Proceedings of the 2004 Winter Simulation Conference
R .G. Ingalls, M. D. Rossetti, J. S. Smith, and B. A. Peters, eds.

A SIMULATION-BASED FIRST-TO-DEFAULT (FTD) CREDIT DEFAULT SWAP (CDS) PRICING APPROACH UNDER JUMP-DIFFUSION

Tarja Joro
Anne R. Niu

University of Alberta School of Business
3-40F Business Building
Edmonton, Alberta T6G 2R6, CANADA

Paul Na

Bayerische Landesbank New York Branch
560 Lexington Avenue
New York, NY 10022, U.S.A.

ABSTRACT

In recent years, the credit derivatives market has grown explosively and credit derivatives have become popular tools for hedging credit risk of financial institutions. Among the more sophisticated credit derivatives are the ones where the contingent payoffs depend on the dependence relationship among several firms in a basket, such as First-to-Default Credit Default Swap. In this paper, we present a simulation-based First-to-Default Credit Derivative Swap pricing approach under jump-diffusion and compare it with the popular default-time approach via Copula.

1 INTRODUCTION

A Credit Default Swap (CDS) is a contract in which one party buys protection for possible losses of reference asset (for example, a bond or a loan) due to a credit event such as default by the issuer. The protection buyer makes periodic payments to the protection seller until either the maturity of the contract or a credit event occurs. Upon the credit event, the seller pays the loss incurred by the credit event to the buyer and the buyer usually makes a final accrual fee payment to the seller. A single-name contract is a CDS that depends on the credit event of one bond or loan (referred to as a “name”).

Multiname contracts are ones where contingent payoffs depend on the credit risk of several names in a basket. For example, in a First-to-Default (FtD) contract, the contingent payment is made by the protection seller to the protection buyer as soon as a default occurs among the names in the basket, and after that the contract expires.

The purpose of this paper is to introduce a jump-diffusion based approach with correlation for pricing FtD and compare it with the popular Copula approach. In Section 2, we discuss the reasoning behind using jump-diffusion. Section 3 looks into the pricing of FtDs and into the role of correlation, whereas Section 4 describes the algorithm. The Copula approach is discussed in Section 5, and Section 6 concludes the paper.

2 MODELING DEFAULT PROBABILITIES

A CDS protects the protection buyer in case of a default of the reference asset. Thus, it is obvious that accurate estimation of default probabilities is crucial to pricing of CDSs. In deriving default probabilities, there are two broad modeling approaches: structural approach (see e.g. Merton (1974); Merton (1976); Black and Cox (1976); Longstaff and Schwartz (1995)), and reduced form approach (see e.g. Duffie and Singleton (1995); Jarrow, Lando and Turnbull (1994); Jarrow and Turnbull (1995); Madan and Unal (1994)). Zhou (1997) characterizes the two approaches as follows:

1. Structural approach proposes that the evolution of the firm's asset value follows a diffusion process, as proposed by Merton (1974). Defaults occur when the value of the asset becomes lower than the debt. According to the structural approach, firms never default by surprise due to the diffusion process, which is continuous.
2. Reduced-form approach assumes that there is no relation between the firm value and default. Default is seen as an unpredictable Poisson event involving a sudden loss in market value. Thus, according to reduced-form approach, firms never default gradually.

As Zhou (1997) argues, in reality default can occur in both ways: firms can default either gradually or by surprise due to unforeseen external shocks. The philosophies behind the structural and reduced form approaches can be combined by using a jump diffusion model that allows both gradual and sudden defaults (see e.g. Merton (1976), Ahn and Thompson (1988), Kou (2001) and Zhou (1997, 2001)).

The jump-diffusion approach overcomes some difficulties encountered in a traditional diffusion-based pricing approach. In particular, a CDS pricing approach based on a diffusion-process produces zero credit spreads for very short maturities. This happens because, if there is a finite distance to the default point (barrier), a continuous process cannot reach it in a very short time period. This is problematic because in reality the credit spreads would not go to zero even for contracts with very short maturities. (See Joro and Na, 2003 for further discussion on pricing CDS with jump-diffusion models).

3 PRICING FTD

Suppose that we have credit exposure to three names: AT&T (T), Ford (F), and Time Warner (TWX) and want to hedge the credit risk (Figure 1). On July 12, 2004, market quotes in basis points (bp, 1% = 100 bp) for 5-year CDS of T, F, and TWX were 330, 180 and 84, respectively. Since it is not possible to know which names will default in advance, full protection requires the purchase of a CDS for all names. However, this can be expensive. In this kind of scenario, FtD is very useful: it offers protection against the first default among the names, and is cheaper than buying three separate CDSs. In fact, the sum of the premiums (quotes) of the three individual CDSs is the theoretical upper bound for FtD price. In addition, FtD provides good investment incentive to the protection seller since by selling FtD, the seller's credit exposure is limited to just one name and FtD typically earns significantly more income than a single CDS.

In our example, intuitively, the quote for FtD should be somewhere between 330 (the worst one: T) and 594 (the sum of all three). It turns out that the dependence relationship plays a big role in determining the quote. Let us examine two correlation scenarios: 5% and 95% (Figures 2a and 2b).

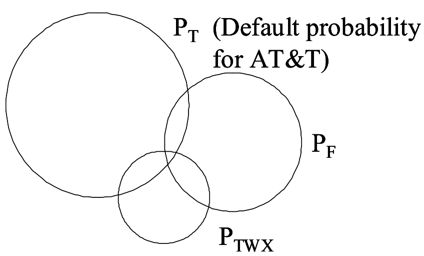

Figure 2a: 5% Correlation for All Pairs

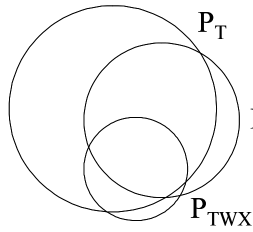

Figure 2b: 95% Correlation for All Pairs

The premium of FtD depends on the area ($P_T \cup P_F \cup P_{TWX}$), i.e. the probability of having at least one default. Clearly, the premium in Figure 2a will be greater than the one in Figure 2b.

When the correlation for all pairs of names in the basket is 1, the FtD premium (S_{FtD}) should be equal to the largest individual premium S_{max}. This essentially means that all names in the basket are identical, and effectively there is exposure to one name only. When the correlation is 0 for all pairs, the premium for FtD contract of n names equals the sum of n individual premiums: $S_{FtD} = \sum_{i=1}^{n} S_i$. Thus, upper and lower limits for the price of a FtD can be expressed as $S_{max} \le S_{FtD} \le \sum_{i=1}^{n} S_i$.

Thus, unlike in the traditional Markowitz portfolio theory, diversity (low correlation) is bad for FtD because FtD get riskier as the correlation goes down.

Clearly, modeling correlation is crucial in accurately pricing multiname products including FtD. However, correlation (default correlation) is rather difficult to model, particularly due to the lack of available market data.

Similar to a single name CDS, a FtD can be priced by equating the sum of present values of the fee leg to the sum of present values of the contingent payment leg.

Suppose that the CDS rate S as a fraction of notional in bp per year is paid at dates $t_1 < t_2 < \ldots < t_n = T$ with $\Delta(t_{i-1}, t_i)$ representing the interval between payments dates (i.e., 0.5

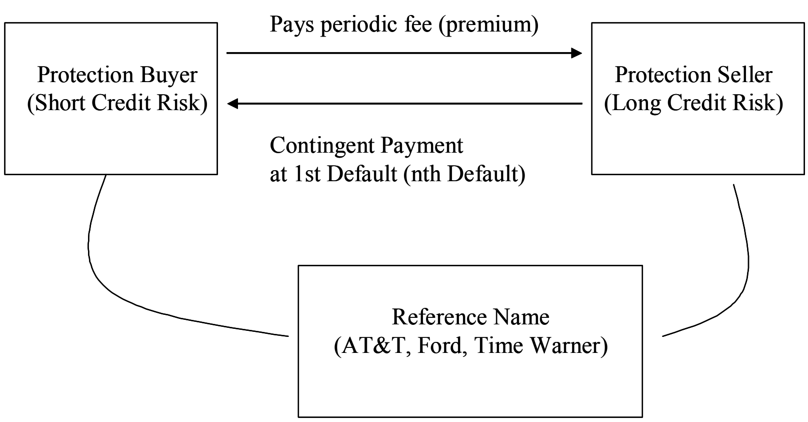

Figure 1: First-to-Default (FtD) Credit Default Swap (CDS)

for semi-annually payments). The sum of present values of fee leg, F, can be written as

$$F = S\sum_{i=1}^{n}\Delta(t_{i-1},t_i)1_{\{\tau_1>t_i\}}e^{-rt_i} = SL$$

where $\tau_1 = \min(\tau_1,\tau_2,\tau_3)$ is the 1st default time.

The sum of present values of fee accruals, A, can be defined as

$$A = S\sum_{i=1}^{n}\frac{\Delta(t_{i-1},t_i)}{2}1_{\{\tau_1\le t_i\}}e^{-rt_i} = SM$$

where it is assumed that the default between the regular fee payments always occurs exactly in the middle. The error from this approximation gets smaller as the time step gets smaller.

The contingent leg payoff, C, can be described as

$$C = (1-R)\sum_{i=1}^{n}1_{\{\tau_1\le t_i\}}e^{-rt_i}.$$

Then,

$$C = F + A = SL + SM = S(L+M).$$

Therefore,

$$S = \frac{C}{(L+M)}.$$

4 JUMP-DIFFUSION STRUCTURAL APPROACH WITH CORRELATION

Following Merton (1976), for all three names, let V follow a jump-diffusion process

$$dV_i = (\mu_i - \lambda_i\kappa_i)V_i dt + \sigma_i V_i dW_i + (J_i - 1)V_i dp_i$$

where $\kappa = E(J-1)$, $ln(J) \sim N(\nu,\gamma^2)$, dp is a Poisson process generating the jumps with the intensity of λ and dW_i represents the standard Brownian motion with correlation structure from equity prices.

Although several models use multivariate normal assumptions, this is problematic in credit risk modeling. In practice, more joint extreme events occur than under normality assumption.

Suppose there exist two random variables X and Y with marginal distributions F_X and F_Y. Then, lower tail dependence is defined as

$$\lambda = \lim_{u\to 0} P\left(Y < F_Y^{-1}(u) \mid X < F_X^{-1}(u)\right).$$

Under multivariate normal distribution, as Nyfeler (2000) shows, λ is 0. In reality, λ should be non-zero, which can be achieved under t-distribution. Thus, we employ t-distribution as follows:

1. Cholesky decomposition A of R (correlation matrix) such that $R = A \cdot A^T$
2. Draw 3-dimentional independent normal random variables $\boldsymbol{z} = (z_1, z_2, z_3)$
3. Draw an independent χ^2_ν random variable s
4. $\boldsymbol{y} = \boldsymbol{z}'A$
5. $\boldsymbol{x} = \boldsymbol{y}\sqrt{\frac{\nu}{s}}$.

For the example given in Section 3, the parameters and correlation matrix for structural approach are given in Tables 1 and 2.

Table 1: Parameters for Asset Value Simulation

	AT&T (T)	Ford (F)	Time Warner (TWX)
Asset in Billions	38	311	118
Default Point	18	201	29
Asset Volatility	14%	6%	25%
Risk free rate	1.90%		
Recovery	50%		
Ln(J)	N(0, 0.0054)		

Table 2: Correlation Matrix

	T	F	TWX
T	1	0.825	0.846
F	0.825	1	0.911
TWX	0.846	0.911	1

As discussed in Section 3, low correlation produces higher premiums. As Table 3 shows, under $\nu = 2$ the fact that there are more extreme events increases default correlations and thus reduces the premium.

Table 3: Results from Jump-Diffusion Approach

ν *(Degrees of Freedom)*	Historical Correlation (as in Table 2)	5% Correlation	95% Correlation
2	393	552	349
30	416	572	365

5 U DEFAULT TIME APPROACH VIA COPULA

A Copula $C(u_1,\ldots,u_n):[0,1]^n \to [0,1]$ is a multivariate distribution function such that its marginal distributions are uniform. Copulas can be used to link marginal distributions with a particular dependence structure.

For given univariate marginal distribution functions $F_1(x_1),\ldots,F_n(x_n)$, $C(F_1(x_1),\ldots,F_n(x_n)) = F(x_1,\ldots,x_n)$ produces a multivariate distribution function through a Copula *C*.

Sklar's Theorem (see Nelsen (1999)) proves that for any multivariate distribution function **F** with marginals $F_1,\ldots,F_n$, there exists a Copula *C* such that $F(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$.

In other words, Sklar's theorem provides that for any multivariate distribution, the univariate marginal distributions and the dependence structure can be separated. This is particularly convenient for pricing multiname products by borrowing dependence structure from equity data and marginal cumulative default probability distribution from single name CDS quotes.

To implement the Copula approach, the following steps are taken (Figure 3):

1. Draw $r_1, r_2, r_3 \sim t_\nu(0,\Sigma)$ with equity correlations.
2. Get uniform random variables through univariate cumulative t-distribution
3. Get the default times $\tau_i = F_i^{-1}(T_\nu(r_i))$ with the credit curve (interpolated term structure of cumulative default probability in bp) in Table 4.

As Figure 3 illustrates, most of time the algorithm returns 11 (no default). Since we are pricing 5-year contracts, anything greater than 5 would result in no default.

The same interpretation as given in Section 4 applies to Table 5.

Table 4: Credit Curves as of 7/12/2004

Year	T	F	TWX
0	0	0	0
1	180	110	48
2	325	181	73
3	470	252	98
4	565	306	133
5	660	360	168
6	686	375	178
7	712	390	188
8	728	399	197
9	744	407	205
10	760	416	214
11	10000	10000	10000

Table 5: Results from Copula Approach

ν	Historical Correlation	5% Correlation	95% Correlation
2	441	510	397
30	487	580	410

6 CONCLUSION

In structural approach, the jump-diffusion nature of default process can be explicitly incorporated into the model. However, the approach is very time consuming and computationally expensive.

The Copula approach is very simple and computationally fast. In Copula approach, the underlying assumption is that all relevant information is already included through the market views in single-name CDS curves.

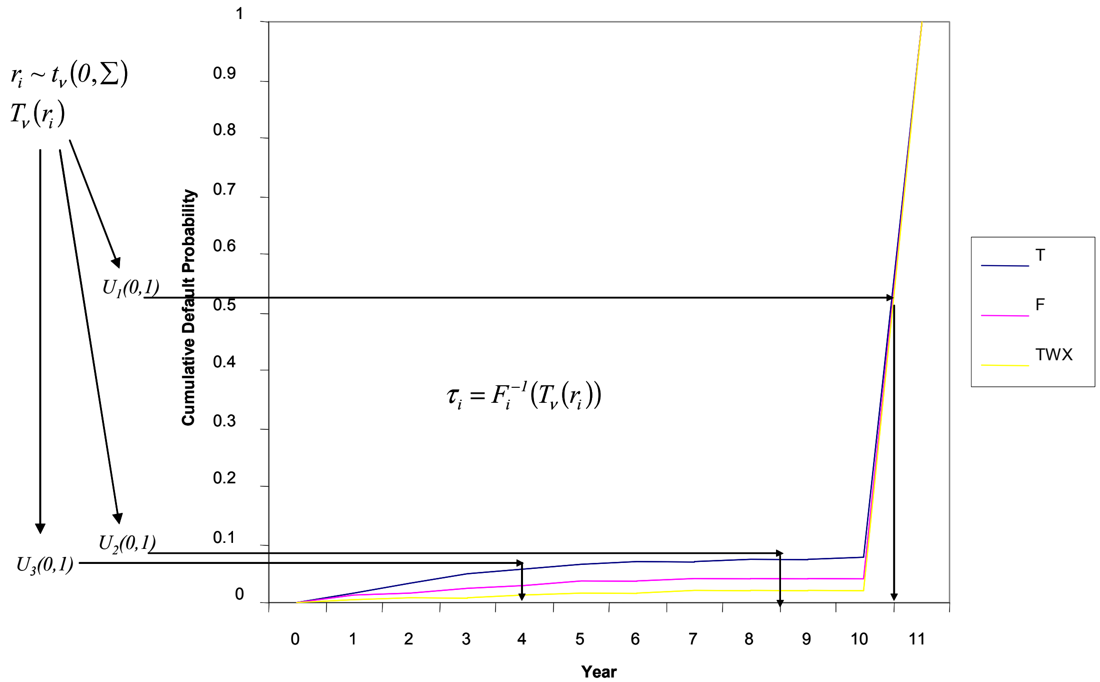

Figure 3: Generating Correlated Default Times from Credit Curve through Copula

ACKNOWLEDGMENTS

Tarja Joro acknowledges financial support from University of Alberta School of Business Canadian Utilities Fellowship. The views expressed in this paper are those of the authors and do not reflect the views of Bayerische Landesbank.

REFERENCES

Ahn, C.M. and H.E. Thompson. 1988. Jump-Diffusion Processes and the Term Structure of Interest Rates, Journal of Finance, 43: 155-174.

Black, F. and J.C. Cox. 1976. Valuing corporate securities: Some effects of bond indenture provisions, Journal of Finance 31: 351-367.

Duffie, D. and K.J. Singleton. 1995. Modeling term structures of defaultable bonds, working paper, Stanford University Business School.

Jarrow, R.A., D. Lando, and S. Turnbull. 1994. A Markov model for the term structure of credit risk spreads, working paper, Cornell University.

Jarrow, R.A. and S. Turnbull. 1995. Pricing derivatives on financial securities, Review of Financial Studies 1: 427-445.

Joe, H. 1997. Multivariate Models and Dependence Concepts, Chapman and Hall, London.

Joro, T. and P. Na. 2003. A Simulation-Based Credit Default Swap Pricing Approach Under Jump-Diffusion, In Proceedings of the 2003 Winter Simulation Conference, ed. S.E. Chick, P.J. Sanchez, D. Ferrin, and D.J. Morrice, 360-363. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers.

Kou, S.G. 2001. A Jump Diffusion Model for Option Pricing with Three Properties: Leptokurtic Feature, Volatility Smile, and Analytical Tractability, working paper, Columbia University, New York.

Li, D.X. 2000. On Default Correlation: A Copula Function Approach, Journal of Fixed Income, 6: 43-54.

Longstaff, F.A. and E.S. Schwartz. 1995. A simple approach to valuing risky and floating rate debt," Journal of Finance 50: 789-819.

Madan, D.B. and H. Unal. 1994. Pricing the risks of default, working paper, The Wharton School of the University of Pennsylvania.

Merton, R.C. 1974. On the Pricing of Corporate Debt: the Risk Structure of Interest Rates, Journal of Finance, 29: 449-470.

Merton, R.C. 1976. Option Pricing when Underlying Stock Returns are Discontinuous, Journal of Financial Economics, 3: 125-144.

Nelsen, R. 1999. An Introduction to Copulas, Springer, New York.

Nyfeler, M.A. 2000. Modelling dependencies in credit risk management, diploma thesis, Swiss Federal Institute of Technology.

Zhou, C. 1997. A Jump-Diffusion Approach to Modeling Credit Risk and Valuing Defaultable Securities, Finance and Economics Discussion Series 1997-15 / Board of Governors of the Federal Reserve System (U.S.).

Zhou, C. 2001. The Term Structure of Credit Spreads with Jump Risk, Journal of Banking and Finance, 25: 2015-40.

AUTHOR BIOGRAPHIES

TARJA JORO is an Assistant Professor in Management Science at the University of Alberta School of Business in Edmonton, Canada. She received her Ph.D. from Helsinki School of Economics. Her research interests are in productivity and efficiency studies and financial engineering. Her email and web address are <tarja.joro@ualberta.ca> and <http://www.bus.ualberta.ca/tjoro>.

PAUL NA is a Vice President and Head of Credit Risk Methodology at Bayerische Landesbank New York Branch. He received his Ph.D. from University of Georgia. His research interests are in credit risk modeling and mutual fund performance. His email is <pna@bayernlbny.com>.

ANNE R. NIU is a doctoral student at the University of Alberta School of Business in Edmonton, Canada. She received undergraduate degrees in Mechanical Engineering and Economics and Master's degree in Management and Economics from Tsinghua University. Her research interests are in supply chain management. Her email is <rniu@ualberta.ca>.