Proceedings of the 2004 Winter Simulation Conference
R .G. Ingalls, M. D. Rossetti, J. S. Smith, and B. A. Peters, eds.

SIMULATION-BASED PRICING OF MORTGAGE-BACKED SECURITIES

Jian Chen

Fannie Mae
3900 Wisconsin Ave. N.W.
Washington, DC 20016, U.S.A.

ABSTRACT

Mortgage-Backed-Securities (MBS), as the largest investment class of fixed income securities, have always been hard to price. Because of the following reasons, normal numerical methods like lattice methods, or finite difference method for solving PDEs are hard to apply: 1) the path dependence of mortgage pool cash flows. 2) the embedded American call option to prepay. 3) the American put option to default. 4) the fact that mortgage borrower do not/cannot exercise these option optimally. And those reasons make Monte Carlo simulation the best approach to price MBS. A standard MBS pricing framework would consists the following parts:1) Interest Rate model. 2) Prepayment model, which consists house turnover model and refinance model. 3) OAS model, which captures risk factors from the market price. Those factors are not accounted for in the previous two models. In order to hedge MBS efficiently and effectively, we need to calculate hedging measures quickly and correct. Chen and Fu (2001, 2002, 2003) has developed some efficient hedging algorithm in the past to perform this task.

1 INTRODUCTION TO MORTGAGE-BACKED SECURITIES

A mortgage-backed security (MBS) is a security collateralized by residential or commercial mortgage loans. An MBS is generally securitized, guaranteed and issued by three major MBS originating agencies: Ginnie Mae, Fannie Mae, and Freddie Mac. The cash flow of an MBS is generally the collected payment from the mortgage borrower, after the deduction of servicing and guaranty fees. However, the cash flows of an MBS are not as stable as that of a government or corporate coupon bond. Because the mortgage borrower has the prepayment option, mainly exercised when moving or refinancing, an MBS investor is actually writing a call option. Furthermore, the mortgage borrower also has the default option, which is likely to be exercised when the property value drops below the mortgage balance, and continuing mortgage payments would not be economically reasonable. In this case the guarantor is writing the borrower a put option, and the guarantor absorbs the cost. However, the borrower does not always exercise the options whenever it is financially optimal to do so, because there are always non-monetary factors associated with the home, like shelter, sense of stability, etc. And it is also very hard for the borrower to tell whether it is financially optimal to exercise these options because of lack of complete and unbiased information, e.g., they may not be able to obtain an accurate home price, unless they are selling it. And there are also some other fixed/variable costs associated with these options, such as the commission paid to the real estate agent, the cost to initialize another loan, and the negative credit rating impact when the borrower defaults on a mortgage. All these factors contribute to the complexity of MBS cash flows. In practice, the cash flows are generally projected by complicated prepayment models, which are based on statistical estimation on large historical data sets. Because of the complicated behaviors of the MBS cash flow, due to the complex relationships with the underlying interest rate term structures, and path dependencies in prepayment behaviors, Monte Carlo simulation is generally the only applicable method to price MBS.

MBS have become increasingly important fixed income instruments, both because of their volume and the role they play in fund investment and portfolio management. The total MBS issuance from the three agencies topped $2,131.9 billion in 2003, which is only second to the US treasury bond market, which is about twice the size. However, if we look at the outstanding MBS volume, which is $3492.1 billion in 2003, and it is rough equal to outstanding treasury securities with volume of $3574.9 billion. Combined with non-agency MBS issues, the MBS capital market is the largest of all fixed income securities, exceeding Treasury securities. (Above numbers are quoted from `<http://www.bondmarkets.com>).`

1.1 MBS Investors

With this huge market volume, any serious investor in the fixed income market need to consider the benefits and risks associated with MBS, in order to make the optimal investment decision. There are several special type of investor worth mentioning because of the specific benefit brought by MBS:

- Pension/Retirement Funds: Because of the long-term investment horizon, and constant cash flow requirement for this type of investors, MBS is an ideal type of investment, with long payback period, and regular amortized payments;
- Mortgage Originators: Because the capital requirement for MBS is much lower than for mortgage loans (8%), this type of financial institution could swap their loans for MBS, and utilize their capital more efficiently;
- Mortgage Servicers: Because the servicing income is highly sensitive to mortgage prepayment, investing in stripped mortgage Principal Only securities would be a natural hedge.

1.2 MBS Structures

There are many different types of MBS. Based on payment structure, they could be classified into three major categories: Pass-through MBS, Collateralized Mortgage Obligations (CMOs), and stripped MBS.

- Pass-through MBS is the most common MBS. As the name suggests, it pass-through principal and interest payment collected from mortgage borrowers, after subtracting the servicing and guaranty fee, to the investor. This is the building block for most other structured MBS.
- Pass-through MBS could be structured into several CMOs, with the most common case, sequentials. In this structure, the original pass-through MBS is divided into several classes, which are called tranches, and generally named A, B, C, D, etc. Tranche A will get the principal payment first, and class B will get principal paid back only when tranche A has been paid off. Tranche C will get principal payment after tranche B is being paid off, so on and so forth. The interest payment for each tranche is proportional to its remaining balance. The reason for this structure is that investors have different preferences. Tranche A is paid off first, with less risk, and shorter life, and the return is lower. While tranche D is paid off after all other tranches have been paid off, so it is more risky with longer life span, the investor would require a higher return on this class of MBS.
- Stripped MBS is different from the above CMOs in this approach: this structure divides the payment of pass-through MBS into two classes: Interest Only (IO), and Principal Only (PO). Each piece is more risky than the original pass-through. In some scenario, if mortgages prepay real fast, the IO investor could not even recover her initial investment. However, as we have pointed before, the PO class could be a natural hedging instrument for the mortgage servicing income.

1.3 Challenges in MBS Analysis

The challenges in MBS analysis lies in two aspects: pricing and hedging.

Pricing of MBS is to estimate the NPV of uncertain future cash flows. This basically involves two fundamental models: interest rate model, which determines the discounting factor and the overall interest rate environment, and prepayment model, which determines the prepayment behavior of the mortgage borrower. Because all agency MBS and most of non-agency MBS are insured against principal loss, so default is generally considered part of prepayment, and it generally only makes 1% of total prepayment, most MBS investors do not model default separately from prepayment. To have a good prepayment model in of paramount importance in the pricing and hedging of MBS.

Hedging of MBS is to estimate the price sensitivity to risk factors, generally interest rates. The most common sensitivity measures used are the following:

- Static duration, which measures the price change with respect to interest rate change, without prepayment consideration;
- Zero duration, which measures the price change with respect to interest rate change, with fixed prepayment assumption;
- Option adjusted duration, which measures the price change with respect to interest rate change, with prepayment rate adjusted for interest rate change;
- Key rates duration, which measures the price change with respect to key interest rates change, with prepayment rate adjusted for interest rate change;
- Principal component duration, which measures the price change with respect to interest rates change driven by the principal components of interest rate, with prepayment rate adjusted for interest rate change.

The above hedging measures are discussed in Chen and Fu (2001, 2002, 2003) in detail. Also Chen and Fu(2003) demonstrated that the principal component duration are much more efficient in MBS hedging.

2 PREPAYMENT MODEL FOR MBS

As discussed earlier, prepayment model is the most important part in pricing MBS, and there are four main types of prepayment functions (Fabozzi (2000)):

- Arctangent Model: (An example from the Office of Thrift Supervision (OTS).)

$$CPR(t) = 0.2406 - 0.1389\arctan(5.9518(1.089 - \frac{WAC}{r_{10}(t)}));$$

- *CPR(S,A,B,M)* Model:

$$CPR(t) = RI(t)AGE(t)MM(t)BM(t);$$

where *RI(t)* is refinancing incentive;
AGE(t) is the seasoning multiplier;
MM(t) is the monthly multiplier, which is constant for a certain month;
BM(t) is the burnout multiplier.

- Prepayment models incorporating macroeconomic factors, i.e., the health of economics, housing market activity, etc.
- Prepayment models for individual mortgages.

For the last two types of prepayment models, we do not have any explicitly stated functional forms, mainly because they are proprietary models in the mortgage industry. But since our approach is general for any type of prepayment function, we can derive the derivatives once we are given an explicit form for the prepayment function.

3 SIMULATION-BASED PRICING OF MBS

In this section, we are going to give the implementation of the algorithm to price an MBS in detail.

3.1 Problem Setting

Generally the price of any security can be written as the net present value (NPV) of its discounted cash flows. Specifying the price of an MBS (here we consider only the pass-through MBS) is as follows:

$$P = E[V] = E\left[\sum_{t=0}^{M} PV(t)\right] = E\left[\sum_{t=0}^{M} d(t)c(t)\right],$$

where
P is the price of the MBS,
V is the value of the MBS, which is a random variable, dependent on the realization of the economic scenario,
PV(t) is the present value for cash flow at time *t,*
d(t) is the discounting factor at time *t*,
c(t) is the cash flow at time *t*,
M is the maturity of the MBS.

Monte Carlo simulation is used to generate cash flows on many paths. By the strong law of large numbers, we have the following:

$$E[V] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} V_i ,$$

where V_i is the value calculated out in path *i*.

The calculation of *d(t)* is found from the short-term (risk-free) interest rate process,

$$\begin{aligned} d(t) &= d(0,1)d(1,2)d(t-1,t) \\ &= \prod_{i=0}^{t-1} \exp(-r(i)\Delta t) = \exp\{-[\sum_{i=0}^{t-1} r(i)]\Delta t\}, \end{aligned} \qquad (1)$$

where
d(i, i+1) is the discounting factor for the end of period *i+1* at the end of period *i*;
r(i) is the short term rate used to generate *d(i, i+1)*, observed at the end of period *i*;
Δt is the time step in simulation, generally monthly, i.e. *Δt*= 1 month.

An interest rate model is used to generate the short term-rate *r(i);* then *d(t)* is instantly available when the short-term rate path is generated.

The difficult part is to generate *c(t)*, the path dependent cash flow of MBS for month *t*, which is observed at the end of month *t*. From chapter 19 of Fabozzi (1993), we have the following formula for *c(t)*:

$$\begin{aligned} &c(t) = MP(t) + PP(t) = TPP(t) + IP(t); \\ &MP(t) = SP(t) + IP(t); \\ &TPP(t) = SP(t) + PP(t); \end{aligned}$$

where
MP(t): Scheduled Mortgage Payment for month *t*;
TPP(t): Total Principal Payment for month *t*;
IP(t): Interest Payment for month *t*;
SP(t): Scheduled Principal Payment for month *t*;
PP(t): Principal Prepayment for month *t*.

These quantities are calculated as follows:

$$\begin{aligned} &MP(t) = B(t-1)\left(\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}}\right); \\ &IP(t) = B(t-1)\frac{WAC}{12}; \\ &PP(t) = SMM(t)(B(t-1) - SP(t)); \\ &B(t) = B(t-1) - TPP(t); \\ &SMM(t) = 1 - \sqrt[12]{1 - CPR(t)}; \end{aligned}$$

where

B(t) is the principal balance of MBS at end of month *t*;
WAC is the weighted average coupon rate for MBS, weighted by the balance of each mortgage;
WAM is the weighted average maturity for MBS, weighted by the balance of each mortgage;
SMM(t) is the single monthly mortality for month *t*, observed at the end of month *t*;
CPR(t) is the conditional prepayment rate for month *t*, observed at the end of month *t*.

In Monte Carlo simulation, along the sample path, *CPR(t)* is the primary variable to be simulated. Everything else can be calculated out once *CPR(t)* is known. Different prepayment models offer different *CPR(t)*, and it is not our goal to derive a new prepayment model or compare existing prepayment models. Instead, our concern is, given a prepayment model, how can we efficiently estimate the price sensitivities of MBS against parameters of interest? Generally different prepayment models will lead to different sensitivity estimates, so it is at the user's discretion to choose an appropriate prepayment function, as our method for calculating the "Greeks" is universally applicable.

3.2 Simulation Framework

In this section we are going to describe the building blocks of our simulation framework: interest rate model and prepayment model. We choose our interest model to be the one-factor Hull-White model (Hull and White 1993), for its simplicity and easy calibration to market term structure. For the prepayment model, we consider a *CPR(S,A,B,M)* model.

3.2.1 Interest Rate Model

In the one-factor Hull-White interest rate model, the underlying process for the short-term rate *r(t)* is given by

$$dr(t) = (\varphi(t) - ar(t))dt + \sigma dB(t),$$

where
B(t) is a standard Brownian motion;
a is the mean reverting speed, constant;
σ is the standard deviation, constant;
$\varphi(t)$ is chosen to fit the initial term structure, which is determined by

$$\varphi(t) = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}),$$

f(0,t) is the instantaneous forward rate, which is determined by

$$R(0,t) = \frac{1}{t}\int_0^t f(0,u)du,$$

Differentiating both sides, with respect to *t*, we have

$$f(0,t) = t\frac{\partial R(0,t)}{\partial t} + R(0,t),$$

where *R(0,t)* is the continuous compounding interest rate from now to time *t*, i.e. the term structure.

In order to simplify the simulation process, the model can be re-parameterized from its original to the following:
$dx(t) = -a(t)x(t)dt + \sigma dB(t), x(0) = 0;$
x(t) is determined by

$$a(t) = r(t) - x(t) = f(0,t) + \frac{\sigma^2}{2a}(1 - e^{-at})^2.$$

The solution for process *x(t)* is given by

$$x(t) = \sigma e^{-at}\int_0^t e^{au}dB(u),$$

which is a Gaussian Markov process, and can also be represented as

$$x(t) = \sigma e^{-at}W(\frac{e^{2at}-1}{2a}),$$

where *{W(t), t≥0}* is also a Brownian motion.
In this case, the interest rate *r(t)* can be represented in the following form:

$$r(t) = (a(t) + g(t)W_{h(t)},$$

where

$$\begin{aligned} a(t) &= f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2; \\ g(t) &= \sigma e^{-at}; \\ h(t) &= \frac{e^{2at}-1}{2a}. \end{aligned} \tag{2}$$

To simulate *r(t)* given by above, we will first simulate *x(t)*, which is a Gaussian Markov process, and then compute the short-term interest rate by (2).

For calculating the price of MBS, the short-term rate is not sufficient; the long-term rate process is also required, especially the 10-year Treasury rate, which is a deterministic function of *r(t)* in the Hull-White model. Generally this is the case for short-term rate models, but not true for more complicated interest rate models, e.g., the HJM (Heath, Jarrow and Morton (1992)) model and the LIBOR forward rate model (Jamshidian(1997)). The long-term rate *R(t,T)* is calculated from the following, :

$$P(t,T) = e^{-R(t,T)(T-t)} = A(t,T)e^{-B(t,T)r(t)};$$
$$B(t,T) = \frac{1-e^{-a(T-t)}}{a};$$
$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-aT}-e^{-at})^2(e^{2at}-1).$$

P(t,T) is the zero coupon bond price at time *t*, with face value \$1, matured at *T*. Thus we can derive the *R(t,T)* as following:

$$R(t,T) = -\frac{\ln A(t,T) - B(t,T)r(t)}{(T-t)}.$$

Figure 1 gives one simulated path of the short rate r(t) and long term rate r10(t), which is later used to generate the prepayment rate.

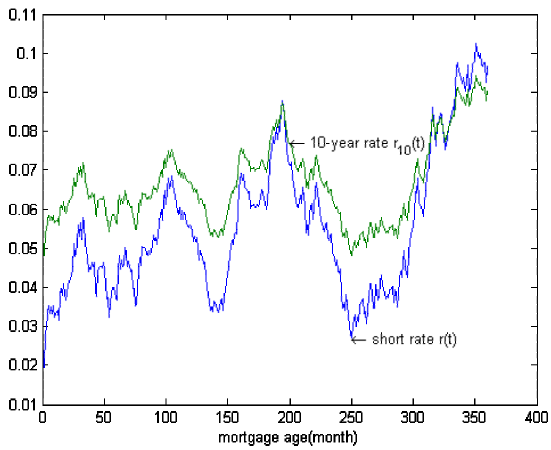

Figure 1: Simulated Interest Rate (1 Path)

3.2.2 Prepayment Model

We use the second type of prepayment function, among the four described in section 3. An example for this type of prepayment model is available from the sample code at `http://www.numerix.com`..

$$CPR(t)=RI(t)AGE(t)MM(t)BM(t);$$

where

$$RI(t)=0.28+0.14\tan^{-1}(-8.571+430(WAC-r_{10}(t-1)));$$
$$AGE(t) = \min(1, \frac{t}{30});$$
$$BM(t) = 0.3 + 0.7\frac{B(t-1)}{B(0)};$$

MM(t) takes the value from [0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.1, 1.18, 1.22, 1.23, 0.98], starting from January, ending in December; $r_{10}(t)$ is the 10-year rate, observed at the end of period *t*, a quantity that is highly correlated with the prevailing 15-year and 30-year fixed mortgage rates.

The MBS we price is a fixed-rate mortgage pool, with a WAC of 6.62%, and pool size of \$4,000,000. Once the 10-year rate is simulated, the prepayment rate can also be determined also with the characteristics of the mortgage pool. Figure 2 shows one path of simulated mortgage prepayment rate.

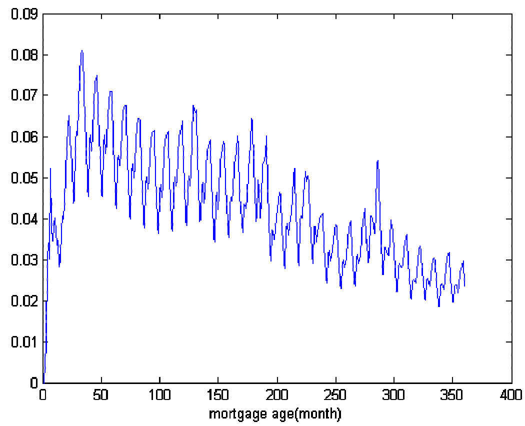

Figure 2: Simulate Mortgage Prepayment Rate (1 Path)

Once the prepayment rate is acquired, the cash flow from the mortgage pool is determined. Figure 3 shows one simulated path of the cash flow and its corresponding net present value. And the sum of the present value would the net present value in this scenario.

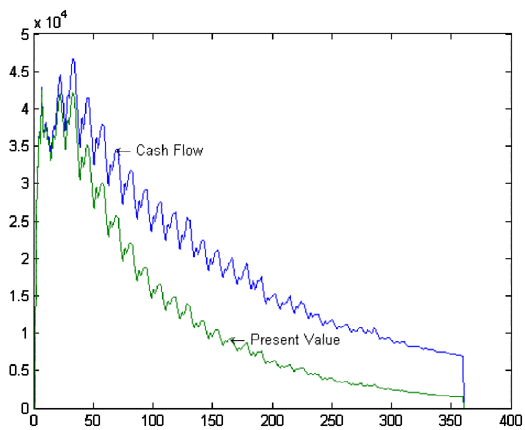

Figure 3: Simulated Cash Flow and Present Value

After we have performed a certain number of simulations, we can calculate the mean of the NPV, and that will be the price of the MBS we priced. Figure 4 shows the histogram of the NPV distribution for 300 simulations. The price of the MBS is $4,439,670.73, and the 95% confidence interval is $36,521.49.

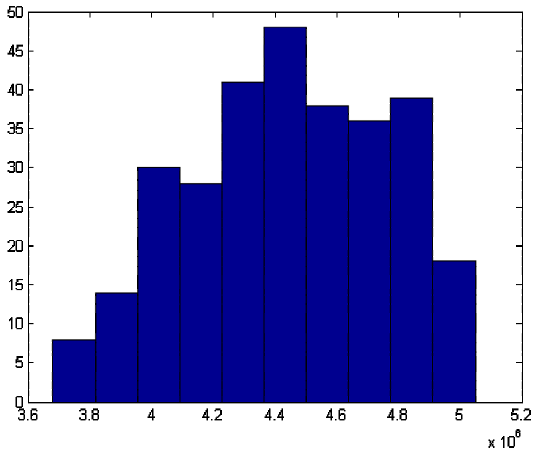

Figure 4: Histogram of the NPV of MBS

3.2.3 OAS Adjustment

When we apply the interest rate and prepayment model to price an MBS, it is generally not in agreement with the market price. In order to adjust our model, such that it could produce a price equal to market price, we need to introduce the option-adjusted spread (OAS). If we believe our prepayment model is accurate, i.e., the cash flow is correct, in order to change the price, we can only adjust the discounting factor, and that is exactly how OAS plays the role. If we change the discounting factor in (1) to the following:

$$d(t) = d(0,1)d(1,2)d(t-1,t)$$
$$= \prod_{i=0}^{t-1} \exp[-r(i)+s)\Delta t] = \exp\{-[\sum_{i=0}^{t-1}(r(i)+s)]\Delta t\},$$

such that

$$E\left[\sum_{t=0}^{M} d(t)c(t)\right] = market\_price\,,$$

then the spread s which make equation (19) holds is called the OAS for this MBS. It is generally solved by some recursive algorithm.

OAS could be viewed as excess return beyond the risk free return, adjusted for the prepayment option. It captures the return required by the investor community, to compensated for risks associated with MBS, after adjustment for prepayment. It could be viewed as the premium for a tiny portion of credit risk from the MBS issuer, and the model uncertainty in the pricing framework, or market liquidity premium.

4 CONCLUSION

As we have pointed before, because of the complicated nature of MBS, pricing and hedging of this type of securities remain difficult. To make things more interesting, there are a lot of new mortgage products, like skip-a-payment mortgage, which the borrower could skip 1 to 10 payments, portable mortgage, which the borrower could take with her when moving, automotive refinance mortgage, which the borrower could automatically get refinance when the mortgage rate drops below a threshold.

As far as industry practitioners are concerned, Monte Carlo simulation is still the only way to price and calculate price sensitivities of MBS. So how to improve simulation accuracy and efficiency will still pose a great challenge for future research.

ACKNOWLEDGMENTS

The author would like to thank Michael C. Fu for his invitation to present this paper at the winter simulation conference. Any opinions expressed in this article are those of the author and do not reflect the views of Fannie Mae.

REFERENCES

Chen, J., and M.C. Fu. 2001. Efficient Sensitivity Analysis for Mortgage-Backed Securities. *Working paper*, University of Maryland, College Park, MD.

Chen, J., and M.C. Fu. 2002. Hedging Beyond Duration and Convexity. In *Proceedings of 2002 Winter Simulation Conference*, ed. E. Yücesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes, 1593-1599. Piscataway, NJ: Institute of Electrical and Electronics Engineers.

Chen, J., and M.C. Fu. 2003. Three essays on mortgage-backed securities: hedging interest rate and credit risks. Doctoral dissertation, Robert H. Smith School of Business, University of Maryland, College Park, MD.

Fabozzi, F. J.. 1993. *Fixed Income Mathematics*. New York, NY: McGraw-Hill.

Fabozzi, F. J.. 2000. *The Handbook of Mortgage-Backed Securities*. New York, NY: McGraw-Hill.

Heath, D., R. A. Jarrow, and A. Morton. 1992. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica* 60 (1): 77-105.

Hull, J., and A. White. 1993. One-Factor Interest-Rate Models and the Valuation of Interest-Rate Derivative Securities. *Journal of Financial and Quantitative Analysis* (28) 2: 235-254.

Jamshidian, F.. 1997. Libor and swap market models and measures. *Finance and Stochastics* (1) 4: 293–330.

AUTHOR BIOGRAPHY

JIAN CHEN <jchen@rhsmith.umd.edu> is currently a financial engineer in Fannie Mae. He received his Ph.D. in computational finance in the Robert H. Smith School of Business, at the University of Maryland. He received his B.S.E.E. and M.S.E.E. from JiaoTong University, in Xi'an and Shanghai, respectively. His research interests include simulation and computational finance, particularly with the pricing and hedging for interest rate derivatives and credit derivatives.