SIMULATION OUTPUT ANALYSIS BASED ON EXCURSIONS

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ABSTRACT

We consider the steady state output analysis problem for a process that satisfies a functional central limit theorem. We construct asymptotically valid confidence intervals for the steady-state mean using normalized excursions of the simulated process. An alternative approach splits the output at the last zero crossing of the normalized path. The methods use a bounded memory of size determined by the user.

1 INTRODUCTION

A basic simulation problem is to construct asymptotically valid confidence intervals for the steady-state mean of a simulated stochastic process. One of the most commonly used and widely applicable methods is the method of batch means. This method can be applied when the simulated process satisfies a functional central limit theorem (described below). The idea is to break up the suitably normalized and centered simulated path into a fixed number of segments (batches), and to compute a statistic based on the increments over the segments; see Goldsman and Schmeiser (1997). Variations include the method of overlapping batch means (Meketon and Schmeiser 1984).

In this paper we consider methods that break the simulated path into random length segments. In some settings there can be computational advantages, as well as advantages in terms of statistical efficiency. The methods are standardized time series methods, but we center the process by running two (or more) independent simulations in parallel and taking their difference. For background on standardized time series methods, see Schruben (1983) or Glynn and Iglehart (1990).

The methods we describe are partly motivated by the problem of simulation-based optimization. Some optimization algorithms use a large collection of estimates based on different parameter values. It must be possible to halt and resume simulations at different parameter values as those parameters become more or less important for identifying the optimum. Thus it is important for the simulations to use

bounded memory (since there may be many of them going on in parallel), and it is not known in advance how long the simulation at a particular parameter value will ultimately run.

In the next section we describe the basic setup and the use of ranked excursions for constructing asymptotically valid confidence intervals. In Section 3 we describe the straight-forward batching method. An alternative approach, based on blocking the output at the last zero crossing, is presented in Section 4. The results of some numerical experiments are presented in Section 5.

2 RANKED EXCURSIONS

Let $Y = [Y(t) : t \geq 0]$ be a process representing the output of a simulation. If the output is a discrete sequence $[Y_0, Y_1, \ldots]$ then set $Y(t) = Y_{\lfloor t \rfloor}$. For each $n \geq 1$, define the scaled process

$$Y_n(t) = \frac{1}{n} \int_{s=0}^{nt} Y(s) \, ds \quad 0 \leq t \leq 1.$$

Let Ω_0 be the space of continuous real-valued functions on the unit interval that vanish at 0. The scaled process Y_n has paths that are in Ω_0 even if the simulation output Y is not continuous.

Suppose that there is a real number μ and a positive number σ such that if we define a sequence of processes X_n, $n \geq 1$, by

$$X_n(t) = \sqrt{n} \left(Y_n(t) - \mu t \right), \quad 0 \leq t \leq 1,$$

then

$$X_n \xrightarrow{d} \sigma B \tag{1}$$

in Ω_0 endowed with the uniform metric as $n \to \infty$, where B is a standard Brownian motion, and $\xrightarrow{d}$ denotes convergence in distribution. In this case we say that the process satisfies a functional central limit theorem.

In discrete-event simulation, we are often interested in estimating the parameters μ and σ. Suppose that one is interested in constructing an asymptotically valid confidence interval for μ using only a bounded amount of memory. In particular, the simulated sample path can not be stored. We now outline such a procedure.

Run two independent simulations, say Y_1 and Y_2. Set

$$A_i(t) = \int_{s=0}^{t} Y_i(s)\, ds \qquad (2)$$

for $i = 1, 2$, and define the centered process $A(t) = A_1(t) - A_2(t)$. Set

$$X_n(t) = \frac{1}{\sqrt{n}} A(nt), \quad 0 \le t \le 1.$$

Let $[\alpha_i, \beta_i]$, $i \ge 1$, denote the intervals of excursion above the minimum for the process A. That is,

$$A(\alpha_i) = A(\beta_i) = \min_{s \le \alpha_i} A(s),$$

and

$$A(s) > A(\alpha_i)$$

for $\alpha_i < s < \beta_i$. Set

$$M_i = \sup\{A(s) - A(\alpha_i) : \alpha_i \le s \le \beta_i\}$$

and

$$l_i = \beta_i - \alpha_i.$$

We call the process A over the interval $[\alpha_i, \beta_i]$ the ith excursion above the minimum. For the ith excursion, consider the pair $v_i = (l_i, M_i(l_i)^{-1/2})$ (we refer to the second component as the normalized excursion height). Fix a number m; the amount of storage will be proportional to m. Let k_n denote the number of excursions above the minimum that have occurred by simulated time n. As the simulation progresses, store the m elements of $\{v_i : 1 \le i \le k_n\}$ with the largest lengths l_i; a priority queue keyed on l_i could be used for this purpose. Let $H_{i,n}$ be the normalized height of the ith longest excursion above the minimum after simulated time n ($= 0$ if $i > k_n$). For the following theorem we fix $m \ge 1$ and consider the m longest excursions.

Theorem 1   *As $n \to \infty$, $H_{i,n}/\sigma$ converges in distribution to the height of a standard Brownian excursion. Furthermore,*

$$\begin{aligned}\left(H_{1,n}, H_{2,n}, \ldots, H_{m,n}, X_n(1)\right) \\ \xrightarrow{d} \sigma \cdot (H_1, H_2, \ldots, H_m, \sqrt{2}N),\end{aligned}$$

where $N \sim N(0, 1)$ and the H_i, the maximum of normalized excursions, are independent.

Theorem 1 is proved in Calvin (2004).

Let B be a standard Brownian motion, and suppose that

$$B(\alpha) = B(\beta) = 0,$$

and

$$B(s) > 0 \quad \forall s \in (\alpha, \beta).$$

Then the process

$$e(t) = \frac{B\left(\alpha + t(\beta - \alpha)\right)}{\sqrt{\beta - \alpha}}, \quad 0 \le t \le 1,$$

is a *standard (positive) Brownian excursion*; see Revuz and Yor (1994). Let M denote the maximum of a standard Brownian excursion. Then

$$P(M \le x) = \sqrt{2}\pi^{5/2}x^{-3} \sum_{n=1}^{\infty} n^2 \exp\left(-\frac{1}{2}\pi^2 \frac{n^2}{x^2}\right).$$

The mean of the distribution is $\sqrt{\pi/2}$.

As a point estimator of μ we take

$$\mu_n = \frac{1}{2n}\left(A_1(n) + A_2(n)\right).$$

Then

$$\sqrt{n}\,(\mu_n - \mu) \xrightarrow{d} \sqrt{2}\,\sigma N(0, 1).$$

Let

$$V_{n,m} = \frac{1}{m} \sum_{i=1}^{m} H_{i,m},$$

and let V_m denote the average of m independent standard excursion heights. Then

$$\frac{\sqrt{n/2}\,(\mu_n - \mu)}{V_{n,m}} \xrightarrow{d} \frac{N(0, 1)}{V_m}. \qquad (3)$$

Let G_m denote the cumulative distribution function of the limit random variable in (3). For $m = 1$ we have

Theorem 2

$$G_1(x) = 1 - \pi x^2 \sum_{n=1}^{\infty} n K_1(\pi n x),$$

where K_1 is the modified Bessel function. The 95th quantile is approximately 1.39739.

It appears difficult to obtain a useful expression for G_m, $m > 1$; simulation can be used to approximate quantiles.

Thus if we keep only the longest excursion above the minimum, then a 90% confidence interval for μ can be constructed as

$$[\mu_n - 1.39 \cdot H_{1,n}/(2\sqrt{n}), \ \mu_n + 1.39 \cdot H_{1,n}/(2\sqrt{n})].$$

3 BATCHING

A natural alternative to the method described in the previous section is to use the increments over a fixed grid. Suppose that we store $\{X_n(i/m) : i = 0, 1, 2, \ldots, m\}$ for some fixed m. Then

$$\frac{\sqrt{n}\,(\mu_n - \mu)}{\left(\frac{1}{m}\sum_{i=1}^{m}(X_n(i/m) - X_n((i-1)/m))^2\right)^{1/2}} \xrightarrow{d} t_m,$$

where t_m denotes a t distribution with m degrees of freedom. In our setup the limit t distribution has one more degree of freedom than for the usual batch means since we have a Brownian motion limit instead of a Brownian bridge.

As n increases, one could "forget" some of the values to maintain a memory of size m. Thus we can construct a method for which the run length would not need to be known in advance, though it would be cumbersome.

4 BLOCKING ON LAST ZERO CROSSING

The method described in the previous section uses evaluations of the normalized path concentrated in a relatively small subset of the unit interval (on average less than half, since the portion after the global minimum is not used). In this section we consider an alternative approach that uses evaluations that are more spread out, though the number of "batches" m can not be increased in the version we describe here.

Let T_n be the location of the last zero crossing of X_n, so that $X_n(T_n) = 0$ and $|X_n(s)| > 0$ for $T_n < s \leq 1$. Let

$$R_n = \frac{\max_{0 \leq s \leq T_n} X_n(s) - \min_{0 \leq s \leq T_n} X_n(s)}{\sqrt{T_n}},$$

and

$$U_n = \frac{|X_n(1)|}{\sqrt{1 - T_n}}.$$

Let R denote the range of a standard Brownian bridge. Then R has the same distribution as the random variable H_1 of Theorem 1 (Vervaat 1979). Let M_1 denote the value at 1 of a standard Brownian meander (roughly speaking, a Brownian motion conditioned to take positive values over $(0, 1]$). Then (Revuz and Yor 1994, p. 468) M_1 has the same distribution as $\sqrt{2\gamma}$, where γ has an exponential distribution with mean 1.

Theorem 3 *As $n \to \infty$,*

$$(R_n, U_n) \xrightarrow{d} \sigma \cdot (H_1, M_1),$$

where H_1 and M_1 are independent, H_1 is the maximum of a standard Brownian excursion, and M_1 is the value at time 1 of a standard Brownian meander.

For scaling function we take $\left(R_n^2 + U_n^2/2\right)^{1/2}$, and so the limit random variable is

$$\frac{N}{\left(H_1^2 + \gamma\right)^{1/2}},$$

where $N \sim N(0, 1)$ and all three random variables are independent. Then for $z > 0$,

$$P\left(\frac{N}{\left(R^2 + \gamma\right)^{1/2}} \leq z\right) =$$

$$= \frac{1}{\sqrt{1 + 2/z^2}} +$$

$$2\sum_{k=1}^{\infty}\left(\frac{2k^2 + 1}{(2k^2 - 1)^2}(1 + 2/z^2)^{-1/2}\right.$$

$$-\frac{4k^2}{z^2(2k^2 - 1)}(1 + 4k^2/z^2)^{-3/2}$$

$$\left.-\frac{2k^2 + 1}{(2k^2 - 1)^2}(1 + 4k^2/z^2)^{-1/2}\right).$$

For a derivation, see Calvin (2004).

5 EXPERIMENTS

This section describes some numerical experiments that compare the characteristics of the excursion and the batching estimators. The simulated model is the Ehrenfest urn model with state space $\{0, 1, \ldots, 8\}$. This is the discrete-time Markov chain with transition probabilities given by: $P_{0,1} = P_{8,7} = 1$, and

$$P_{i,i+1} = \frac{8 - i}{8} = 1 - P_{i,i-1}, \quad 0 < i < 8.$$

We ran $1,000$ independent replications, each replication having a run length of $1,000,000$ transitions. In Table 1 we plot the coverage of nominal 90% confidence intervals, and the average and standard deviation of the confidence interval half-widths.

In Table 2 we plot similar results for the excursion-based estimator. It appears that coverage degrades, while the size and variability of the half widths decrease.

Table 1: Coverage, Average and Standard Deviation of Half-Widths for Confidence Intervals Based on m Batches

m	coverage	avg. hw	std. of hw
1	0.893	0.0137	0.0091
2	0.894	0.0068	0.0024
3	0.900	0.0057	0.0015
4	0.891	0.0052	0.0012
5	0.899	0.0050	0.0011
6	0.899	0.0049	0.0011
7	0.911	0.0048	0.0010
8	0.902	0.0047	0.0010

Table 2: Coverage, Average and Standard Deviation of Half-Widths for Confidence Intervals Based on m Excursions

m	coverage	avg. hw	std. of hw
1	0.887	0.00447	0.001015
3	0.883	0.00444	0.000564

6 CONCLUSIONS

We have described a method to construct asymptotically valid confidence intervals for the time-average of a simulated process using a bounded memory of size selected by the user. The method requires (at least) two independent simulations to be run in parallel to center the output in order to avoid storing the entire simulation data. Compared with a straightforward batching scheme, the method appears to have relatively small confidence intervals of low variability for the same memory size (though the batching scheme has excellent coverage properties).

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