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SELECTION AND MULTIPLE-COMPARISON PROCEDURES FOR REGENERATIVE SYSTEMS

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ABSTRACT

We propose two-stage methods for selection and multiple comparisons with the best (MCB) of steady-state performance measures of regenerative systems. We assume the systems being compared are simulated independently, and the methods presented are asymptotically valid as the confidence-interval width parameter shrinks and the first-stage run length grows at a rate that is at most the inverse of the square of the confidence-interval width parameter. When the first-stage run length is asymptotically negligible compared to the total run length, our procedures are asymptotically efficient. We provide an asymptotic comparison of our regenerative MCB procedures with those based on standardized time series (STS) methods in terms of mean and variance of total run length. We conclude that regenerative MCB methods are strictly better than STS MCB methods for any fixed number of batches, but the two become equivalent as the number of batches grows large.

1 INTRODUCTION

In many simulation studies the analyst wants to compare different systems to determine which is best relative to some performance measure. For example, one may be faced with 10 possible designs for a computer system, and the goal is to determine which has the highest steady-state availability.

In this paper we assume that there are $k < \infty$ systems, where each system i has unknown steady-state mean μ_i and unknown time-average variance constant (TAVC) σ_i^2. We allow for the σ_i^2, $i = 1,\ldots,k$, to be unequal. We assume that bigger μ_i is better, and the goal is to use simulation to identify the system i with the largest μ_i.

We attack this problem when the systems being simulated are regenerative. Loosely speaking, a stochastic process is said to be regenerative if it has a sequence of increasing times (known as *regeneration points*) at which the process probabilistically restarts; see Shedler (1993) for background on regenerative processes. An example is a positive-recurrent Markov chain living on a discrete state space, and returns to a fixed state constitute a sequence of regeneration points. The sample path between two successive regeneration points is called a *regenerative cycle*, and the sequence of such cycles are independent and identically distributed (i.i.d.). The *regenerative method* (Iglehart 1978) exploits this structure to construct asymptotically valid confidence intervals for steady-state performance measures.

In this paper we develop two-stage selection and multiple-comparison procedures for comparing the steady-state means of regenerative systems. In general the aim of selection procedures is to identify with pre-specified probability $1-\alpha$ the system with the largest mean. Our selection procedure is developed under Bechhofer's (1954) "indifference zone" formulation, in which we assume we are indifferent to systems whose means are within δ of one other. In addition, our procedures implement multiple comparisons with the best (MCB; Hsu 1984), where the goal is to construct simultaneous confidence intervals for $\mu_i - \max_{\ell\neq i}\mu_\ell$, $i = 1,\ldots,k$. Thus, MCB intervals provide bounds on how close each system's mean performance is to that of the best of the rest. Bechhofer, Santner, and Goldsman (1995), Swisher, Jacobson, and Yucesan (2003), and Kim and Nelson (2006b) provide overviews of selection and multiple-comparison methods.

The MCB intervals produced by our two-stage procedures have a pre-specified width parameter $\delta > 0$, which we also use as an indifference-zone parameter of our selection procedure. We consider both absolute-width and relative-width parameters δ for MCB. Our methods are asymptotically valid as $\delta \to 0$ when the first-stage run length grows at rate $\delta^{-\lambda}$ with $0 < \lambda \leq 2$. When $\lambda < 2$, the first-stage run length is asymptotically negligible compared to δ^{-2}, and the procedures are asymptotically efficient (Chow and Robbins 1965) in the sense that the total run length is the same (to first order) as when the variances are known.

Our methods build on the work of Rinott (1978), Mukhopadhyay (1979), and Matejcik and Nelson (1995). Rinott (1978) develops two-stage selection procedures to compare means of independent normal populations using i.i.d. sampling within each population. For the case of $k = 2$ normal populations, Mukhopadhyay (1979) modifies Rinott's procedure to be asymptotically efficient by letting the first-stage sample size grow at rate $1/\delta$. Matejcik and Nelson (1995) extend Rinott's method to also construct MCB intervals.

Iglehart (1977) proposes heuristic two-stage and fully sequential selection methods for regenerative systems, but does not prove the validity of his approach. Also, his procedure is based on the "cycle time scale" (i.e., the method determines the number of cycles to simulate), whereas our two-stage selection and MCB methods use the "natural time scale" in the sense that our approaches determine the total run length needed.

Instead of assuming the systems are regenerative, as we do here, some other papers require a functional central limit theorem (FCLT) to establish the asymptotic validity of MCB and selection procedures for steady-state simulations. (A generalization of a standard central limit theorem, an FCLT specifies that a centered and scaled version of a process converges in distribution to a Brownian motion; see Billingsley 1999). For example, Damerdji and Nakayama (1999) consider the MCB problem for steady-state simulations, and their two-stage procedures use standardized time series (STS) methods (Schruben 1983, Glynn and Iglehart 1990). Also, Kim and Nelson (2006a) analyze fully sequential selection procedures for steady-state simulations under an FCLT assumption. While the class of stochastic processes that satisfy an FCLT (and thus for which STS methods apply) is quite large, Glynn and Iglehart (1993) provide an example of a regenerative process that does not satisfy an FCLT. (On the other hand, there are also non-regenerative processes for which a FCLT holds.)

We provide an asymptotic comparison of our regenerative MCB procedures with those based on STS methods (Damerdji and Nakayama 1999) in terms of the average and variability of total run length. We conclude that regenerative MCB methods are strictly better than STS MCB methods for any fixed number of batches, but the two become equivalent as the number of batches grows large.

The rest of the paper has the following organization. We describe our notation and assumptions in Section 2. Section 3 presents a two-stage procedure to produce simultaneous MCB confidence intervals having an absolute-width parameter δ, and Section 4 extends the method to also be a selection procedure. Section 5 presents an MCB procedure with relative-width parameter δ. In Section 6 we compare our regenerative MCB procedures with STS MCB methods. Section 7 contains some concluding remarks. Proofs of results from this paper are given in Nakayama (2006).

2 NOTATION AND ASSUMPTIONS

Assume that we have $k < \infty$ systems to compare, where the evolution of each system i is a stochastic process $X_i = [X_i(t) : t \geq 0]$ living on a state space S_i and having probability measure P_i. Let $f_i : S_i \to \Re$ be a reward function on S_i, and for each $t > 0$, define

$$\mu_i(t) = \frac{1}{t} \int_0^t f_i(X_i(s))\, ds,$$

which is the time-average reward of process i over the interval $[0,t]$. We assume the following:

Assumption 1 *For each system i, there exists a sequence of times $A_{i,-1} = 0 \leq A_{i,0} < A_{i,1} < \cdots$ such that X_i is regenerative with respect to the sequence $(A_{i,j} : j \geq 0)$ under measure P_i.*

For each $j \geq 1$, $[X_i(s) : A_{i,j-1} \leq s < A_{i,j}]$ is the jth regenerative cycle of system i. Define $N_i(t) = \sup\{j \geq 0 : A_{i,j} \leq t\}$, which is the number of regenerative cycles that process i completes by time t. Let $\tau_{i,j} = A_{i,j} - A_{i,j-1}$ be the length of the jth regenerative cycle for process i, and define

$$Y_{i,j} = \int_{A_{i,j-1}}^{A_{i,j}} f_i(X_i(t))\, dt,$$

which is the cumulative reward over the jth regenerative cycle of the ith process. Let E_i denote the expectation operator induced by probability measure P_i. The following ensures the validity of the regenerative method in our context (Glynn and Iglehart 1993):

Assumption 2 *For each system i, $E_i[\tau_{i,1}] < \infty$, and there exists a finite constant μ_i such that $E_i[Y_{i,1} - \mu_i \tau_{i,1}] = 0$ and $0 < E_i[(Y_{i,1} - \mu_i \tau_{i,1})^2] < \infty$.*

In this case, $\mu_i = E_i[Y_{i,1}]/E_i[\tau_{i,1}]$, and it can be shown that our assumptions imply $\mu_i(t) \Rightarrow \mu_i$ as $t \to \infty$, where $\Rightarrow$ denotes convergence in distribution (Billingsley 1999); i.e., μ_i is the long-run time-average reward, which we assume is unknown. For each $i = 1, \ldots, k$, let (i) denote the (unknown) system with the ith smallest mean, so $\mu_{(1)} \leq \mu_{(2)} \leq \cdots \leq \mu_{(k)}$. Our goal is to identify the system (k) having the largest $\mu_{(k)}$.

Let $\sigma_i^2 = E_i[(Y_{i,1} - \mu_i \tau_{i,1})^2]/E_i[\tau_{i,1}]$, which is the TAVC for the process X_i under reward function f_i. Specifically, σ_i^2 is the variance constant appearing in the central limit theorem (CLT)

$$\sqrt{t}\,(\mu_i(t) - \mu_i) \Rightarrow N(0, \sigma_i^2)$$

as $t \to \infty$, which holds under Assumptions 1 and 2, where $N(a,b)$ denotes a normal random variable with mean a and variance b; see Glynn and Iglehart (1993). We assume the σ_i^2 are unknown, and we allow for the σ_i^2, $i = 1, 2, \ldots, k$, to be unequal. For each $t > 0$, let

$$V_i(t) = \frac{1}{t} \sum_{j=1}^{N_i(t)} \left[Y_{i,j} - \mu_i(t) \tau_{i,j} \right]^2, \tag{1}$$

which is an estimator of σ_i^2 based on a simulation of system i up to time t, and Glynn and Iglehart (1993) show that $V_i(t)$ is weakly consistent; i.e., $V_i(t) \Rightarrow \sigma_i^2$ as $t \to \infty$.

3 ABSOLUTE-WIDTH MCB

The following two-stage procedure constructs MCB intervals whose absolute-width parameter is $\delta > 0$.

Procedure A

1. Specify the number of systems $2 \le k < \infty$, the confidence level $1-\alpha$ with $0 < \alpha < 1$, the desired absolute-width parameter δ of the MCB confidence intervals, and the first-stage run length T_0 for each system.
2. Independently simulate each system for a run length of T_0.
3. For each system i, compute the total run length required as

$$T_i(\delta) = \max\left(T_0, \frac{\gamma^2 V_i(T_0)}{\delta^2} \right), \tag{2}$$

   where the constant $\gamma = \sqrt{2} z_{(1-\alpha)^{1/(k-1)}}$, the constant z_β satisfies $\Phi(z_\beta) = \beta$ for $0 < \beta < 1$, and Φ is the distribution function of a standard (mean 0 and variance 1) normal distribution.
4. For each system i, continue to simulate from time T_0 to $T_i(\delta)$, where the k systems are simulated independently, and form the point estimator $\widetilde{\mu}_i(\delta) = \mu_i(T_i(\delta))$ of μ_i.
5. Use the absolute-width parameter δ to construct the simultaneously confidence intervals

$$I_i(\delta) = \left[-\left(\widetilde{\mu}_i(\delta) - \max_{\ell \ne i} \widetilde{\mu}_\ell(\delta) - \delta \right)^-, \left(\widetilde{\mu}_i(\delta) - \max_{\ell \ne i} \widetilde{\mu}_\ell(\delta) + \delta \right)^+ \right]$$

   for $i = 1, \ldots, k$, which are the MCB confidence intervals for $\mu_i - \max_{\ell \ne i} \mu_\ell$, $i = 1, \ldots, k$, respectively, where $-(\beta)^- = \min(\beta, 0)$ and $(\beta)^+ = \max(\beta, 0)$ for $\beta \in \Re$.

Before presenting some asymptotic properties of Procedure A, we first define the constant $\bar{\gamma} = \sqrt{2}\bar{z}_{k-1,1-\alpha}$, where $\bar{z}_{p,\beta}$ is the upper-β equicoordinate point of a p-variate standard normal distribution with unit variances and common correlation coefficient $1/2$. Table B.1 of Bechhofer, Santner, and Goldsman (1995) provides values of $\bar{z}_{p,\beta}$ for various p and β. It can be shown that $\bar{\gamma} < \gamma$ for each $k \ge 3$ and $0 < \alpha < 1$. We then have the following.

Theorem 1 *If Assumptions 1 and 2 hold and Procedure A is used with first-stage run length $T_0 = \zeta \delta^{-\lambda}$, where $\zeta > 0$ and $0 < \lambda \le 2$ are any constants, then the following hold:*

(i) $\lim_{\delta \to 0} P\{\mu_i - \max_{\ell \ne i} \mu_\ell \in I_i(\delta), i = 1, \ldots, k\} > 1 - \alpha$.

(ii) $\delta^2 T_i(\delta) \Rightarrow t_i$ *as* $\delta \to 0$, *where*

$$t_i = \begin{cases} \gamma^2 \sigma_i^2 & \text{if } 0 < \lambda < 2, \\ \max(\zeta, \gamma^2 \sigma_i^2) & \text{if } \lambda = 2. \end{cases} \tag{3}$$

(iii) *In addition, suppose $0 < \lambda < 2$, and replace γ in (2) and (3) with $\bar{\gamma}$. Then (i) and (ii) still hold, and $T_i(\delta)/(\bar{\gamma}\sigma_i/\delta)^2 \Rightarrow 1$ as $\delta \to 0$. Moreover, assume 0 and $\tau_{i,1}$ are the first two regeneration times of X_i, and $E_i[Y_{i,1}(|f_i|)^8 + \tau_{i,1}^8] < \infty$, where $Y_{i,1}(|f_i|) = \int_0^{\tau_{i,1}} |f_i(X_i(s))|\, ds$. Then $E[T_i(\delta)/(\bar{\gamma}\sigma_i/\delta)^2] \to 1$ as $\delta \to 0$.*

Part (i) shows the asymptotic validity of the MCB intervals. Part (ii) establishes that the total run length for system i asymptotically equals t_i/δ^2 to first order, as $\delta \to 0$. Part (iii) shows that when the first-stage run length is negligible compared to the total run length (i.e., $\lambda < 2$), replacing γ by $\bar{\gamma}$ results in Procedure A being asymptotically efficient for $k \ge 3$ systems in the sense that the total run length is the same as what it would be if σ_i^2 were known (Section 2.6 of Bechhofer, Santner, and Goldsman 1995). Glynn and Iglehart (1986) show that the moment conditions in (iii) ensure that $[V_i(t) : t \ge 0]$ is uniformly integrable.

4 SELECTION PROCEDURE

As Matejcik and Nelson (1995) note, MCB procedures can often be modified to be selection procedures as well under the "indifference zone" formulation of Bechhofer (1954). Specifically, suppose we use the MCB width parameter δ also as an indifference parameter, in the sense that we assume we are indifferent to systems whose means are within δ of one another. Thus, our goal is to identify a system i such that $\mu_i \ge \mu_{(k)} - \delta$.

Let $[1], [2], \ldots, [k]$ be defined such that $\widetilde{\mu}_{[1]}(\delta) \le \widetilde{\mu}_{[2]}(\delta) \le \cdots \le \widetilde{\mu}_{[k]}(\delta)$, so system $[i]$ has the ith smallest sample mean at the end of the second stage in Procedure A. We then modify Procedure A as follows:

Procedure A2

Use steps 1–5 of Procedure A and add the following step:

6. Select system $[k]$ as the best.

Let $\mu = (\mu_1,\ldots,\mu_k)$, and when δ is the indifference parameter, define

$$\Omega(\delta) = \{\mu = (\mu_1,\ldots,\mu_k) : \mu_{(k)} - \delta > \mu_{(k-1)}\},$$

which is the set of configurations of means such that only the best system is desirable. We then define a *correct selection* to be the event

$$\mathrm{CS}_\mu(\delta) = \left\{\widetilde{\mu}_{(k)}(\delta) > \widetilde{\mu}_i(\delta), \forall i \neq (k)\right\} \text{ when } \mu \in \Omega(\delta).$$

When analyzing the asymptotic properties of Procedure A2, we need to modify our problem formulation to avoid theoretical trivialities. In Procedure A2 the systems and their means are fixed as $\delta \to 0$, so, because the total run length $T_i(\delta) \geq T_0 = \zeta/\delta^2 \to \infty$ as $\delta \to 0$, we have that $\widetilde{\mu}_i(\delta) \to \mu_i$ with probability 1 by the strong law of large numbers. Hence, the probability of correct selection approaches 1 as $\delta \to 0$. To avoid this uninteresting theoretical result, we let the vector μ of means vary as δ shrinks, which leads to the probability of correct selection converging to a value less than 1. However, this significantly complicates the analysis (Damerdji et al. 1996), so we make the following simplifying assumption:

Assumption 3 *For each system i, there exists a process $Z_i = [Z_i(t) : t \geq 0]$ such that $f_i(X_i(t)) = \mu_i + Z_i(t)$ for all $t \geq 0$, where the distribution of Z_i does not depend on μ_i, and $Z_1,\ldots,Z_k$ are independent.*

We can think of Z_i as a "noise process" added to the mean μ_i we are trying to estimate. Kim and Nelson (2006b) make a similar simplifying modeling assumption for their selection procedures for steady-state simulations. The following result establishes the asymptotic validity of our combined selection and MCB procedure.

Theorem 2 *If Assumptions 1–3 hold and Procedure A2 is used with first-stage run length $T_0 = \zeta\delta^{-\lambda}$, where $\zeta > 0$ and $0 < \lambda \leq 2$ are any constants, then*

$$\lim_{\delta\to 0} \inf_{\mu\in\Omega(\delta)} P\left\{CS_\mu(\delta),\ \mu_i - \max_{\ell\neq i}\mu_\ell \in I_i(\delta),\ i = 1,\ldots,k\right\} > 1-\alpha.$$

5 RELATIVE-WIDTH MCB

Procedure A produces MCB intervals with absolute-width parameter δ. However, in many situations, one desires confidence intervals having a pre-specified relative precision, e.g., $\pm 10\%$. Below we present a procedure to do this.

For $i = 1,\ldots,k$, define $\langle i\rangle$ to be the system with the ith smallest sample mean after simulating each of the processes for run length T_0. Thus, $\mu_{\langle 1\rangle}(T_0) \leq \mu_{\langle 2\rangle}(T_0) \leq \cdots \leq \mu_{\langle k\rangle}(T_0)$.

Procedure R

1. Specify the number of systems $2 \leq k < \infty$, the confidence level $1-\alpha$ with $0 < \alpha < 1$, the desired relative-width parameter δ of the MCB confidence intervals, and the first-stage run length T_0 for each system.
2. Independently simulate each system for a run length of T_0.
3. For each system i, compute the total run length required as
$$T_{i,r}(\delta) = \max\left(T_0, \frac{\gamma^2 V_i(T_0)}{\delta^2\,\varepsilon_i^2(T_0)}\right), \qquad (4)$$
where the constant γ is the same as in (2), $\varepsilon_i(T_0) = \mu_{\langle k\rangle}(T_0) - \mu_i(T_0)$ if $\mu_i(T_0) < \mu_{\langle k\rangle}(T_0)$, and $\varepsilon_i(T_0) = \mu_{\langle k\rangle}(T_0) - \mu_{\langle k-1\rangle}(T_0)$ if $\mu_i(T_0) = \mu_{\langle k\rangle}(T_0)$.
4. For each system i, continue to simulate from time T_0 to $T_{i,r}(\delta)$, where the k systems are simulated independently, and form the point estimator $\widehat{\mu}_i(\delta) = \mu_i(T_{i,r}(\delta))$ of μ_i.
5. Use the relative-width parameter δ to construct the simultaneously confidence intervals
$$J_i(\delta) = \left[-\left(\widehat{\mu}_i(\delta) - \max_{\ell\neq i}\widehat{\mu}_\ell(\delta) - \delta d_i(\delta)\right)^-, \left(\widehat{\mu}_i(\delta) - \max_{\ell\neq i}\widehat{\mu}_\ell(\delta) + \delta d_i(\delta)\right)^+\right],$$
for $i = 1,\ldots,k$, which are the MCB confidence intervals for $\mu_i - \max_{\ell\neq i}\mu_\ell$, $i = 1,\ldots,k$, respectively, where $d_i(\delta) = |\widehat{\mu}_i(\delta) - \max_{\ell\neq i}\widehat{\mu}_\ell(\delta)|$.

Theorem 3 *Suppose Assumptions 1 and 2 hold and $\mu_{(k-1)} < \mu_{(k)}$. If Procedure R is used with first-stage run length $T_0 = \zeta\delta^{-\lambda}$, where $\zeta > 0$ and $0 < \lambda \leq 2$ are any constants, then the following hold:*

(i) $\lim_{\delta\to 0} P\{\mu_i - \max_{\ell\neq i}\mu_\ell \in J_i(\delta), i = 1,\ldots,k\} > 1-\alpha$.

(ii) $\delta^2 T_{i,r}(\delta) \Rightarrow t_{i,r}$ *as* $\delta \to 0$, *where*

$$t_{i,r} = \begin{cases} (\gamma\sigma_i/\varepsilon_i)^2 & \text{if } 0 < \lambda < 2, \\ \max(\zeta, (\gamma\sigma_i/\varepsilon_i)^2) & \text{if } \lambda = 2, \end{cases} \qquad (5)$$

$\varepsilon_i = \mu_{(k)} - \mu_i$ if $i \neq (k)$, and $\varepsilon_i = \mu_{(k)} - \mu_{(k-1)}$ if $i = (k)$.

(iii) In addition, suppose $0 < \lambda < 2$, and replace γ in (4) and (5) with $\bar{\gamma}$. Then (i) and (ii) still hold, and $T_{i,r}(\delta)/(\bar{\gamma}\sigma_i/\delta)^2 \Rightarrow 1$ as $\delta \to 0$. Moreover, under the moment conditions in Theorem 1(iii), $E[T_{i,r}(\delta)/(\bar{\gamma}\sigma_i/\delta)^2] \to 1$ as $\delta \to 0$.

6 ASYMPTOTIC COMPARISON OF REGENERATIVE MCB AND STS MCB

We now compare our regenerative MCB methods and the STS MCB procedures of Damerdji and Nakayama (1999), the latter of which requires each system to satisfy a functional central limit theorem (FCLT), which we now describe. For each system i and $T > 0$, define the processes $\bar{X}_{i,T} = [\bar{X}_{i,T}(t) : t \geq 0]$ and $W_{i,T} = [W_{i,T}(t) : t \geq 0]$ with

$$\begin{aligned} \bar{X}_{i,T}(t) &= \frac{1}{T}\int_0^{tT} f_i(X_i(s))\,ds, \\ W_{i,T}(t) &= \sqrt{T}\,[\bar{X}_{i,T}(t) - \mu_i t]. \end{aligned}$$

Then we assume the following FCLT holds for each system i:

$$W_{i,T} \Rightarrow \sigma_i B_i \tag{6}$$

as $T \to \infty$, where B_i is a standard Brownian motion. Because we simulated $X_1, \ldots, X_k$ independently, $B_1, \ldots, B_k$ are independent.

When using $m \geq 1$ batches, each STS method has a corresponding function g_m, whose square, when applied to $\bar{X}_{i,T}$, provides an estimate of the process's TAVC. Specifically, we let

$$V'_{i,m}(T_0) = T_0 g_m^2(\bar{X}_{i,T_0})$$

be the STS estimator of σ_i^2 based on the first stage of length T_0. To facilitate the comparison of STS MCB procedures with regenerative MCB methods, we will assume that STS functions g_m are scaled such that $E[g_m^2(B)] = 1$, where B is a standard Brownian motion.

The STS MCB method for constructing intervals with absolute-width parameter δ is the same as Procedure A in Section 3 except (2) is changed to

$$T'_{i,m}(\delta) = \max\left(T_0, \frac{\gamma'^2 V'_{i,m}(T_0)}{\delta^2}\right), \tag{7}$$

where γ' is a constant to be discussed shortly, and we replace $\widetilde{\mu}_i(\delta)$ in steps 4 and 5 with $\widetilde{\mu}'_i(\delta) = \mu_i(T'_{i,m}(\delta))$. Similar modifications can be made to Procedure R of Section 5 to construct STS MCB intervals with relative-width parameter δ. (Damerdji and Nakayama 1999 actually define their two-stage STS MCB procedures to determine the total number of batches to simulate for each system, but these can easily be modified to determine instead the total run length required, as in (7).)

We now provide more details on the constant γ' in (7). Suppose that we want to construct STS MCB intervals having asymptotic joint confidence level at least $1-\alpha$ when using STS function g_m based on $m \geq 1$ batches. Then $\gamma' = \gamma'(k, 1-\alpha, g_m)$ in (7) is chosen to satisfy

$$E\left[\prod_{i=1}^{k-1} \Phi\left(\frac{\gamma'}{\left[(1/g_m^2(B_i)) + (1/g_m^2(B_k))\right]^{1/2}}\right)\right] = 1-\alpha,$$

where $B_1, \ldots, B_k$ are independent standard Brownian motions. In contrast our regenerative MCB method requires the constant $\gamma = \gamma(k, 1-\alpha)$ in (2), which satisfies

$$\left[\Phi\left(\frac{\gamma}{\sqrt{2}}\right)\right]^{k-1} = 1-\alpha,$$

so $\gamma = \sqrt{2} z_{(1-\alpha)^{1/(k-1)}}$.

If we use the STS function $g_{\mathrm{bm},m}$ corresponding to batch means (BM) with a fixed number $m \geq 2$ of batches, the parameter γ' is the solution to

$$E\left[\prod_{i=1}^{k-1} \Phi\left(\frac{\gamma'}{\left[(m-1)/\chi_i^2 + (m-1)/\chi_k^2\right]^{1/2}}\right)\right] = 1-\alpha,$$

where $\chi_1^2, \ldots, \chi_k^2$ are independent χ^2 random variables, each with $m-1$ degrees of freedom. In this case the parameter $\gamma'(k, 1-\alpha, g_{\mathrm{bm},m})$ is exactly Rinott's (1978) constant in his two-stage selection procedure for comparing independent normal populations when the first-stage sample size for each population is m. Wilcox (1984) and Bechhofer, Santner, and Goldsman (1995) provide tables of values for γ'.

We now provide a comparison of the STS and regenerative MCB methods in terms of their total run lengths. We also compare the methods in terms of the *potential* total run lengths, which we define as the second terms in the maxima in (7) and (2). Specifically, these are

$$\begin{aligned} \bar{T}'_{i,m}(\delta) &= \frac{\gamma'^2 V'_{i,m}(T_0)}{\delta^2}, \\ \bar{T}_i(\delta) &= \frac{\gamma^2 V_i(T_0)}{\delta^2}, \end{aligned}$$

for the STS and regenerative MCB methods, respectively. We analyze the ratios $R_{i,m}(\delta) = T'_{i,m}(\delta)/T_i(\delta)$ and $\bar{R}_{i,m}(\delta) = \bar{T}'_{i,m}(\delta)/\bar{T}_i(\delta)$.

Theorem 4 *Suppose an STS MCB procedure is applied with any STS function g_m based on a fixed number $m \geq 1$ of batches and scaled such that $E[g_m^2(B)] = 1$. Also,* *suppose the first-stage run length for both the regenerative and STS MCB methods is* $T_0 = \zeta\delta^{-\lambda}$ *for any constants* $\zeta > 0$ *and* $0 < \lambda \leq 2$. *Then the following hold:*

(i) $\delta^2 T'_{i,m}(\delta) \Rightarrow t'_{i,m}$ *as* $\delta \to 0$, *where*

$$t'_{i,m} = \begin{cases} \gamma'^2 \sigma_i^2 g_m^2(B_i) & \text{if } 0 < \lambda < 2, \\ \max[\zeta, \gamma'^2 \sigma_i^2 g_m^2(B_i)] & \text{if } \lambda = 2. \end{cases}$$

(ii) $\gamma'(k, 1-\alpha, g_m) > \gamma(k, 1-\alpha)$.

(iii) $R_{i,m}(\delta) \Rightarrow R_{i,m}$ *and* $\bar{R}_{i,m}(\delta) \Rightarrow \bar{R}_{i,m}$ *as* $\delta \to 0$, *where*

$$R_{i,m} = \frac{t'_{i,m}}{t_i}, \quad \bar{R}_{i,m} = \frac{\gamma'^2 g_m^2(B_i)}{\gamma^2},$$

and t_i *is defined in (3).*

(iv) *If* $\lambda = 2$ *and* $\{V'_{i,m}(t) : t > 0\}$ *is uniformly integrable, then* $\lim_{\delta\to 0} E[R_{i,m}(\delta)] > 1$. *If* $\lambda < 2$ *and* $\{V'_{i,m}(t)/V_i(t) : t > 0\}$ *is uniformly integrable, then*

$$\lim_{\delta\to 0} E[R_{i,m}(\delta)] = r_{i,m} \equiv \left(\frac{\gamma'(k,1-\alpha,g_m)}{\gamma(k,1-\alpha)}\right)^2 > 1.$$

(v) *If* $\{V'_{i,m}(t)/V_i(t) : t > 0\}$ *is uniformly integrable, then* $\lim_{\delta\to 0} E[\bar{R}_{i,m}(\delta)] = r_{i,m}$.

Theorem 4 shows that in terms of average total run length, regenerative MCB methods are strictly better than STS MCB methods. As for the variability, Theorem 1(ii) establishes that the total run length $T_i(\delta)$ for regenerative MCB is asymptotically equal to $t_i\delta^{-2}$ to first order as $\delta \to 0$, where t_i is a degenerate random variable. In contrast, by Theorem 4(i), STS MCB methods have a total run length $T'_{i,m}(\delta)$ that is asymptotically equivalent to $t'_{i,m}\delta^{-2}$ to first order, where $t'_{i,m}$ is nondegenerate. Therefore, the run length for regenerative MCB has no random variability of order δ^{-2} but it does for STS MCB, so regenerative MCB methods have asymptotically less variable run lengths than STS methods.

We now examine what happens when we let the number m of batches grow large.

Theorem 5 *Under the same assumptions as in Theorem 4, the following hold:*

(i) $\gamma'(k, 1-\alpha, g_m) \to \gamma(k, 1-\alpha)$ *as* $m \to \infty$.

(ii) $\lim_{m\to\infty} \lim_{\delta\to 0} \delta^2 T'_{i,m}(\delta) \Rightarrow t_i$, *where* t_i *is defined in (3).*

(iii) $R_{i,m} \Rightarrow 1$ *and* $\bar{R}_{i,m} \Rightarrow 1$ *as* $m \to \infty$.

(iv) $E[R_{i,m}] \to 1$ *and* $E[\bar{R}_{i,m}] \to 1$ *as* $m \to \infty$.

From Theorems 4 and 5, we now can make the following comparisons of the STS MCB procedures applied to the limiting Brownian motions and our regenerative MCB methods. As the number of batches grows large, STS MCB methods become comparable on average to our regenerative MCB methods in terms of the asymptotic total run length. But for any fixed number of batches, STS MCB methods are inferior to regenerative MCB methods. The reason for this is that for a fixed number m of batches, the STS estimator of the TAVC is not consistent as the first-stage run length grows, whereas the regenerative estimator $V_i(t)$ in (1) of the TAVC is consistent. However, this comparison does not address which approach is better in the small-sample context, where it is possible that STS MCB procedures outperform regenerative ones. See Nakayama (2006) for an empirical comparison.

As an illustration, we list below some values of $\gamma'(k, 1-\alpha, g_{\mathrm{bm},m})$ (taken from Wilcox 1984) for the BM MCB methods and $\gamma(k, 1-\alpha)$ for the regenerative MCB methods for $k = 5$ systems and confidence level $1-\alpha = 0.9$:

$$\begin{aligned} \gamma'(5, 0.9, g_{\mathrm{bm},10}) &= 3.137, \\ \gamma'(5, 0.9, g_{\mathrm{bm},30}) &= 2.855, \\ \gamma'(5, 0.9, g_{\mathrm{bm},50}) &= 2.811, \\ \gamma(5, 0.9) &= 2.748. \end{aligned}$$

Observe that $\gamma' > \gamma$ for each m and the two values become close as m gets large, which is consistent with Theorems 4(ii) and 5(i). For comparison, $\bar{\gamma} = \bar{\gamma}(k, 1-\alpha)$ in Theorems 1(iii) and 3(iii) has $\bar{\gamma}(5, 0.9) = 2.599$.

7 CONCLUSIONS

We presented two-stage procedures selection and MCB procedures for steady-state performance measures when simulating regenerative systems. Procedure A produces asymptotically valid MCB intervals having absolute-width parameter δ, and Procedure A2 modifies A to also be a selection method. Procedure R constructs relative-width MCB intervals. When the first-stage run length is asymptotically negligible compared to δ^{-2}, our methods are asymptotically efficient in the sense that the total run lengths are equivalent (to first order) to what they would be if the variances were known. In terms of the average and variability of total run lengths, our regenerative MCB procedures are asymptotically more efficient than the STS MCB methods of Damerdji and Nakayama (1999), but they become equivalent as the number of batches grows large.

Although this paper focused on comparing steady-state means of regenerative systems, the selection and MCB procedures presented hold more generally. Specifically, suppose that each system i has a parameter θ_i, which is not necessarily the steady-state mean of system i, and we want to compare the k systems in terms of $\theta_1, \ldots, \theta_k$. Assume that we have for each parameter θ_i an estimation process $\theta_i(t)$ that satisfies a CLT, and there is a weakly consistent estimator of the variance parameter appearing in the CLT. Then we show in Nakayama (2006) how to develop analogous selection and MCB procedures in this setting. This general framework encompasses comparing means of independent (not necessarily normally distributed) populations, functions of means, quantiles, steady-state means of non-regenerative systems, and functions of steady-state means. Moreover, the estimation process $\theta_i(t)$ need not converge at rate $t^{-1/2}$. For details, see Nakayama (2006).

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