$E[\exp((\theta^* - \theta)X_1 + (\eta^* - \eta)\tau_1)I(\tau_1 \geq c)], \qquad (30)$$

and $\tilde{F}_\tau(\cdot)$ is the marginal distribution of τ under $\tilde{P}$.

Note that $\int_{c\in\Re^+} h_c(\theta, \eta) dc$ equals

$$\int_{c\in\Re^+} \int_{x\in\Re, t\in(c,\infty)} \exp(-\theta x - \eta(t - c) d\tilde{F}(x,t) dc. \qquad (31)$$

Now, by simply changing the order of integration, recalling that $d\tilde{F}(x,t) = \exp(\theta^* x + \eta^* t) dF(x,t)$, (31) can be seen to equal

$$\frac{1}{\eta}(\exp(\psi(\theta^* - \theta, \eta^*)) - \exp(\psi(\theta^* - \theta, \eta^* - \eta))).$$

For $(\theta,\eta) \in \mathcal{D}, \eta \leq 0$, $h_c(\theta,\eta)$ is a non-increasing function (from (30)) and integrable function of c and hence is directly Reiman integrable (see, e.g., Asmussen 2003). For $(\theta,\eta) \in \mathcal{D}, \eta > 0$, it is a bounded continuous function of c upper bounded by (from (29)) $\tilde{E}[\exp(-\theta X_1) I(\tau_1 \geq c)]$ equals

$$\exp(\psi(\theta^* - \theta, \eta^*)) P_{(\theta^* - \theta, \eta^*)}(\tau_1 \geq c)$$

Since the upper bound is non-increasing and integrable, it follows that $h_c(\theta,\eta)$ is directly Reimann integrable. The result follows from the key renewal theorem as τ is spread-out under Assumption 2. ■

It follows from Lemma 6 that $W_c = \theta^* X_{N(c)+1} + \eta^*(T_{N(c)+1} - c)$ converges to a rv W_∞ such that

$$\tilde{E}(\exp(-W_\infty)) = \Gamma(\theta^*, \eta^*) = \frac{\lambda}{\eta^*}(\exp(\psi(0,\eta^*)) - 1).$$

Proof.[**Lemma 3**] To see (10), note that, its left hand side is bounded above by

$$\tilde{E}[e^{-\theta^* X_{N(c)+1}} \times$$

$$(I(|\tilde{T}_{\beta(c)}| \geq \nu\sqrt{\log\beta(c)}) + I(|W_c| \geq \nu\beta(c))$$
$$+ I(|\tilde{Y}_{\beta(c)}| \geq \beta(c)^{-1/16}, |\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}))].$$

Using Cauchy-Shwartz, we see that this in turn is bounded above by the sum of

$$\tilde{E}(e^{-2\theta^* X_{N(c)+1}})^{1/2} \tilde{P}(|\tilde{T}_{\beta(c)}| \geq \nu\sqrt{\log\beta(c)})^{1/2}, \quad (32)$$

$$\tilde{E}(e^{-2\theta^* X_{N(c)+1}})^{1/2} \tilde{P}(W_c \leq -\nu\log\beta(c))^{1/2}, \quad (33)$$

and

$$\tilde{E}(e^{-2\theta^* X_{N(c)+1}})^{1/2}$$
$$\tilde{P}(|\tilde{Y}_{\beta(c)}| \geq \beta(c)^{-1/16}, |\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)})^{1/2}. \quad (34)$$

Since,

$$e^{-2\theta^* X_{N(c)+1}} \leq \sum_{i=1}^{N(c)+1} e^{-2\theta^* X_i},$$

it follows that

$$\tilde{E}(e^{-2\theta^* X_{N(c)+1}}) \leq \tilde{E}(N(c)+1)\tilde{E}[e^{-2\theta^* X_1}]$$

is finite. Using renewal theory arguments, we showed in the previous section that $\tilde{E}(e^{-2\theta^* X_{N(c)+1}})$ converges to a constant as $c \to \infty$. To see that (32) is $o(c^{-1/2})$, note the well known fact that (see, e.g., Feller Vol. 2 1971)

$$\tilde{P}(|\tilde{T}_n| \geq x) = \bar{\Phi}(x) + \frac{\phi(x)Q(x)}{n^{1/2}} + o(n^{-1/2}),$$

where $\bar{\Phi}(\cdot)$ denotes the tail cumulative distribution function of standard Gaussian random variable, $\phi(\cdot)$ denotes its density and $Q(x)$ is a polynomial in x. Since $\bar{\Phi}(x) \sim \frac{1}{\sqrt{2\pi}x}\exp(-\frac{x^2}{2})$, it follows that

$$\bar{\Phi}(\nu\sqrt{\log\beta(c)})^{1/2} \sim \frac{1}{\nu^{1/2}(2\pi\log\beta(c))^{1/4}}\beta(c)^{-\nu^2/4}.$$

Therefore (32) is $o(c^{-1/2})$ for $\nu \geq \sqrt{2}$.

To see that (33) is $o(c^{-1/2})$, first note that

$$\tilde{P}(W_c \leq -\nu\log\beta(c)) \leq \tilde{P}(\theta^* X_{N(c)+1} \leq -\nu\log\beta(c)).$$

Note that $X_{N(c)+1}$ has an exponentially decaying left tail. This ensures that ν can be selected to be sufficiently large so that (33) is $o(c^{-1/2})$.

To see that (34) is $o(c^{-1/2})$ consider

$$\tilde{P}(|\tilde{Y}_{\beta(c)}| \geq \beta(c)^{-1/16}, |\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}). \quad (35)$$

As we mentioned earlier, $\tilde{T}_{\beta(c)} < \sqrt{\nu\log\beta(c)}$ implies $T_{\beta(c)} < c$ and then $\tilde{Y}_{\beta(c)}$ is distributed as

$$\frac{S_{N(c - T_{\beta(c)})}}{\tilde{\sigma}(X)\beta(c)^{1/2}}.$$

Hence, (35) equals

$$\tilde{P}(|S_{N(c-T_{\beta(c)})}| \geq \tilde{\sigma}(X)\beta(c)^{1/2}\beta(c)^{-1/16},$$
$$|\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}).$$

Note that $|\tilde{T}_{\beta(c)}| < \nu\sqrt{\log\beta(c)}$ implies that for $\epsilon > 0$ and all c sufficiently large,

$$\lambda^{-1}c^{5/8}(1-\epsilon) \le c - T_{\beta(c)} \le \lambda^{-1}c^{5/8}(1+\epsilon).$$

Equation (34) now follows from Lemma 7 given below. ■

Lemma 7 *For $\epsilon > 0$ and $K > 0$,* $\lim_{c\to\infty} c^{1/2}$ *times*

$$\tilde{P}\left(\sup_{t\in(\lambda^{-1}c^{5/8}(1-\epsilon),\lambda^{-1}c^{5/8}(1+\epsilon))} |S_{N(t)}| \ge Kc^{7/16}\right) = 0$$

Proof. It is easy to find constants $0 < \alpha_1 < \alpha_2$ so that

$$\lim_{c\to\infty} c^{1/2}\tilde{P}(N(\lfloor\lambda^{-1}c^{5/8}(1-\epsilon)\rfloor) \le \alpha_1 c^{5/8}) = 0$$

and

$$\lim_{c\to\infty} c^{1/2}\tilde{P}(N(\lfloor\lambda^{-1}c^{5/8}(1+\epsilon)\rfloor) \ge \alpha_2 c^{5/8}) = 0$$

(see, e.g., Glynn and Whitt 1994). Therefore, it suffices to show that

$$\lim_{c\to\infty} c^{1/2}\tilde{P}(\sup_{n\in[\alpha_1 c^{5/8},\alpha_2 c^{5/8}]} |S_n| \ge Kc^{7/16}) = 0.$$

Note that $\tilde{P}(\sup_{n\in[\alpha_1 c^{5/8},\alpha_2 c^{5/8}]} |S_n| \ge Kc^{7/16})$ is bounded above by

$$\sum_{n=\lfloor\alpha_1 c^{5/8}\rfloor}^{\lceil\alpha_2 c^{5/8}\rceil} \tilde{P}(|S_n| \ge Kc^{7/16}). \quad (36)$$

From moderate deviations theory it can be seen that for a sequence $a_n \to \infty$ and $na_n \to \infty$, for $0 < \delta < 1$, and all n sufficiently large,

$$\tilde{P}(|S_n|\sqrt{a_n/n} \ge K) \le e^{-\frac{K^2}{2a_n\tilde{\sigma}(X)^2}(1-\delta)}.$$

(See Dembo and Zeitouni Theorem 3.7.1). Therefore, in our settings, for sufficiently large c,

$$\tilde{P}(|S_{\lfloor\alpha_1 c^{5/8}\rfloor}| \ge Kc^{7/16}) \le e^{-\frac{K^2c^{1/8}}{2\alpha_1^{1/2}\tilde{\sigma}(X)^2}(1-\delta)}$$

so that (36) may be bounded from above by

$$\left((\alpha_2-\alpha_1)c^{5/8}+2\right)e^{-\frac{K^2c^{1/8}}{2\alpha_1^{1/2}\tilde{\sigma}(X)^2}(1-\delta)}.$$

The result follows. ■

Proof.[**Lemma 5**] The result follows immediately from the following conditional local limit result:

$$\sup_{|s|,|t|<\nu\sqrt{\log n}}\left(\frac{\tilde{f}_{\tilde{S}_n|\tilde{T}_n=t}(s)}{\phi_{\tilde{S}|\tilde{T}=t}(s)} - 1\right) \to 0 \quad (37)$$

as $n \to \infty$, for any $\nu > 0$. To see (37), note that under Assumption 2, and if all the joint moments of X_1 and τ_1 exist, then the local limit theorems in Bhattacharya and Rao (1976) imply that

$$\tilde{f}_n(s,t) = \phi_\rho(s,t) + \phi_\rho(s,t)\frac{\sum_{k=3}^d Q_k(s,t)}{n^{(k-2)/2}} + o(\frac{1}{n^{(d-2)/2}}), \quad (38)$$

where $Q_k(\cdot,\cdot)$ are polynomials in s and t and $o(\frac{1}{c^{(d-2)/2}})$ term is independent of s and t.

Note that

$$\begin{aligned}\phi_\rho(s,t) &= \frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}[s^2-2\rho st+t^2]} \\ &\ge \frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}[s-t]^2}.\end{aligned}$$

Therefore, for $|s|,|t| < \nu\sqrt{\log n}$, $\phi_\rho(s,t)$ is bounded from below by an order $n^{-2\nu^2/(1-\rho^2)}$ term. In (38), selecting d sufficiently large, we get

$$\sup_{|s|,|t|<\nu\sqrt{\log n}}\left(\frac{\tilde{f}_n(s,t)}{\phi(s,t)} - 1\right) \to 0$$

as $n \to \infty$.

Let $\tilde{f}_n(\cdot)$ denote the marginal pdf of $\tilde{T}_n$ under $\tilde{P}$ and $\phi(\cdot)$ denote the standard Gaussian density function with mean zero and unit variance. From Bhattacharya and Rao (1976) we observe that

$$\tilde{f}_n(t) = \phi(t) + \phi(t)\frac{\sum_{k=3}^d \tilde{Q}_k(t)}{n^{(k-2)/2}} + o(\frac{1}{n^{(d-2)/2}}), \quad (39)$$

where $\tilde{Q}_k(\cdot)$ are polynomials in t and and $o(\frac{1}{n^{(d-2)/2}})$ term is independent of t. Therefore, by selecting d appropriately large,

$$\sup_{t<\nu\sqrt{\log n}}\left(\frac{\tilde{f}_n(t)}{\phi(t)} - 1\right) \to 0$$

as $n \to \infty$.

Equation (37) follows from (38) and (39). ■

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