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PRICING DERICATIVE SECURITIES IN INCOMPLETE MARKETS

Sergey Sarykalin
Stan Uryasev

Department of Industrial and Systems Engineering
303 Weil Hall
University of Florida
Gainesville, FL 32611, U.S.A.

ABSTRACT

We propose the algorithms for pricing American and European options in incomplete markets. We consider a non-self-financing replicating portfolio and minimize the hedging error consisting of the self-financing error of the portfolio dynamics and the error of the option's payoff replication. We treat the pricing problem as regression with constraints and reduce it to a quadratic minimization problem. The algorithms of pricing American and European options differ in imposing one additional type of constraints. Prices of options for different initial and strike prices can be found in one optimization run. The algorithms create a table representing the option price as a function of time and the underlying stock price for the whole lifetime of the option. We illustrate the numerical performance of the algorithms with options on futures contracts in natural gas market.

1 INTRODUCTION

The classical Black and Scholes theory (Black and Scholes, 1973) of options pricing is based on perfect replication of an option payoff by a self-financing portfolio consisting of a stock and a bond (the market is assumed to be complete). However, real markets, such as electricity and natural gas markets, may have various sources of incompleteness, leading to significant errors in option prices based on the classical algorithms.

One of the common approaches to pricing options in incomplete markets is considering a replicating portfolio and minimizing the square of the replication error at expiration, see for example (Follmer and Schied, 2002). Pricing is done under various assumptions on underlying processes; for instance, for the Markov underlying stock process, recursive expressions for optimal replication strategy can be derived for European style options (Bertsimas et all., 2001).

2 DESCRIPTION OF THE ALGORITHM

We consider a dynamic replication strategy by rebalancing positions of the stock and the bond. The strategy is not self-financing, i.e. hedging and replication errors may have nonzero values at different time intervals. We minimize the total mean square error of hedging by adjusting positions of the stock and the bond. We use a historical bootstrap simulation procedure to model the evolution of the underlying stock process. The drift coefficient and the volatility of the "fan" of historical sample-paths is adjusted to match forecasted drift coefficient and volatility.

The model can fit a broad class of underlying stock processes. The type of the stock process is incorporated by including a set of constraints on functions representing the stock and the bond parts of the replicating portfolio. These constraints describe properties of the functions, such as monotonic behavior and convexity (properties implied by the underlying process).

The hedging error minimization is reduced to a quadratic optimization problem. We use a two dimensional grid in the space of time and the stock price. The hedging positions of the stock and the bond are functions of time and the stock prices. The resulting strategy creates a two dimensional table providing positions of the stock and the bond for all possible stock price movements at discrete times. The algorithm estimates prices of options for different initial and strike prices in one optimization run. We consider both European and American options. American options are priced by imposing additional constraints on the replicating portfolio (option price should be greater than or equal to the immediate exercise value of the option).

3 NUMERICAL RESULTS

We have conducted various numerical experiments for pricing American options in different markets. Numerical experiments demonstrated reasonable performance of the suggested algorithm.

This section illustrates the proposed algorithm with the results of pricing of options in the natural gas market. We priced American put options on futures contracts on natural gas with different strikes (expiration 5/24/2001). Tables 1 and 2 show calculated prices of options with 16 and 9 days to expiration, respectively. We used 30 historical futures sample paths to model the underlying futures price process.

Table 1: Numerical Results for American Put Options on Futures Contracts. Initial Futures Price is \$4.24, Time to Expiration is 16 Days. Strike = Strike Price of the Option, Actual = Actual Price of the Option (P_A), Obtained = Calculated Price of the Option (P_O), Accuracy = $2(P_A - P_O)/(P_A + P_O)$, Strike-Initial = the Difference between the Strike Price and the Initial Price

Strike	Actual	Obtained	Accuracy	Strike-Initial
3.6	0.006	0.050	157.45%	-0.639
3.7	0.007	0.063	159.71%	-0.539
3.8	0.047	0.077	48.63%	-0.439
3.9	0.067	0.100	39.23%	-0.339
4.0	0.097	0.125	25.46%	-0.239
4.2	0.175	0.191	8.74%	-0.039
4.4	0.287	0.285	-0.63%	0.161
4.6	0.431	0.426	-1.28%	0.361
4.8	0.598	0.592	-0.97%	0.561
5.0	0.778	0.783	0.69%	0.761
5.2	0.969	0.976	0.75%	0.961
5.4	1.164	1.171	0.59%	1.161
5.6	1.363	1.370	0.52%	1.361
5.9	1.661	1.670	0.53%	1.661

To demonstrate the typical structure of the solution across time, we present calculation results for the option with strike \$5.20 and 9 days to expiration. Table 3 shows the two dimensional table with value of this option as a function of time and the underlying futures price. It is straightforward to obtain the exercise policy from Table 3: the option should (should not) be exercised if the value of the replicating portfolio is equal to (is less than) the immediate exercise value of the option. The corresponding exercise policy is presented in Table 4.

Table 2: Numerical Results for American Put Options on Futures Contracts. Initial Futures Price is \$4.40, Time to Expiration is 9 Days. Strike = Strike Price of the Option, Actual = Actual Price of the Option (P_A), Obtained = Calculated Price of the Option (P_O), Accuracy = $2(P_A - P_O)/(P_A + P_O)$, Strike-Initial = the Difference between the Strike Price and the Initial Price

Strike	Actual	Obtained	Accuracy	Strike-Initial
3.6	0.006	0.024	120.79%	-0.794
3.7	0.007	0.033	130.52%	-0.694
3.8	0.017	0.045	90.68%	-0.594
3.9	0.025	0.058	79.66%	-0.494
4.0	0.045	0.073	47.46%	-0.394
4.2	0.082	0.112	30.84%	-0.194
4.4	0.169	0.173	2.40%	0.006
4.6	0.295	0.267	-9.85%	0.206
4.8	0.448	0.425	-5.32%	0.406
5.0	0.624	0.608	-2.63%	0.606
5.2	0.814	0.807	-0.83%	0.806
5.4	1.009	1.007	-0.20%	1.006
5.6	1.207	1.207	0.00%	1.206
5.9	1.506	1.507	0.07%	1.506

Table 3: The Option Price is a Function of Time and the Underlying Futures Price. Results for the American Put Option with Initial Price \$4.40, Strike Price \$5.20, and Time to Expiration 9 Days

Futures price, \$	Option price, \$									
5.94	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0	0
5.66	0.13	0.13	0.13	0.13	0.13	0.13	0.11	0.06	0.03	0
5.39	0.43	0.3	0.23	0.23	0.20	0.18	0.16	0.15	0.10	0
5.13	0.43	0.31	0.31	0.31	0.29	0.26	0.24	0.21	0.10	0.07
4.88	0.43	0.43	0.43	0.43	0.43	0.41	0.40	0.36	0.32	0.32
4.65	0.61	0.61	0.61	0.61	0.61	0.60	0.59	0.57	0.55	0.55
4.43	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.77	0.77
4.21	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
4.01	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19
3.82	1.38	1.38	1.38	1.38	1.38	1.38	1.38	1.38	1.38	1.38
3.64	1.57	1.57	1.57	1.57	1.57	1.57	1.57	1.56	1.56	1.56
3.46	1.74	1.74	1.74	1.74	1.74	1.74	1.74	1.74	1.74	1.74
Time to expiration (days)	9	8	7	6	5	4	3	2	1	0

Table 4: Exercise Policy: "Keep" = Keep the Option; "ex"= Exercise the Option. Results for the American Put Option with Initial Price $4.40, Strike Price $5.20, and Time to Expiration 9 Days

Futures price, $	Decision (keep/exercise)									
5.94	keep	keep	keep	keep	keep	keep	keep	keep	keep	keep
5.66	keep	keep	keep	keep	keep	keep	keep	keep	keep	keep
5.39	keep	keep	keep	keep	keep	keep	keep	keep	keep	keep
5.13	keep	keep	keep	keep	keep	keep	keep	keep	keep	keep
4.88	keep	keep	keep	keep	keep	keep	keep	keep	ex	ex
4.65	keep	keep	keep	keep	keep	keep	keep	keep	ex	ex
4.43	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
4.21	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
4.01	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
3.82	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
3.64	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
3.46	ex	ex	ex	ex	ex	ex	ex	ex	ex	ex
Time to expiration (days)	9	8	7	6	5	4	3	2	1	0

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AUTHOR BIOGRAPHIES

SERGEY SARYKALIN is a PhD student in the Department of Industrial and Systems Engineering at the University of Florida. E-mail: `<saryk@ufl.edu>`.

STAN URYASEV, Ph. D., is an Associate Professor in Department of Industrial and Systems Engineering and a Director of Risk Management and Financial Engineering Lab at the University of Florida. His research interests include Financial Engineering applications (portfolio optimization, trading strategies, credit cards scoring), Risk Management (Conditional Value-at-Risk, credit risk), and military applications. E-mail: `<uryasev@ufl.edu>`, website: `<www.ise.ufl.edu/uryasev>`.