Proceedings of the 2004 Winter Simulation Conference
R. G. Ingalls, M. D. Rossetti, J. S. Smith, and B. A. Peters, eds.

RANDOMIZED QUASI-MONTE CARLO: A TOOL FOR IMPROVING THE EFFICIENCY OF SIMULATIONS IN FINANCE

Christiane Lemieux

Department of Mathematics and Statistics
University of Calgary
Calgary, AB, T2N 1N4, CANADA

ABSTRACT

Quasi-Monte Carlo (QMC) methods have been used in a variety of problems in finance over the last few years, where they often provide more accurate estimators than the Monte Carlo (MC) method. These results have led many researchers to try to find reasons for the success of QMC methods in finance. A general explanation is that financial problems often have a structure that interacts in a constructive way with the point set used by the QMC method, thus resulting in estimators with reduced error. This positive interaction can be amplified by various fine-tuning techniques, which we review in the first part of this paper. Leaving aside these techniques, we then choose a few randomized QMC methods and test their "robustness" by comparing their performance against MC on different financial problems. Our results suggest that the chosen methods are efficient in a broad sense for financial simulations.

1 INTRODUCTION

Since the seminal work of Boyle (1977) where the Monte Carlo (MC) method was introduced for pricing options in finance, this method has proven to be useful for a wide variety of financial problems. One often cited advantage of MC is its "robustness", that is, its application is generally not restricted to special types of models or problems. One area where it was thought for a long time that MC could not be used is the pricing of *American options*. But several people have shown in the last few years that MC could be used for this problem as well. We refer the reader to Glasserman (2004) for a thorough treatment of the use of MC in finance.

As an alternative to MC, quasi-Monte Carlo (QMC) methods can be used for financial problems. These methods use a *highly uniform point set* (HUPS) to perform sampling instead of using random sampling like in MC. An early reference where these methods are used in finance is the work of Paskov and Traub (1995), who give numerical evidence demonstrating the superiority of QMC methods on a mortgage-backed security problem. At the time, this caused some surprise since this problem deals with a function defined over a 360-dimensional space and back then, it was believed that QMC could only outperform MC when the dimension was not too large (e.g, smaller than 20, say). That belief came from the fact that QMC methods are deterministic and the known error bounds for the QMC approximation

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{u}_i) \tag{1}$$

of

$$\mu = \int_{[0,1)^s} f(\mathbf{u})d\mathbf{u} \tag{2}$$

based on a HUPS $P_n = \{\mathbf{u}_i, i = 1, \ldots, n\}$ behave like $O(n^{-1}\log^s n)$. Thus when the dimension $s = 360$, the number of points n needs to be unrealisatically large in order for the QMC error bound to be better than the $n^{-1/2}$ probabilistic bound associated with MC.

Before going further, let us give an example to illustrate how the formulation (2) is general enough to include most financial problems. Consider an *Asian option* pricing problem, where the goal is to evaluate

$$\mathrm{E}(e^{-rT}\max(A_T - K, 0)), \tag{3}$$

where $A_T = \frac{1}{s}\sum_{j=1}^{s} S(t_j)$, $0 \le t_1 < \ldots < t_s = T$, $S(t)$ is the value of the underlying asset at time t, r is the risk-free interest rate, T is the expiration time of the option, and K its strike price. Assume the underlying asset follows the Black-Scholes model, that is, under the *risk-neutral measure* (see, e.g., Glasserman (2004)),

$$dS(t) = rS(t)dt + \sigma S(t)dB(t),$$

where $B(\cdot)$ is a standard Brownian motion. Therefore, $S(t)$ has a lognormal distribution whose parameters depend on

r, t, $S(0)$, and the volatility σ of the underlying asset. More precisely, we can write

$$S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)}, \tag{4}$$

and since $B(t) \sim N(0, \sqrt{t})$ (where $N(\mu, \sigma)$ denotes the normal distribution with mean μ and variance σ^2), (4) can also be written as $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z}$, where $Z \sim N(0, 1)$. Now, if Z is generated by inversion, we can write $Z = \Psi^{-1}(U)$, where $\Psi(\cdot)$ is the cumulative distribution function of a $N(0, 1)$, and $U \sim U[0, 1)$. So each $S(t_j)$ can be written as a function of $u_j \in [0, 1)$, which implies we can write $e^{-rT}\max(A_T - K, 0) = f(u_1, \ldots, u_s)$ for some function f, and thus

$$\mathrm{E}(e^{-rT}\max(A_T - K, 0)) = \int_{[0,1)^s} f(\mathbf{u})d\mathbf{u}.$$

The work of Paskov and Traub led many researchers to try to understand better why QMC methods could do well even in large dimensions. The notion of *effective dimension* has been introduced as a result of these investigations. Roughly, for a problem with an effective dimension d, good results can be obtained as long as the QMC method used is based on a HUPS that has good d-dimensional projections.

From a more general point of view, what has been emerging from these investigations is that QMC methods can be successful when the underlying point set interacts in a constructive way with the problem at hand. This interaction can actually be enhanced by either choosing a HUPS specifically for a given problem, or trying to reformulate the function to be integrated so that good properties of the HUPS can be exploited. An example of the latter is the use of *Brownian bridge* techniques, which were introduced by Caflisch and Moskovitz (1995). Examples of the former include recent work by Wang and Sloan (2003a), where *lattice rules* are chosen in a component-wise fashion using a criterion that depends on the integrand. More details on these *enhancement* or *fine-tuning techniques* are given in the first part of this paper. In the second part, we take a different point of view and try to see if these enhancement techniques are necessary for QMC methods to be efficient for financial simulations. To do so, we choose a few QMC constructions and test their "robustness" by using them on different financial problems.

More precisely, the remainder of this paper is organized as follows: in Section 2, we briefly recall basic facts about RQMC methods. The concept of effective dimension is discussed in Section 3, along with other related definitions. Section 4 reviews methods that can be used to enhance QMC, by exploiting the interaction between the function to be integrated and the HUPS used. Numerical results illustrating the efficiency of a few selected RQMC methods for financial simulations are presented in Section 5. Concluding comments and ideas for future research are outlined in Section 6.

2 RANDOMIZED QUASI-MONTE CARLO

We mentioned in the introduction that when QMC methods are used to approximate (2), bounds on the deterministic error can be found. These bounds are valid for functions satisfying strong regularity conditions (e.g., f must be of *bounded variation*: see Niederreiter (1992) and Owen (2004) for the details), and are not useful in practice because they are very difficult to compute and too conservative.

One way of obtaining error estimates for QMC methods is to randomize the underlying HUPS. More precisely, let $\mathbf{v}$ be a uniform random vector in some space Ω. Then choose a randomization function $r : \Omega \times [0, 1)^s \rightarrow [0, 1)^s$ and construct the randomized version $\tilde{P}_n = \{\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_n\}$ of P_n, defined by $\tilde{\mathbf{u}}_i = r(\mathbf{v}, \mathbf{u}_i)$. For example, with the Cranley-Patterson rotation (Cranley and Patterson, 1976), $\Omega = [0, 1)^s$ and $r(\mathbf{v}, \mathbf{u}_i) = (\mathbf{u}_i + \mathbf{v}) \bmod 1$.

The function r should be chosen so that (i) $r(\mathbf{v}, \mathbf{u})$ is uniformly distributed over $[0, 1)^s$ for any $\mathbf{u}$, and (ii) $\tilde{P}_n$ has the same highly uniform properties as P_n.

Once a randomization is chosen, the variance of the resulting estimator $\sum_{i=1}^n f(\tilde{\mathbf{u}}_i)/n$ can be estimated by generating m i.i.d. randomized point sets $\tilde{P}_n$.

For more on randomization techniques and standard constructions for QMC methods, we refer the reader to Owen (1998), L'Ecuyer and Lemieux (2002), and Glasserman (2004).

3 EFFECTIVE DIMENSION AND RELATED CONCEPTS

Let us first briefly introduce some notation. For $I = \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, s\}$, let $-I = \{1, \ldots, s\} \setminus I$ and $\mathbf{u}_I = (u_{j_1}, \ldots, u_{j_t})$. For each I, define the *ANOVA component*

$$f_I(\mathbf{u}) = \int_{[0,1)^{s-t}} f(\mathbf{u})d\mathbf{u}_{-I} - \sum_{J \subset I} f_J(\mathbf{u}).$$

We can then write $f(\mathbf{u}) = \sum_I f_I(\mathbf{u})$, and we have that

$$\int_{[0,1)^s} f_I(\mathbf{u})f_J(\mathbf{u})d\mathbf{u} = \begin{cases} 0 & \text{if } I \neq J \\ \sigma_I^2 & \text{if } I = J. \end{cases}$$

Therefore $\mathrm{Var}(f) = \sigma^2 = \sum_{I \neq \emptyset} \sigma_I^2$ and thus the σ_I^2/σ^2 – called *sensitivity indices* in Sobol' (2001) – can be seen as a measure of the relative importance of the ANOVA components f_I. We refer the reader to Owen (1998) for more on ANOVA decompositions.

To analyze the interaction between f and a given point set, it is useful to talk about the projections of P_n: for $I = \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, s\}$, let $P_n(I) = \{(u_{i,j_1}, \ldots, u_{i,j_t}), i = 1, \ldots, n\}$. Now, for P_n to approximate (2) with small variance, the intuition is that the projections $P_n(I)$ corresponding to important subsets I – as measured by σ_I^2/σ^2 – should be highly uniform. For less important subsets I, the quality of $P_n(I)$ is not as crucial.

Often, problems in finance are such that the nominal dimension s of the corresponding function f is large, but the important components f_I are such that $|I|$ is small. The notion of *effective dimension* captures this idea: following Caflisch, Morokoff and Owen (1997), we say that f has an *effective dimension* d_{tr} *in the truncation sense (in proportion* p*)* if $\sum_{I:I\subseteq\{1,\ldots,d_{\text{tr}}\}} \sigma_I^2 \geq p\sigma^2$, and f has an *effective dimension of* d_{su} *in the superposition sense* if $\sum_{I:|I|\leq d_{\text{su}}} \sigma_I^2 \geq p\sigma^2$.

For example, for the Asian call option described by (3) with the parameters $s = 32, \sigma = 0.3, r = 0.05, S(0) = 50, K = 45$, Lemieux and Owen (2001) estimate a lower bound of 0.97 on $\sum_{I:|I|\leq 2} \sigma_I^2/\sigma^2$. Hence this 32-dimensional problem has an effective dimension of 2 in the superposition sense in proportion 0.97. They use quasi-regression to compute these lower bounds, and the standard approach outlined in the introduction to generate the underlying asset price's paths. Alternatively, Sobol' (1993) and Archer, Saltelli and Sobol' (1997) estimate the sensitivity indices by directly estimating σ_I^2, an approach also used by Wang and Fang (2003), and Wang and Sloan (2003b). As discussed in the two latter papers, this approach works well to estimate the effective dimension in the truncation sense, but is cumbersome for estimating the effective dimension in the superposition sense.

Numerical results presented in these two papers suggest that many problems in finance have a low effective dimension in the superposition sense (with paths generated using the standard approach), which is consistent with results given in Paskov (1997), and Caflisch, Morokoff and Owen (1997). Wang and Sloan (2003b) also present an interesting theoretical analysis aimed at explaining this phenomenon.

4 ENHANCING QUASI-MONTE CARLO

The fact that many financial problems naturally have a low effective dimension in the superposition sense implies that HUPS with good low-dimensional projections – i.e., $P_n(I)$ is highly uniform whenever $|I|$ is small – can provide accurate estimates for these problems. However, many constructions for HUPS are such that the quality of $P_n(I)$ deteriorates as the smallest index j_1 in I increases. Sobol' and Halton sequences are examples of such constructions. Hence for RQMC estimators based on these HUPS to perform well, the problem at hand must have a small effective dimension in the truncation sense. This fact has motivated the introduction of techniques aimed at reducing this type of effective dimension, which we now discuss.

4.1 Brownian Bridges and Dimension Reduction

The *Brownian bridge* (BB) technique for QMC integration was first introduced by Caflisch and Moskowitz (1995), and then generalized by Morokoff and Caflisch (1997). In the standard approach to generate a path – which we outlined in the introduction – the coordinates $u_1, \ldots, u_s$ of $\mathbf{u}$ are successively used to generate the observations $B(t_1), \ldots, B(t_s)$ of the asset's underlying Brownian motion. If instead one tries to use the first few coordinates of $\mathbf{u}$ to specify as much as possible the behavior of $B(\cdot)$, then hopefully, this should reduce the effective dimension of the problem in the truncation sense. BB does that by first generating $B(t_s)$, then $B(t_{\lfloor s/2 \rfloor})$, then $B(t_{\lfloor s/4 \rfloor})$ and $B(t_{\lfloor 3s/4 \rfloor})$, and so on. This can be done easily since the Brownian bridge property tells us that for any $u < v < w$, we have that $B(v)|(B(u) = a, B(w) = b)$ has a normal distribution with mean $a(w-v)/(w-u)+b(v-u)/(w-u)$ and variance $(v - u)(w - v)/(w - u)$.

This technique can be generalized by observing that the standard method to generate $B(\cdot)$ can be written as $W = A\mathbf{z}$, where $W = (B(t_1), \ldots, B(t_s))^T$, $\mathbf{z} = (z_1, \ldots, z_s)^T$, and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

if we assume that $t_j - t_{j-1} = 1$ for each $j = 1, \ldots, s$, and the $z_j = \Phi^{-1}(u_j)$ are i.i.d. standard normal variables. Replacing A by a matrix V such that $VV^T = AA^T =: \Sigma$ is called a *generalized Brownian bridge technique* in Morokoff and Caflisch (1997). For example, Acworth, Broadie and Glasserman (1997) use a principal components analysis (PCA) to define V, that is, they take $V = PD^{1/2}$, where P's columns are formed by the eigenvectors of the covariance matrix Σ, and D is a diagonal matrix containing the corresponding eigenvalues of Σ in decreasing order. In their numerical results, PCA outperforms BB, and in addition, it can be used for multi-assets problems, whereas BB can only be used in this case if the assets are uncorrelated. PCA requires more computation time however, but Åkesson and Lehoczy (2000) propose a modification to PCA that reduces it.

More recently, Imai and Tan (2002) have proposed to use a matrix V of the form $V = AH$, where A is the lower triangular matrix obtained from the Cholesky decomposition of Σ, and H is an orthogonal matrix chosen so as to minimize the effective dimension of the problem in the truncation sense. A feature of this technique not present in BB and PCA is that the chosen matrix V depends on the problem. In the examples they provide, their technique slightly outperforms PCA, but they do not discuss the relative importance of the overhead computation that is required.

As mentioned before, the above techniques are aimed at reducing the effective dimension in the truncation sense, and are thus mostly useful for HUPS whose projections $P_n(I)$ deteriorate as the smallest index j_1 in I increases. It is important to know that some HUPS do not have this undesirable feature; they are *dimension-stationary*, that is, for any $I = \{j_1, \ldots, j_t\} \subseteq \{1, \ldots, s\}$ and $l \leq s - j_t$, we have that $P_n(I) = P_n(I+l)$. So for instance, a dimension-stationary point set is such that $P_n(\{1,3,4\}) = P_n(\{2,4,5\}) = \ldots = P_n(100, 102, 103\})$, and so on. By using *recurrence-based point sets* (L'Ecuyer and Lemieux 2002), which include *Korobov* and *polynomial Korobov* lattice point sets, it is easy to construct HUPS that are dimension-stationary. In addition, these point sets can handle problems with infinite dimension.

4.2 Customized Sampling Method

To study problems having a small effective dimension d_{tr} in the truncation sense, one can use *weighted* spaces of functions. In such spaces, a weight γ_j is associated with each dimension $j = 1, \ldots, s$. The properties of functions with small d_{tr} can then be captured by using weights, say, of the form $\gamma_j = \alpha\tau^j$, where α and $\tau < 1$ are parameters to be determined. Wang and Sloan (2003a) use this approach to construct lattice rules in a *component-by-component* fashion (see, e.g., Sloan, Kuo and Joe (2002)). In their work, the parameters α and τ are problem-dependent, which is why we call this approach a "customized sampling method".

For problems with a small d_{su} but for which d_{tr} is not necessarily small, working with weighted spaces of this form is not appropriate. However, one can still choose a HUPS based on a criterion that incorporates information on the problem. For example, a preliminary estimation of the sensitivity indices could be performed to guide the definition of a criterion of the form $M_{t_1,\ldots,t_d}$ or $\Delta_{t_1,\ldots,t_d}$ discussed in L'Ecuyer and Lemieux (2002).

5 NUMERICAL RESULTS

In the two previous sections, we reviewed approaches that can be used to enhance QMC methods by exploiting specific properties of the problem at hand. Although these fine-tuning techniques are certainly useful, in some cases it may not be feasible to use them. For example, in a general-purpose simulation software that includes RQMC methods, it may not be realistic (or safe) to make these techniques available. From this point of view, it seems of interest to select a few RQMC methods and see how "robust" they are, that is, if they can outperform MC on a variety of problems, and without using these enhancement techniques.

The purpose of this section is to investigate this point, and we do this by comparing the performance of three RQMC methods against MC on three problems. The RQMC methods chosen are Sobol', Korobov (Kor), and polynomial Korobov (PKor) rules. The Sobol' sequence is implemented as in Lemieux, Cieslak and Luttmer (2002), and the two Korobov methods are based on parameters chosen (for each n) via the criteria $M_{32,24,12,8}$ and $\Delta_{32,24,12,8}$ (L'Ecuyer and Lemieux 2002). All three methods are randomized by a shift (which is digital for Sobol' and polynomial Korobov), and thus properties (i) and (ii) of Section 2 holds for all our RQMC methods. The problems we consider consist in pricing (1) a digital option; (2) an American option; (3) a mortgage-backed security.

5.1 Digital Options

We chose this problem because it has been shown (Papageorgiou (2002)) that the Brownian bridge technique is worsening the estimation here, which is another reason to investigate the effectiveness of "plain" RQMC methods (i.e., that do not use enhancement techniques). The payoff of a digital call option is given by

$$C_D = \frac{1}{s}\sum_{j=1}^{s}(S_{t_j} - S_{t_{j-1}})^0_+ S_{t_j},$$

where $t_j = jT/s$ for $j = 1, \ldots, s$, and $(x)^0_+$ is equal to 1 if $x > 0$, and is 0 otherwise. Thus, the value of this type of option is determined more heavily by local trends of the underlying asset rather than by its global trend, which might be a reason for the failure of the Brownian bridge technique on this problem.

Table 1 gives results for different values of s and n, and with $T = 1$, $r = 0.045$, $\sigma = 0.3$, and $S(0) = 100$. The number of randomizations m was set to 25 for these results. For each pair (s, n), we give for each method the estimator for μ on the first line, and its standard error on the second line.

As we can see in this table, the three RQMC methods consistently succeed in reducing the variance for this problem, by factors ranging between about 30 and 4000. Note also that the PKor method is often the best method.

5.2 American Options

The options that we discussed so far were both *European* options, that is, the holder can only exercise the option at expiration time. An *American option* gives its holder the right to exercise *before* expiration time. More precisely, in this paper we assume that with an American option, the holder

Table 1: Digital Option

(s,n)	MC	Sobol'	Kor.	PKor
(64,1024)	52.718	52.691	52.698	52.690
	9.88e-2	1.08e-2	9.60e-3	5.07e-3
(64,4096)	52.737	52.689	52.691	52.691
	5.35e-2	1.99e-3	2.60e-3	8.05e-4
(128,1024)	52.235	52.228	52.237	52.230
	8.13e-2	1.52e-2	1.04e-2	3.32e-3
(128,4096)	52.272	52.234	52.233	52.233
	3.59e-2	5.42e-3	4.40e-3	4.59e-3
(256,1024)	51.954	51.912	51.916	51.905
	8.36e-2	1.33e-2	1.18e-2	3.30e-3
(256,4096)	51.947	51.923	51.908	51.918
	3.15e-2	5.38e-3	3.41e-3	4.32e-3

can exercise at a set of specific dates $t_1 < t_2 < \ldots < t_b = T$, which are usually equally spaced in time (such options are sometimes called *Bermudan options* in the literature). This complicates the pricing problem considerably because in order to estimate the value of the option, we can no longer simply run several realization paths of the underlying asset and compute the average actualized payoff: we must also determine *when* should the option be exercised. Formally, an American option based on a vector $\mathbf{S} = (S_1(\cdot), \ldots, S_d(\cdot))$ of d underlying assets and with payoff function $C(t, \mathbf{S}(t))$ has value

$$\mu = \max_{1 \le j \le b} \mathrm{E}(e^{-rt_j} C(t_j, \mathbf{S}(t_j))). \tag{5}$$

Many researchers have studied American option pricing in the last few years, and how MC could be used for that problem. We refer the reader to Glasserman (2004) for a detailed review of this research area. Here, we focus on the regression-based (REG) technique proposed by Longstaff and Schwartz (2001), which produces a low-biased estimate. To the best of our knowledge, Chaudhary (2004) is the only work that studies the use of QMC methods for this approach. This author also proposes a space-effective implementation of the Brownian bridge technique within this approach.

We now describe the REG approach, giving only the information necessary to understand our RQMC implementation. We refer the reader to Longstaff and Schwartz (2001); Clément, Lamberton and Protter (2002); Glasserman (2004) for additional information, including motivation for this method and how it relates to other methods.

The method uses n realization paths $\{\mathbf{S}^i(t), t = 0, t_1, \ldots, t_b; i = 1, \ldots, n\}$ of the underlying assets. It then estimates for each path i when is the optimal exercise time t_i^*. This is done by proceeding backward from T as follows: set $t_i^* = T$, then at time $t = t_{b-1}, t_{b-2}, \ldots, t_1$, set $t_i^* = t$ if $C(t, \mathbf{S}^i(t)) > \hat{F}(\mathbf{S}^i(t))$, where $\hat{F}(t, \mathbf{S}^i(t))$ is an estimate of the continuation value of the option at time t given $\mathbf{S}^i(t)$. This estimate is obtained by regression of the actualized payoffs (from time t_j^*, for each path j) against the current value of the assets over the paths that are *in-the-money*, that is, such that $C(t, \mathbf{S}^i(t)) > 0$. More precisely, a finite set of multivariate basis functions $\{\psi_l(\cdot), l = 0, 1, \ldots, M\}$ is chosen, and the regression coefficients are estimated as

$$(\hat{\beta}_0, \ldots, \hat{\beta}_M)^T = (\Psi^T \Psi)^{-1} \Psi^T (y_1, \ldots, y_{n^*})^T,$$

where n^* is the number of paths that are in-the-money at time t, $y_i = C(t, \mathbf{S}^i(t))$, and $\Psi_{i,l} = \psi_l(\mathbf{S}^i(t))$ for $i = 1, \ldots, n^*, l = 0, \ldots, M$. Then $\hat{F}(t, \mathbf{S}^i(t))) = \sum_{l=0}^{M} \hat{\beta}_l \psi_l(\mathbf{S}^i(t))$.

Once the optimal exercise times are estimated for each path, the option's value is estimated by

$$\hat{\mu}_{n,\mathrm{reg}} = \frac{1}{n} \sum_{i=1}^{n} e^{-rt_i^*} C(t_i^*, \mathbf{S}(t_i^*)).$$

Note that this approach – as well as most MC-based approaches for American option pricing – uses information across all paths to compute an estimate for (5). More precisely, the regression coefficients – and thus the estimated optimal exercise times t_i^* – are determined from all paths. Hence the estimator $\hat{\mu}_{n,\mathrm{reg}}$ does not satisfy the general definition (1). What does it imply? Formally, it means that if we use an RQMC method to compute $\hat{\mu}_{n,\mathrm{reg}}$ – i.e., the n paths are obtained from using the n points of a HUPS in dimension $s = bd$ – then we cannot use standard results on the variance of RQMC estimators to predict the behavior of $\hat{\mu}_{n,\mathrm{reg}}$. However, we can still hope that the high uniformity of the underlying HUPS will result in an estimator with reduced variance. The results below indicate that this seems to be the case, although the variance reduction factors are not as large as for the other two problems considered in this paper.

We now report results for three kinds of American options studied in Longstaff and Schwartz (2001). In each case, we assume the vector of underlying assets $\mathbf{S}(\cdot)$ follows a multivariate geometric Brownian motion, that is, for $l = 1, \ldots, d$,

$$dS_l(t) = S_l(t)(r - \delta_l)dt + \sum_{k=1}^{d} M_{l,k} dB_k(t),$$

where δ_l is the continuous dividend rate for $S_l(\cdot)$, $B_1(\cdot), \ldots, B_d(\cdot)$ are independent standard Brownian motions, and the matrix M whose (l,k)th entry is $M_{l,k}$ is such that $C = MM^T$, where C is the instantenous covariance matrix of $\mathbf{S}(\cdot)$. In our numerical experiments, we assume $C_{l,l} = \sigma^2$ for $l = 1, \ldots, d$, and $C_{k,l} = \rho\sigma^2$ for $k \neq l$. All results were obtained with $n = 4096$ and $m = 25$, and using the same basis functions as Longstaff and Schwartz. Each entry in the forthcoming tables contains an estimate for μ on the first line, and its standard error on the second line. Results for the corresponding European options are also given.

In Table 2, we look at a simple put option on one asset paying no dividend, with $T = 2$ and 50 exercise periods per year. Hence $s = 100$ for this problem. The other parameters are $\sigma = 0.2$, $r = 0.06$, and $K = 40$.

Table 2: American Put on Single Asset

$S(0)$	MC	Sobol'	Kor	PKor
36	American			
	4.861 9.87e-3	4.847 6.23e-3	4.849 5.10e-3	4.843 5.89e-3
	European			
	3.786 1.43e-2	3.757 5.08e-3	3.763 6.30e-3	3.770 4.77e-3
40	American			
	2.911 9.96e-3	2.899 5.27e-3	2.894 4.80e-3	2.896 5.31e-3
	European			
	2.379 1.19e-2	2.347 6.08e-3	2.353 5.23e-3	2.363 5.00e-3
44	American			
	1.721 6.36e-3	1.704 3.61e-3	1.693 5.12e-3	1.701 3.74e-3
	European			
	1.446 9.97e-3	1.423 6.57e-3	1.427 4.38e-3	1.434 5.21e-3

The results in Table 2 show that all three RQMC methods consistently reduce the variance compared to MC, by factors ranging between 1.5 and 4 for the American options, and 2.3 and 9 for the European options.

In Table 3, we consider an American-Bermudan-Asian option on one asset paying no dividend, and with $T = 2$ years. The payoff of this option at time t is given by $\max(0, A_t - K)$, where K is the strike price and A_t is the arithmetic average of the underlying asset during the period three months prior to time 0 up until time t. The option can be exercised any time after time 0.25, and time is discretized into 100 steps per year to approximate the continuous average and exercise opportunities. Thus the dimension s is 200 for this example. The results in Table 3 are given for different pairs $(A_0, S(0))$, and for $\sigma = 0.2$, $r = 0.06$, and $K = 100$.

Here again, all three RQMC methods consistently reduce the variance compared to MC, by factors ranging between approximately 1.6 and 15 for the American options, and 1.6 and 37 for the European options.

The last example considered is a call option on the maximum of five assets. These five assets are identically distributed, and are assumed to be independent in Table 4, while in Table 5 we assume $\rho = 0.5$ in the instantaneous covariance matrix C. There are three exercise periods per year and $T = 3$ in this example, so the dimension s is

Table 3: American-Bermudan-Asian Call on Single Asset

$(A_0, S(0))$	MC	Sobol'	Kor	PKor
(90,80)	American			
	0.955 1.05e-2	0.981 7.37e-3	0.978 8.30e-3	0.977 5.52e-3
	European			
	0.947 1.04e-2	0.972 7.22e-3	0.970 8.11e-3	0.968 5.56e-3
(90,120)	American			
	22.407 3.62e-2	22.417 1.12e-2	22.404 9.59e-3	22.406 9.32e-3
	European			
	21.248 4.10e-2	21.243 6.92e-3	21.230 7.62e-3	21.230 6.76e-3
(100,100)	American			
	8.663 2.60e-2	8.665 1.20e-2	8.669 1.00e-2	8.657 7.95e-3
	European			
	8.193 2.82e-2	8.207 8.93e-3	8.190 9.24e-3	8.186 8.57e-3
(110,90)	American			
	4.185 1.82e-2	4.216 8.37e-3	4.206 8.20e-3	4.203 7.94e-3
	European			
	3.950 1.98e-2	3.981 8.59e-3	3.963 8.56e-3	3.968 8.89e-3
(110,110)	American			
	17.349 2.80e-2	17.353 1.16e-2	17.347 1.17e-2	17.352 1.07e-2
	European			
	15.407 3.60e-2	15.411 8.05e-3	15.397 8.10e-3	15.394 7.79e-3

45 here. The dividend rate δ is set to 0.1 for each asset, $\sigma = 0.2$, $r = 0.05$, and $K = 100$.

Once again, all three RQMC methods succeed in reducing the variance compared to MC, whether the assets are correlated or not. For correlated assets, we see in Table 5 that the standard error of the MC estimators is often quite large (over 0.1), while RQMC methods have standard errors that are always below 0.05.

5.3 Mortgage-Backed Securities

This problem has been studied several times in the QMC literature, including in Paskov (1997), Caflisch, Morokoff and Owen (1997), and Tezuka (2001), to which we refer the reader for more details. In short, the goal here is to estimate a quantity of the form

$$\mu = \mathrm{E}\left[\sum_{j=1}^{s} d_j c_j\right], \tag{6}$$

Table 4: Call on Maximum of Five Uncorrelated Assets

$S(0)$	MC	Sobol'	Kor.	PKor
90	American			
	16.763 5.15e-2	16.696 2.88e-2	16.697 2.55e-2	16.710 3.22e-2
	European			
	14.626 5.55e-2	14.569 1.84e-2	14.574 2.98e-2	14.551 1.66e-2
100	American			
	26.231 6.41e-2	26.170 3.22e-2	26.176 2.12e-2	26.200 3.52e-2
	European			
	23.090 6.67e-2	23.029 2.45e-2	23.024 3.44e-2	23.002 2.15e-2
110	American			
	36.832 7.09e-2	36.765 3.79e-2	36.809 2.58e-2	36.815 3.62e-2
	European			
	32.718 7.74e-2	32.656 2.97e-2	32.662 3.84e-2	32.633 2.42e-2

Table 5: Call on Maximum of Five Correlated Assets

$S(0)$	MC	Sobol'	Kor.	PKor
90	American			
	13.489 7.70e-2	13.567 2.60e-2	13.553 3.21e-2	13.579 2.99e-2
	European			
	12.130 8.43e-2	12.303 2.18e-2	12.320 2.89e-2	12.299 1.84e-2
100	American			
	19.451 1.04e-1	19.630 2.91e-2	19.537 4.48e-2	19.541 3.61e-2
	European			
	17.516 1.05e-1	17.722 2.25e-2	17.726 2.90e-2	17.702 1.72e-2
110	American			
	26.560 1.06e-1	26.687 4.28e-2	26.542 3.93e-2	26.601 4.05e-2
	European			
	23.768 1.24e-1	23.996 2.29e-2	23.993 2.80e-2	23.963 1.88e-2

which represents the value at time 0 of a security backed by mortgages of length s months. Typically, s is chosen to be 360, corresponding to mortgages of 30 years. The variables d_j and c_j represent the discount factor and cash flow for month j, respectively. In turn, these quantities depend on random factors such as interest and prepayment rates. At the end though, the sum in (6) can be written as a function of $i_1, \ldots, i_{360}$, where i_j is the interest rate for month j, and μ in (6) can be written as an integral of the form (2).

The resulting function f in (2) has an effective dimension d_{su} in the superposition sense that depends on the parameters for the interest rate model. Caflisch, Morokoff and Owen (1997) give two sets of parameters for which f is "nearly linear" (i.e., $d_{\text{su}} = 1$ in proportion $p \approx 1$), and "non-linear" (i.e., $d_{\text{su}} = 1$ only in proportion $p \approx 0.94$). Those are the two sets we use in our experiments below.

As we can see in Table 6, all three RQMC methods reduce the variance substantially for this problem, both in the nearly linear and non-linear cases. More precisely, reduction factors between 34 and 4100 are obtained for this problem. The PKor method is the best method in all cases, and performs especially well for the nearly linear example.

Table 6: Mortgage-Backed Security

n	MC	Sobol'	Kor.	PKor
	nearly linear			
1024	131.782 3.85e-2	131.788 2.18e-3	131.788 2.05e-3	131.788 8.22e-4
4096	131.790 2.03e-2	131.787 6.84e-4	131.787 7.58e-4	131.787 3.16e-4
	non-linear			
1024	130.705 2.97e-2	130.706 4.91e-3	130.710 5.09e-3	130.713 2.32e-3
4096	130.711 1.47e-2	130.712 1.78e-3	130.714 1.69e-3	130.712 1.61e-3

6 CONCLUSION

The notion of effective dimension and other concepts related to it have led to the discovery of many fine-tuning techniques for the application of RQMC methods in finance in the last few years. In this paper, we briefly reviewed recent developments in this area of research. As a consequence of this recent trend, it is interesting to see if these fine-tuning techniques are necessary for RQMC methods to perform better than MC. We attempted to answer this question by giving numerical results comparing the performance of three RQMC methods against MC on three different financial problems. Although the number of points was relatively small (1024 or 4096) and the dimension relatively large (ranging between 45 and 360), our numerical results suggest that the chosen RQMC methods consistently provide estimators with smaller variance than MC. The gains are sometimes modest (factor of 1.5), but in some cases are quite large (over 4000).

There is still a lot of ongoing research addressing the problem of American option pricing. Following the work presented here, some possible next steps would be to study the combination of RQMC with other variance reduction techniques, such as importance sampling and control variates. Ideas developed in Broadie and Glasserman (1997), Avramidis and Hyden (1999), and Morani (2003) might be useful here. Boyle, Kolkiewicz and Tan (2001) have studied the use of RQMC methods within algorithms designed to produce high-biased estimators, such as the *stochastic mesh* of Broadie and Glasserman (1997): we believe it would be interesting to compare alternative ways of incorporating RQMC methods within this algorithm. Finally, it would be useful to compare the efficiency of the estimators obtained here with other low-biased estimators, such as the one proposed by Boyle, Kolkiewicz and Tan (2003), which is based on a *low-discrepancy mesh* and uses ideas developed by Avramidis and Hyden (1999).

ACKNOWLEDGMENTS

This work was supported by NSERC grant #RGP238959. The author also wishes to thank Hardeep S. Gill for his help with the numerical experiments.

REFERENCES

Acworth, P., M. Broadie, and P. Glasserman. 1997. A comparison of some Monte Carlo and quasi-Monte Carlo techniques for option pricing. In *Monte Carlo and Quasi-Monte Carlo Methods 1996*, ed. H. Niederreiter, P. Hellekalek, G. Larcher and P. Zinterhof, Volume 127 in Lecture Notes in Statistics, 1–18. Springer-Verlag.

Åkesson, F., and J. P. Lehoczy. 2000. Path generation for quasi-Monte Carlo simulation of mortgage-backed securities. *Management Science*, 46: 1171–1187.

Archer, G. E. B., A. Saltelli, and I. M. Sobol'. 1997. Sensitivity measures, ANOVA-like techniques and the use of bootstrap. *Journal of Statistical Computation and Simulation*, 58: 99–120.

Avramidis, A. N., and P. Hyden. 1999. Efficiency improvements for pricing American options with a stochastic mesh. In *Proceedings of the 1999 Winter Simulation Conference*, ed. P. A. Farrington, H. B. Nembhard, D. T. Sturrock, and G. W. Evans, 344–350. Piscataway, New Jersey: IEEE Press.

Boyle, P. 1977. Options: a Monte Carlo approach. *Journal of Financial Economics* 4: 323–338.

Boyle, P., A. W. Kolkiewicz, and K. S. Tan. 2001. Pricing American derivatives using simulation: a biased low approach. In *Monte Carlo and Quasi-Monte Carlo Methods 2000*, ed. K.-T. Fang, F. J. Hickernell and H. Niederreiter, 181–200, Springer.

Boyle, P., A. W. Kolkiewicz, and K. S. Tan. 2003. An improved simulation method for pricing high-dimensional American derivatives. *Mathematics and Computers in Simulation*, 62: 315–322.

Broadie, M., and P. Glasserman. 1997. A stochastic mesh method for pricing high-dimensional American options. Unpublished manuscript.

Caflisch, R. E., W. J. Morokoff, and A. B. Owen. 1997. Valuation of mortgage-backed securities using Brownian bridges to reduce effective dimension. *The Journal of Computational Finance*, 1: 27–46.

Caflisch, R. E., and B. Moskowitz. 1995. Modified Monte Carlo methods using quasi-random sequences. In *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, ed. H. Niederreiter and P. J.-S. Shiue, Volume 106 of Lecture Notes in Statistics, 1–16. New York: Springer-Verlag.

Chaudhary, S. K. 2004. Acceleration of Monte Carlo methods using low discrepancy sequences. Forthcoming doctoral dissertation, Department of Mathematics, University of California, Los Angeles, California.

Clément, E., D. Lamberton, and P. Protter. 2002. An analysis of a least squares regression method for American option pricing. *Finance and Stochastics*, 6: 449 –471.

Cranley, R., and T. N. L. Patterson. 1976. Randomization of number theoretic methods for multiple integration. *SIAM Journal on Numerical Analysis*, 13: 904–914.

Glasserman, P. 2004. *Monte Carlo methods in financial engineering*. New York: Springer-Verlag.

Imai, J., and K. S. Tan. 2002. Enhanced quasi-Monte Carlo methods with dimension reduction. In *Proceedings of the 2002 Winter Simulation Conference*, ed. E. Yücesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes, 1502–1510. Piscataway, New Jersey: IEEE Press.

L'Ecuyer, P., and C. Lemieux. 2002. Recent advances in randomized quasi-Monte Carlo methods. In *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications*, ed. M. Dror, P. L'Ecuyer and F. Szidarovszki, 419–474. Boston: Kluwer Academic Publishers.

Lemieux, C., M. Cieslak, and K. Luttmer. 2002. RandQMC user's guide: A package for randomized quasi-Monte Carlo methods in C. Technical report 2002-712-15, Department of Computer Science,University of Calgary.

Lemieux, C., and A. B. Owen. 2001. Quasi-regression and the relative importance of the ANOVA components of a function. In *Monte Carlo and Quasi-Monte Carlo Methods 2000*, ed. K.-T. Fang, F. J. Hickernell and H. Niederreiter, 331–344, Berlin: Springer-Verlag.

Longstaff, F. A., and E. S. Schwartz. 2001. Valuing American options by simulations: A simple least-squares approach. *Review of Financial Studies*, 14: 113–147.

Morani, N. 2003. Pricing American options: A variance reduction technique for the Longstaff-Schwartz algorithm. Research report 2003-256, CERMICS-ENPC, Champs-sur-Marne, France.

Morokoff, W. J., and R. E. Caflisch. 1997. Quasi-Monte Carlo simulation of random walks in finance. In *Monte Carlo and Quasi-Monte Carlo Methods 1996*, ed. H. Niederreiter, P. Hellekalek, G. Larcher and P. Zinterhof, Volume 127 of Lecture Notes in Statistics, 340–352. Springer-Verlag.

Niederreiter, H. 1992. *Random number generation and quasi-Monte Carlo methods*, Volume 63 of *SIAM CBMS-NSF Regional Conference Series in Applied Mathematics*. Philadelphia: SIAM.

Owen, A. B. 1998. Latin supercube sampling for very high-dimensional simulations. *ACM Transactions on Modeling and Computer Simulation*, 8: 71–102.

Owen, A. B. 2004. Multidimensional variation for quasi-Monte Carlo. Manuscript.

Papageorgiou, A. 2002. The Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo Integration. *Journal of Complexity*, 18: 171–186.

Paskov, S. 1997. New methodologies for valuing derivatives. In *Mathematics of Derivative Securities*, ed. S. Pliska and M. Dempster, 545–583. Cambridge University Press, UK.

Paskov, S., and J. Traub. 1995. Faster valuation of financial derivatives. *Journal of Portfolio Management* 22: 113–120.

Sloan, I. H., and F. Y. Kuo, and S. Joe. 2002. Constructing randomly shifted lattice rules in weighted Sobolev spaces. *SIAM Journal on Numerical Analysis*, 40: 1650–1665.

Sobol', I. M. 1993. Sensitivity estimates for nonlinear mathematical models. *Mathematical Modeling and Computational Experiment*, 1: 407–414.

Sobol', I. M. 2001. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. *Mathematics and Computers in Simulation*, 62: 255–263.

Tezuka, S. 2001. Quasi-Monte Carlo – discrepancy between theory and practice. In *Monte Carlo and Quasi-Monte Carlo Methods 2000*, ed. K.-T. Fang, F. J. Hickernell and H. Niederreiter, 124–140, Berlin: Springer-Verlag.

Wang, W., and K.-T. Fang. 2003. The effective dimension and quasi-Monte Carlo integration. *Journal of Complexity*, 19: 101–124.

Wang, W., and I.H. Sloan. 2003a. Efficient weighted lattice rules with applications to finance. Applied Mathematics Report AMR03/03, University of New South Wales, Australia.

Wang, W., and I.H. Sloan. 2003b. Why are high-dimensional finance problems of low effective dimension? Applied Mathematics Report AMR03/20, University of New South Wales, Australia.

AUTHOR BIOGRAPHY

CHRISTIANE LEMIEUX is an assistant professor in the Departments of Mathematics & Statistics and Computer Science at the University of Calgary. She completed her Ph.D. in computer science at the University of Montreal in 2000. Her research interests are quasi-Monte Carlo methods, simulation, integration, and financial mathematics. Her e-mail address is <`lemieux@math.ucalgary.ca`>. and her web page is <`www.math.ucalgary.ca/~lemieux`>.