Proceedings of the 2005 Winter Simulation Conference
M. E. Kuhl, N. M. Steiger, F. B. Armstrong, and J. A. Joines, eds.

SUBCONTRACTING IN A MAKE-TO-STOCK PRODUCTION SYSTEM, IPA GRADIENTS FOR AN SFM

Sameh Al-Shihabi

Industrial Engineering Department
University of Jordan
Amman 11942, JORDAN

ABSTRACT

This work extends the single product, single facility and single warehouse Make-to-Stock (MTS) model (Zhao and Melamed 2004), by allowing subcontracting. In addition to the inventory triggering level employed in their model, a new subcontracting triggering level is used to manage the subcontracted amount. A stochastic fluid model (SFM) is used to model the MTS system with subcontracting. In the current model, the subcontractor is assumed capable of satisfying any difference between the demand and the in-house production to keep the inventory at the subcontracting triggering level. The sensitivity of the inventory on-hand, backorders and amounts subcontracted with respect to the two employed triggering levels are found using infinitesimal perturbation analysis (IPA).

1 INTRODUCTION

Manufacturing systems keep ample inventory to quickly respond to demand's variability. A famous example is the Make-to-Stock MTS production system where a plant replenishes a pile of stock that is used to satisfy an external demand. Subcontracting and outsourcing are new practices that companies are increasingly implementing to assist in satisfying such variability in demand (Tully 1994 and Abernathy et. al. 2000).

One model that accounts for both the in-house production and subcontracting (Bradley 2004); where, a Brownian motion model is used to represent the production system as M/M/1 queue and the optimum levels of production capacity, inventory triggering level and subcontracting triggering level are found.

The subcontracting mechanism starts once the inventory level reaches the subcontracting triggering level while the production replenishment mechanism is initiated once the inventory drops below the in-house production triggering level. The use of such a dual base-stock policy is found optimal for a continuous time M/M/1 system (Bradley 2004). Two triggering levels can also be used to control the inventory replenishment process at Caterpillar (Rao et al. 2002).

In this work, we extend the previous MTS model (Zhao and Melamed 2004) such that the stock could be replenished from the in-house IH products in addition to the out-house OH products supplied by the subcontractor. Unlike Bradley model (Bradley 2004), the subcontractor is assumed to have enough production capabilities to compensate any difference between the demand and the IH supply. Any demand and supply distributions are accepted by this work. Assuming an infinite subcontracting capacity might be feasible if the difference between demand and internal supply is low. Additionally, cases where the external market is capable of satisfying such a difference or for a factory having a single shift per day that makes them three shifts might also be covered in the proposed model.

In their model, Zhao and Melamed use a stochastic fluid model SFM to represent the flow of demand and production within their proposed system (Zhao and Melamed 2004). Motivation, advantages and literature about this shift in modeling from a discrete event dynamic system DEDS and diffusion models to SFM is thoroughly discussed in their work.

One motivation behind this modeling shift is the online calculation of the Infinitesimal Perturbation Analysis IPA gradients using the SFM. The sample path derivatives of the average inventory $L_I(\theta)$, average backorders $L_B(\theta)$ and average amounts subcontracted $L_\gamma(\theta)$ are found with respect to the two decision variables involved; the IH production triggering level (S) and the subcontracting triggering level (Z).

Any of these derivatives involves the calculation of $E[\frac{dL(\theta)}{d\theta}]$ and for this derivative to be representative of $\frac{dE[L(\theta)]}{d\theta}$; the real derivative , it needs to be unbiased, or $E[\frac{dL(\theta)}{d\theta}] = \frac{dE[L(\theta)]}{d\theta}$. As summarized by Zhao and Melamed (Zhao and Melamed 2004), for the expected derivate to be unbiased estimator, two conditions need to be met (Rubinstein and Shapiro 1993):

1. For each $\theta \in \Theta$, the IPA derivatives $L'(\theta)$ exists with probability 1 (w.p.1).
2. W.p.1, $L(\theta)$ is Liptschitz continuous in Θ, and the random Liptschitz constants $K(\theta)$, have finite first moments.

2 MTS MODEL WITH SUBCONTRACTING

New notation is added to Zhao and Melamed model to represent the new terms related to subcontracting (Zhao and Melamed 2004). Some of the new symbols used are adopted from Wardi et al. (2002). Incoming demand is satisfied through the on-hand inventory (OI) available while for any shortage encountered, the Warehouse WH backorders all of the shortage if this allowed. The backorders are either satisfied through the In-House (IH) products produced by the Manufacturing Plant (MP) or from the Out-House (OH) products supplied by the Subcontractor (SUBC). Infinite raw material is available for the MP and its production capacity is only dependent on its internal production capabilities such as machines and workforce.

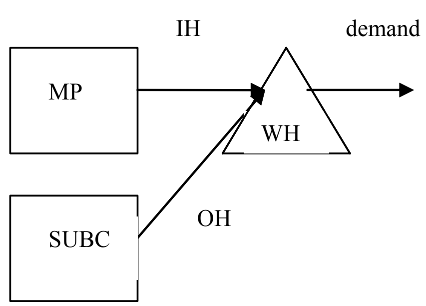

Figure 1: The flow of Material within the MTS System

Figure 1 shows the proposed system and the process of material flow. The base stock level S is used to control the MP production while the subcontracting triggering level Z is used to control the amount bought from SUBC. It is assumed that the holding cost is less than the backordering cost such that S>0. Extra cost is paid for any product subcontracted and this cost exceeds the production and holding costs for the IH products such that Z<S.

A first come first served FCFS service discipline is used to satisfy the continuous coming demand. The SUBC is assumed to have unlimited capacity that can keep the inventory at level Z when needed. The decision maker might allow backorders which depends on the cost's structure composed of holding, backordering and subcontracting costs.

The incoming demand at time t- $\alpha(t)$ - is satisfied from the on-hand inventory otherwise the demand is backordered if backordering is allowed. The conservation relation

$$S = I(t) - B(t) + X(t), \tag{1}$$

is always satisfied. The backordered amount at time t is given by

$$B(t) = \left[X(t) - S\right]^{+}, \tag{2}$$

while the on-hand inventory is

$$I(t) = \left[S - X(t)\right]^{+}, \tag{3}$$

and the outstanding orders by:

$$X(t) \in \left[0, S - Z\right]. \tag{4}$$

The production process can be described as follow, for $X(t) < S - Z$, only IH products replenishes the WH while for $X(t) = S - Z$, both the MP and SUBC replenishes the inventory. The amount replenished $\rho(t)$ by the MP is given by:

$$\rho(t) = \begin{cases} \min\{\mu(t), \alpha(t)\} & \text{if X(t) = 0} \\ \mu(t) & \text{if X(t) > 0} \end{cases} \tag{5}$$

while the amount subcontracted $\gamma(t)$ at $X(t) = S - Z$ is:

$$\gamma(t) = [\alpha(t) - \mu(t)]^{+}. \tag{6}$$

It needs to be noted that subcontracting initiates once $X(t) = S - Z$, however, it terminates once $\alpha(t) \le \mu(t)$.

The dynamics of the outstanding orders are governed by the following stochastic differential equations:

$$\frac{dX(t)}{dt^{+}} = \begin{cases} 0 & X(t) = 0 \text{ and } \alpha(t) \le \mu(t) \\ \alpha(t) - \mu(t) & X(t) = 0 \text{ and } \alpha(t) > \mu(t) \\ \alpha(t) - \mu(t) & 0 < X(t) < S - Z \\ 0 & X(t) = \text{S - } Z \text{ and } \alpha(t) \ge \mu(t) \\ \alpha(t) - \mu(t) & X(t) = \text{S - } Z \text{ and } \alpha(t) < \mu(t) \end{cases}. \tag{7}$$

To simplify the analysis of the sample path, let the sample path be divided into K regions where the start of each of these regions corresponds to the first point that $I(t) = S$ after a partially full inventory.

The full inventory regions are characterized by their start points and end point $F_k = [\xi_k(\theta), \tau_k]$, $k = 1,2,\cdots,K$. The start points can be dependant on S and Z while the end points correspond to instances showing changes in the sign of $\alpha(t)-\mu(t)$. The sample path starts with a full inventory.

Subcontracting takes place during some of these K periods as shown in period 2 of figure 2. The index set: $\Phi(\theta)=\{k \in \{1,...,K\}: \text{subcontracting takes place}\}$, shows which partial periods showed subcontracting. $N_T(\theta)$ represent the number of such periods. Within each of these periods, let the number of times during which subcontracting took place be M_k.

The start and end of each of these periods is defined by $\lfloor u_{k,m}(\theta), v_{k,m} \rfloor$. Not all starts depend on the S and Z values as will be shown later and none of the end points is dependent on any parameter. Lets also define $\Re_k$ to represent the first point of time before subcontracting takes place in period k during which the inventory reaches its full level. Lets also assume the existence of $N_T(\theta)$ such points.

We restate assumptions 1-4 taken from Zhao and Melamed (2004):

Assumption 1

1. *The processes* $\{\alpha(t)\}$ *and* $\{\mu(t)\}$ *have right-continuous sample paths that are piecewise continuously differentiable, w.p.1.*
2. *Each of the processes,* $\{\alpha(t)\}$ *and* $\{\mu(t)\}$*, has a finite number of discontinuities in any finite time interval, w.p.1.*
3. *No multiple events occur simultaneously, w.p.1*
4. *The processes,* $\{\alpha(t)\}$ *and* $\{\mu(t)\}$*, are both independent of the parameters S and Z.*

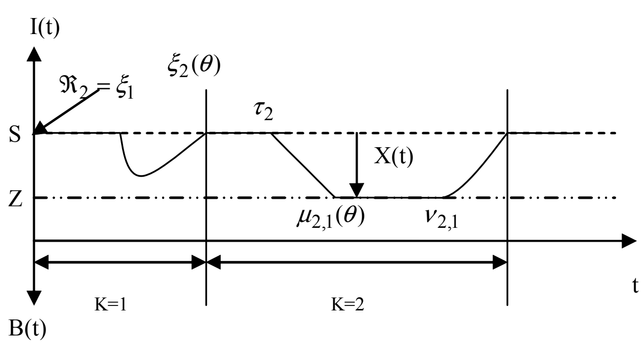

Figure 2: A Hypothetical Sample Path Showing Subcontracting and No Backorders for Z>0.

The measure of performance that we are interested in, are the following:

1. The average subcontracted amount

$$L_\gamma(\theta) = \frac{1}{T}\int_0^T \gamma(\theta,t)\cdot dt . \tag{8}$$

2. The average inventory on-hand

$$L_I(\theta) = \frac{1}{T}\int_0^T I(\theta,t)\cdot dt . \tag{9}$$

3. The average backorders

$$L_B(\theta) = \frac{1}{T}\int_0^T B(\theta,t)\cdot dt . \tag{10}$$

where the parameters of interest are S and Z.

3 IPA DERIVATIVES

This section starts with the derivatives of the subcontracted amount followed by the on-hand inventory and the backordered amount. A total of six derivatives are needed to fully describe the model's sample path derivatives with respect to the two decision variables.

3.1 Subcontracted Amount Derivatives

Subcontracting takes place once $X(t) = S - Z$. For this reason, it is only needed to find the IPA derivatives for regions corresponding to this condition. The sample path derivatives $L'_\gamma(S)$ and $L'_\gamma(Z)$ do exist w.p.1 since only the start points of the subcontracting regions do depend on the decision parameters, however, the amounts subcontracted and the end points of these regions are only dependent on the difference between $\{\alpha(t)\}$ and $\{\mu(t)\}$. As stated in assumption 1, $\{\alpha(t)\}$ and $\{\mu(t)\}$ themselves are independent on both S and Z. $\Phi(\theta)$ is assumed independent on $\Phi(\theta)$ in the following derivations.

Proposition 1 *For every* $S \in \Theta$,

$$L'_\gamma(S) = -\frac{1}{T}\cdot N_T(S) . \tag{11}$$

Proof *The subcontracting derivative is*

$$L'_\gamma(S) = \frac{1}{T}\cdot \sum_{k\in\Phi(S)} \sum_{m=1}^{M_k} \frac{d}{dS} \int_{\mu_{k,m}(S)}^{v_{k,m}} \alpha(t)-\beta(t)\cdot dt . \tag{12}$$

- **Case I:** let $L'_\gamma(S,k,m=1)$ represent the subcontracted amount derivatives of any $k \in \Phi(\theta)$ and for m=1,

$$L'_\gamma(S,k,m=1) = (\alpha(\mu_{k,1}(S)) - \beta(\mu_{k,1}(S))) \cdot \frac{d\mu_{k,1}(S)}{dS} - (\alpha(\nu_{k,1}) - \beta(\nu_{k,1})) \cdot \frac{d\nu_{k,1}}{dS} + \int_{\mu_{k,1}(S)}^{\nu_{k,1}} \frac{d}{dS}[\alpha(t) - \beta(t)] \cdot dt \quad . \quad (13)$$

it can be easily noted that the last two terms are independent of S. The value of the first term is found by looking at the integration of the term

$$\Lambda_{S \to Z} = \int_{\tau_k}^{\mu_{k,1}(S)} \alpha(t) - \beta(t) \cdot dt = S - Z \, . \quad (14)$$

Differentiating the above term with respect to S results in

$$(\alpha(\mu_{k,1}(S)) - \beta(\mu_{k,1}(S))) \cdot \frac{d\mu_{k,1}(S)}{dS} = 1.0 \, . \quad (15)$$

- **Case II:** For $m > 1$

$$\Lambda_{Z \to Z} = \int_{\nu_{k,m-1}}^{\mu_{k,m>1}(S)} \alpha(t) - \beta(t) \cdot dt = 0 \, . \quad (16)$$

which results in

$$(\alpha(\mu_{k,m>1}(S)) - \beta(\mu_{k,m>1}(S))) \cdot \frac{d\mu_{k,m>1}(S)}{dS} = 0.0 \quad . \quad (17)$$

or equivalently

$$L'_\gamma(S,k,m>1) = 0.0 \, . \quad (18)$$

Summing all the terms for $k \in \Phi(S)$ result on proposition 1.

Proposition 2 *For every $Z < S$,*

$$L'_\gamma(Z) = \frac{1}{T} \cdot N_T(Z) \, . \quad (19)$$

Proof *following the same steps of proposition 1 proof, the only difference would be the deferential of equation 14 with respect to Z:*

$$(\alpha(\mu_{k,1}(Z)) - \beta(\mu_{k,1}(Z))) \cdot \frac{d\mu_{k,1}(Z)}{dZ} = -1.0 \, , \quad (20)$$

while the rest of derivatives in equation 21 are 0.

$$L'_\gamma(Z) = \frac{1}{T} \cdot \sum_{k \in \Phi(Z)} \sum_{m=1}^{M_k} \frac{d}{dZ} \int_{\mu_{k,m}(Z)}^{\nu_{k,m}} \alpha(t) - \beta(t) \cdot dt \, . \quad (21)$$

Summing all these terms would result in equation 19.

The sign difference in equations 11 and 19 is intuitive given that an increase in S would delay the subcontracting for the first subcontracting period. An increase in the value of Z expedites subcontracting. However, later subcontracting periods, are independent of both.

These formulas are similar to the loss volume derivatives with respect to the buffer size (Wardi et al 2002). However, the buffer size can be considered as the limits of the outstanding orders and it can be controlled by changing the values of: S and Z, where the amount subcontracted replaces the loss volume in their work. The unbiasedness proofs are similar to that of Wardi et al. (2002).

3.2 On-hand Inventory and Backorder Derivatives

Depending on the value of Z chosen, the MTS model presented in this paper might and might not be subject to backordering. The IPA derivatives with respect to S are presented first followed by those of Z. The case of having $Z > 0$ is considered in this model; however, it is not optimal to have such a situation; instead, the subcontracting triggering level can be reduced to $Z = 0$ and the production triggering level is reduced to $S - Z$.

Proposition 3 *For $Z \geq 0$,*

$$L'_I(S) = \frac{1}{T} \cdot \sum_{\forall k \in \Phi(S)} \left[\mu_{k,1}(S) - \Re_k\right], \quad (22)$$

and

$$L'_B(S) = 0 \, , \quad (23)$$

while for $Z < 0$

$$L'_I(S) = \frac{1}{T} \cdot \sum_{\forall k \in \Phi(S)} \int_{\Re_k}^{\mu_{k,1}(S)} 1 \cdot \{I(S,t) > 0\} dt \quad (24)$$

and

$$L'_B(S) = \frac{1}{T} \cdot \sum_{\forall k \in \Phi(S)} \int_{\Re_k}^{\mu_{k,1}(S)} 1 \cdot \{B(S,t) > 0\} dt . \qquad (25)$$

Proof *For $t \in [0,T]$, the OH inventory derivative is given by*

$$L'_I(S) = \frac{1}{T} \cdot \frac{d}{dS} \int_0^T I(t,S) \cdot dt . \qquad (26)$$

The sample path is divided based on the value of $\frac{dI}{dS}$ whether it is 1.0 or 0.0. The above derivative can be re-written as

$$L'_I(S) = \sum_{k \in \Phi(S)} \begin{matrix} [\frac{d}{dS} \int_{R_k}^{\mu_{k,1}} I(t,S) \cdot dt \\ + \frac{d}{dS} \int_{\mu_{k,1}}^{R_{k+1}} I(t,S) \cdot dt] \end{matrix} , \qquad (27)$$

which is reduced to

$$L'_I(S) = \sum_{k \in \Phi(S)} \begin{matrix} [\int_{R_k}^{\mu_{k,1}} \frac{d}{dS} I(t,S) \cdot dt \\ + \int_{\mu_{k,1}}^{R_{k+1}} \frac{d}{dS} I(t,S) \cdot dt] \end{matrix} . \qquad (28)$$

since any change in the end point of a region is cancelled by the change in the start point of the following region. Additionally the integration's start and end points are not functions of S.

At any point $\Re_k, k \in \Phi(S)$, $\frac{dI(S,\Re_k)}{dS} = 1.0$ and $\frac{dX(S,\Re_k)}{dS} = 0.0$. Additionally, for any $\mu_{k1}(S), k \in \Phi(S)$, $\frac{dI(S,\mu_{k1}(S))}{dS} = 0.0$ and $\frac{dX(S,\mu_{k1}(S))}{dS} = 1.0$ results in canceling the second term in Equation 28 resulting in:

$$L'_I(S) = \frac{1}{T} \cdot \sum_{k \in \Phi(S)} \int_{R_k}^{\mu_{k,1}} 1.0 \cdot dt . \qquad (29)$$

For $Z < 0$, the OH derivative is only active if $I(t) \geq 0.0$. As shown in Zhao and Melamed (2004), and assuming that at time H –time points where the OH inventory becomes 0- changes due to $I(H,t) \cdot \frac{dH}{dS} = 0.0$ since $I(H,t) = 0.0$.

The derivative of $L^{'}_B(S)$ is quite intuitive, as $\frac{dB(S,t)}{dS} = 1.0$ for the regions where the subcontracting didn't take place yet.

Proposition 4 *For $Z \geq 0$,*

$$L'_I(Z) = \frac{1}{T} \cdot \sum_{j=1}^{NT(\theta)} \left[\Re_{j+1} - \mu_{j,1}(\theta)\right], \qquad (30)$$

and

$$L'_B(Z) = 0 . \qquad (31)$$

While for $Z < 0$

$$L'_I(Z) = \frac{1}{T} \cdot \sum_{j=1}^{NT(\theta)} \int_{\mu_{j,1}(\theta)}^{\Re_{j+1}} 1 \cdot \{I(Z,t) > 0\} dt , \qquad (32)$$

and

$$L'_B(Z) = \frac{1}{T} \cdot \sum_{j=1}^{NT(\theta)} \int_{\mu_{j,1}(\theta)}^{\Re_{j+1}} 1 \cdot \{B(Z,t) > 0\} dt . \qquad (33)$$

Proof *Similar to proof 2, however, it needs to be noted that derivatives' signs shift once $X(t) = S - Z$ for the first time in any period $k \in \Phi(Z)$. That is $\frac{dI(\mu_{k,1}(Z))}{dS} = 0.0$ and $\frac{dI(\mu_{k,1}(Z))}{dZ} = 1.0$.*

Proving the unbiasedness of the above IPA derivatives is similar to the proof of proposition 3 of Zhao and Melamed (2004).

4 CONCLUSIONS

This work adds a new control level to trigger subcontracting in the MTS production model of Zhao and Melamed (Zhao and Melamed 2004). A subcontractor is capable of maintaining the amount of outstanding orders at S-Z. The SFM of Zhao and Melamed is used to find the IPA derivative of the measures of performance with respect to the two triggering levels (Zhao and Melamed 2004).

The derivatives of the amounts subcontracted were similar to the loss derivatives with respect to buffer size of Wardi et al. (2002). The on hand inventory derivatives with respect to S and Z were also derived.

A direct extension to this work might include the case of limited subcontracting capacity as in Bradley (2004). Extending the above model to include transshipments is also possible.

REFERENCES

Abernathy, F., J. Dunlop, J. Hammond and D. Weil. 2000. Control your inventory in a world of lean retailing. *Harvard Business Review November-December: 169-176.*

Bradley, J. 2004. A Brownian approximation of a production-inventory system with a manufacturer that subcontracts. *Operations Research 52(5): 765-785.*

Rao, U., A. Scheller-Wolf and S. Tayur. 2000. Development of a rapid-response supply chain at Caterpillar. *Operations Research 48(2): 189-204.*

Rubinstein, R.Y., and A. Shapiro. 1993. Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method. John Wiley and Sons, New York, New York.

Tully, S. You'll never guess who really makes ….1994. *Fortune, October (3) : 124-128.*

Wardi, Y., B. Melamed, C.G. Cassandras, and C.G. Panayiotou. 2002. On-line IPA gradient estimators in stochastic continuous fluid models. *Journal of Optimization Theory and Applications 115(2): 369-405.*

Zhao, Y., and B. Melamed. 2004. Make-to-stock systems with backorders: IPA gradients. *Proceedings of the Winter Simulation Conference 2004:559-267.*

AUTHOR BIOGRAPHY

SAMEH AL-SHIHABI is an assistant professor in the Department of Industrial Engineering at the University of Jordan. His research interests include simulation optimization and combinatorial optimization. His e-mail address is s.shihabi@ju.edu.edu.