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# MAKE-TO-STOCK SYSTEMS WITH BACKORDERS: IPA GRADIENTS

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## ABSTRACT

We consider single-stage, single-product Make-to-Stock systems with random demand and random service (production) rate, where demand shortages at the inventory facility are backordered. The Make-to-Stock system is modeled as a stochastic fluid model (SFM), in which the traditional discrete arrival, service and departure stochastic processes are replaced by corresponding stochastic fluid-flow rate processes. IPA (Infinitesimal Perturbation Analysis) gradients of various performance metrics are derived with respect to parameters of interest (here, base-stock level and production rate), and are showed to be unbiased and easy to compute. The (random) IPA gradients are obtained via sample path analysis under very mild assumptions, and are inherently nonparametric in the sense that no specific probability law need be postulated. The formulas derived can be used in simulation, as well as in real-life systems.

## 1 INTRODUCTION

The objective of this paper is to derive IPA gradients (derivatives) of the random variables that model performance metrics of interest in Make-to-Stock systems, and to show them to be unbiased. Specifically, let $L(\theta)$ be a random variable, parameterized by a generic real-valued parameter $\theta$ from a closed and bounded set $\Theta$. The IPA gradient of $L(\theta)$ with respect to $\theta$ is the random variable $L^{'}(\theta) = \frac{d}{d\theta}L(\theta)$, provided it exists almost surely. Furthermore, $L(\theta)$ is said to be *unbiased*, if the expectation and differentiation operators commute, namely, $E[\frac{d}{d\theta}L(\theta)] = \frac{d}{d\theta}E[L(\theta)]$; otherwise, it is said to be *biased*.

Sufficient conditions for unbiased IPA derivatives are given in the following lemma.

**Lemma 1** *(see Rubinstein and Shapiro (1993), Lemma A2, p. 70)*

*An IPA derivative $L'(\theta)$ is unbiased, if*

**(a)** *For each $\theta \in \Theta$, the IPA derivatives $L'(\theta)$ exist with probability 1 (w.p.1).*

**(b)** *W.p.1, $L(\theta)$ is Lipschitz continuous in $\Theta$, and the (random) Lipschitz constants, $K(\theta)$, have finite first moments.*

Comprehensive discussions of IPA derivatives and their applications can be found, for example, in Glasserman (1991) and Ho and Cao (1991).

Most papers on stochastic production-inventory systems (and Make-to-Stock systems in particular) postulate specific probability laws that govern the underlying stochastic processes (e.g., Poisson demand arrivals and exponential service times). For simple systems, such as the one-stage Make-to-Stock variety, closed-form formulas of key performance metrics (e.g., statistics of inventory levels and lost sales or backorders), have been derived as functions of control parameters. For example, Zipkin (1986) and Karmarkar (1987) obtain the optimal control of such systems with respect to batch sizes and re-order points by standard optimization techniques. For more complex Make-to-Stock systems, such as the multi-stage serial variety, closed-form formulas are not available. Buzacott, Price, and Shanthikumar (1991) carried out sample path analysis for a 2-stage production system, governed by the continuous-time base-stock policy. Diffusion models and deterministic fluid models have been proposed in order to mitigate the analytical and computational complexity of performance evaluation and optimal control. For example, Wein (1992) used a diffusion process to model a multi-product, single-server Make-to-Stock system, while Veatch (2002) discussed diffusion and fluid-flow models of serial Make-to-Stock systems. Note, however, that diffusion models require a heavy traffic condition to be valid approximations (Wein 1992). In a similar vein, while deterministic fluid-flow models provide valuable insights into the control rules of such systems, deterministic modeling may well result in substantial numerical errors (Veatch 2002).

Simulation has been widely used to study the performance of complex production systems. Under periodic-review policies, system performance is evaluated only at specific review times. For example, Glasserman and Tayur (1995) considered a class of production systems under periodic-review modified base-stock policy, and estimated via simulation its performance metrics and expected IPA derivatives. Fu (1994) derived IPA derivatives for production systems under periodic-review with the $(s, S)$ policy. In contrast, discrete-event simulation can track system performance continuously. For instance, Caramanis and Liberopoulos (1992) developed a numerical technique based-on discrete-event simulation and IPA to design a near optimal flow controller for failure-prone manufacturing systems. All in all, most papers on stochastic production-inventory systems postulate a specific underlying probability law, and focus on off-line control and optimization algorithms.

The stochastic fluid model (SFM) improves on the aforementioned models in that it models fluid-flow queueing systems subject to randomness. Like in traditional queueing systems, an SFM consists of a buffer and a server. However, the operation of the system differs from its traditional counterpart in that the workload moves like (continuous) fluid rather than (discrete) jobs. More specifically, a fluid stream is injected into a buffer according to some stochastic arrival-rate process, and is discharged by a server according to some stochastic service-rate process.

Finally, we point out that, ceteris paribus, SFM systems enjoy important advantages over their discrete counterparts. First, IPA gradients in SFM setting are *unbiased*, while their counterparts in discrete queueing systems are by and large *biased* (Heidelberger et al. 1988). Second, IPA gradients in SFM setting are nonparametric in the sense that no probability law need be postulated, so that they may be computed both in simulation and real-life systems. Consequently, IPA gradients, derived in SFM setting, can provide important information and insights into their *discrete counterparts*, by applying gradient formulas obtained in SFM setting to traditional queueing systems.

Motivated by the considerations above, Wardi et al. (2002) derived IPA gradients for the loss volume and buffer-workload time average, for simple queues in SFM setting; each of these metrics was differentiated with respect to buffer size, a parameter of the arrival rate process and a parameter of the service rate process. The paper showed the IPA gradients to be unbiased, easily computable and non-parametric. Most recently, Paschalidis et al. (2004) treated multi-stage production-inventory systems with continuous-time base-stock policy in SFM setting, and computed IPA gradients of the time averaged inventory level and service level with respect to base-stock levels, and used them to determine optimal base-stock levels at each stage.

In this paper, we derive the IPA gradients of the time averaged inventory level and backorders for the class of single-product, single-stage Make-to-Stock systems. Our proof methodology differs markedly from Paschalidis et al. (2004), and we further derive new IPA formulas for the time averaged inventory level and backorders with respect to a production rate parameter. Our ultimate goal is to use the gradient information for control and optimization of supply chains, which will be the subject of further research.

Throughout the paper, we use the following notational conventions and terminology. The indicator function of set $A$ is denoted by $1_A$, and we denote $x^+ = \max\{x, 0\}$. A function $f(x)$ is said to be *locally differentiable at* $x$ if it is differentiable in a neighborhood of $x$; it is said to be *locally independent of* $x$ if it is constant in a neighborhood of $x$.

The rest of the paper is organized as follows. Section 2 presents the production-inventory model under study. Section 3 derives IPA gradient formulas for the MTS systems with backorders. Finally, Section 4 summarizes the results and discusses future work.

## 2 MAKE-TO-STOCK SYSTEMS WITH BACKORDERS

Consider the traditional single-stage, single-product Make-to-Stock system (MTS system, for short), consisting of a production facility and an inventory facility. The two facilities interact: the latter sends orders to the former, while the former produces stock to replenish the latter. The production facility is comprised of a queue that houses a production server (a single machine, a group of machines or a production line), preceded by an infinite buffer that holds outstanding production orders. We assume that the production facility has an unlimited supply of raw material, so it never starves. The inventory facility satisfies incoming demands on a first come first serve (FCFS) basis, and is controlled by a continuous-time base-stock policy with some base-stock level $S > 0$. More specifically, the inventory and production facilities are coupled: the inventory facility places orders as discrete jobs in the production facility's buffer, while the production facility restocks the inventory facility. The demand process consists of an interarrival-time process of demands and their random magnitudes. Demands arrive at the inventory facility and are satisfied from inventory on hand (if available). Otherwise, the shortage is backordered and the demand waits in a FCFS buffer at the inventory facility until the production facility replenishes the inventory facility with the shortage amount.

We now proceed to map the traditional discrete MTS system into an SFM version, shown in Figure 1. Here, the level-related stochastic processes are fluid volumes, where $I(t)$ is the volume of inventory on-hand at time $t$, $B(t)$ is the volume of backorders at time $t$, and $X(t)$ is the volume of outstanding orders at time $t$. In a similar vein, Traffic-related stochastic processes model random flow rates, where $\alpha(t)$ is the rate of incoming demands at time $t$. $\mu(t)$ is the production rate at time $t$, $\lambda(t)$ is the rate of incoming outstanding orders at time $t$, and $\rho(t)$ is the traffic rate of product replenishment at time $t$. Similarly to Wardi et al. (2002), we define two types of *events* along a sample path. We say that an *exogenous event* occurs whenever a jump occurs in the sample path of $\{\alpha(t)\}$ or $\{\mu(t)\}$, and that an *endogenous event* occurs whenever a time interval is inaugurated, in which $X(t) = 0$ or $X(t) = S$.

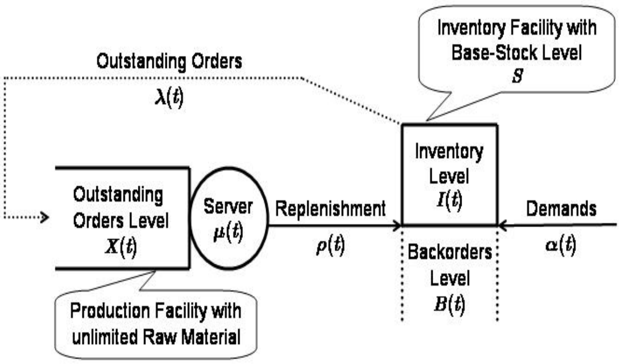


Figure 1: The Queue-Inventory System with Backorders

We impose the following mild regularity conditions (cf. Wardi et al. (2002)).

**Assumption 1**

**(a)** *The processes $\{\alpha(t)\}$ and $\{\mu(t)\}$ have right-continuous sample paths that are piecewise continuously-differentiable, w.p.1.*

**(b)** *Each of the processes, $\{\alpha(t)\}$ and $\{\mu(t)\}$, has a finite number of discontinuities in any finite time interval, w.p.1.*

**(c)** *No multiple events occur simultaneously, w.p.1.*

Let $[0, T]$ be a finite time interval. In this paper, we will be interested in the following performance metrics: the time average of the fluid volume of inventory on-hand over the interval $[0, T]$, given by

$$L_I = \frac{1}{T}\int_0^T I(t)\,dt, \tag{1}$$

and the time average of the fluid volume of backorders over the interval $[0, T]$, given by

$$L_B = \frac{1}{T}\int_0^T B(t)\,dt. \tag{2}$$

The parameters of interest are the base-stock level of the inventory facility, $S$, and a parameter of the production rate process $\{\mu(t)\}$ (to be defined later). Observe that the metrics $L_I$ and $L_B$ are random variables for each $T$. However their dependence on the sample path and on $T$ is suppressed to simplify the notation.

Let $\theta \in \Theta$ denote a generic parameter of interest with a closed and bounded domain, $\Theta$. We write $S(\theta)$, $\mu(\theta, t)$, $L_I(\theta)$, $L_B(\theta)$ and so on, to explicitly display the dependence of a performance random variable on its parameter of interest. Our objective is to derive closed-form formulas for the IPA gradients $L'_I(\theta) = \frac{d}{d\theta}L_I(\theta)$ and $L'_B(\theta) = \frac{d}{d\theta}L_B(\theta)$ in SFM setting, using sample path analysis, and to show them to be unbiased.

The interval $[0, T]$ can be partitioned into two types of alternating periods:

1. *Surplus periods* are periods during which $B(\theta, t) = 0$.
2. *Shortage periods* are periods during which $B(\theta, t) > 0$.

For each $\theta \in \Theta$, let $\mathcal{I}_j(\theta)$, $j = 1, \ldots, J(\theta)$, be the successive surplus periods in $[0, T]$, and let $\mathcal{B}_k(\theta)$, $k = 1, \ldots, K(\theta)$, be the successive shortage periods in $[0, T]$, so that

$$[0, T] = \left(\bigcup_{j=1}^{J(\theta)} \mathcal{I}_j(\theta)\right) \bigcup \left(\bigcup_{k=1}^{K(\theta)} \mathcal{B}_k(\theta)\right), \tag{3}$$

where $|J(\theta) - K(\theta)| \le 1$. We can now rewrite Eqs. (1) - (2) as

$$L_I(\theta) = \frac{1}{T}\int_0^T I(\theta, t)\,dt = \frac{1}{T}\sum_{j=1}^{J(\theta)} \int_{\mathcal{I}_j(\theta)} I(\theta, t)\,dt, \tag{4}$$

$$L_B(\theta) = \frac{1}{T}\int_0^T B(\theta, t)\,dt = \frac{1}{T}\sum_{k=1}^{K(\theta)} \int_{\mathcal{B}_k(\theta)} B(\theta, t)\,dt. \tag{5}$$

We next proceed to derive IPA gradients for the MTS system, assuming throughout the following initial conditions: $I(0) = S$ (full inventory), B(0) = 0 (no backorders), and $X(0) = 0$ (empty outstanding order buffer).

## 3 IPA DERIVATIVES

In the SFM version of the MTS model with backorders, $\{\lambda(t)\}$ is given by

$$\lambda(t) = \alpha(t), \quad t \ge 0, \tag{6}$$

and $\{\rho(t)\}$ is given by

$$\rho(t) = \begin{cases} \mu(t), & \text{if X(t)} > 0 \\ \min\{\mu(t), \lambda(t)\}, & \text{if X(t)} = 0 \end{cases} \tag{7}$$

This model further satisfies the conservation relation

$$I(t) - B(t) + X(t) \;=\; S, \tag{8}$$

where the inventory volume process is given by

$$I(t) \;=\; [S - X(t)]^+, \tag{9}$$

the backorder volume process is given by

$$B(t) \;=\; [X(t) - S]^+, \tag{10}$$

and the dynamics of the volume process of outstanding orders, $\{X(t)\}$, in the production facility are governed by the one-side stochastic differential equation,

$$\frac{dX(t)}{dt^+} = \begin{cases} 0, & X(t) = 0 \text{ and } \alpha(t) \le \mu(t), \\ \alpha(t) - \mu(t), & \text{otherwise.} \end{cases} \tag{11}$$

Observe that the stochastic derivative above does not depend on $S$, but only on $\alpha(t)$ and $\mu(t)$.

### 3.1 IPA Gradients with Respect to the Base-Stock Level

In this section we treat the IPA derivatives, $L_I^{'}(\theta)$ and $L_B^{'}(\theta)$, where $\theta$ is the base-stock level, $S(\theta) = \theta$, $\theta \in \Theta$. The following assumptions are made throughout this section.

**Assumption 2**

**(a)** *The processes $\{\alpha(t)\}$ and $\{\mu(t)\}$ are independent of the parameter $\theta$.*

**(b)** *The random derivatives $L'_I(\theta)$ and $L'_B(\theta)$ exist w.p.1. Note that this assumption already follows from Assumption 1 in the special case that $\{\alpha(t)\}$ and $\{\mu(t)\}$ have piecewise-constant sample paths.*

Notice that the time points at which $\{I(\theta, t)\}$ reaches $S$ or 0 are generally functions of $\theta$. However, the time points at which $\{I(\theta, t)\}$ ceases to be full (equivalently, $\{X(t)\}$ ceases to be zero) are locally independent of $\theta$, because they correspond to a jump or a change of sign in $\{\alpha(t) - \mu(t)\}$, and this difference process is independent of $\theta$ by (a) of Assumption 2.

Let $\mathcal{I}_j(\theta) = [G_j(\theta), H_j(\theta))$, $j = 1, \ldots, J(\theta)$, denote the $j$-th surplus period in $[0, T]$, where $G_j(\theta)$ is its start point and $H_j(\theta)$ is its end point (note the dependence on the base-stock level, $\theta$, except for the initial $G_1 = 0$). We use the convention that $H_{J(\theta)}(\theta) = T$ when $I(\theta, T) > 0$ (i.e., when a surplus period is still in progress at time $T$). A generic sample path is depicted in Figure 2.

By Assumption 1 and 2, the sample paths of $G_j(\theta)$ and $H_j(\theta))$ are locally differentiable functions of $\theta$, w.p.1.; furthermore, the number of surplus periods, $J(\theta)$, is locally independent of $\theta$, w.p.1.

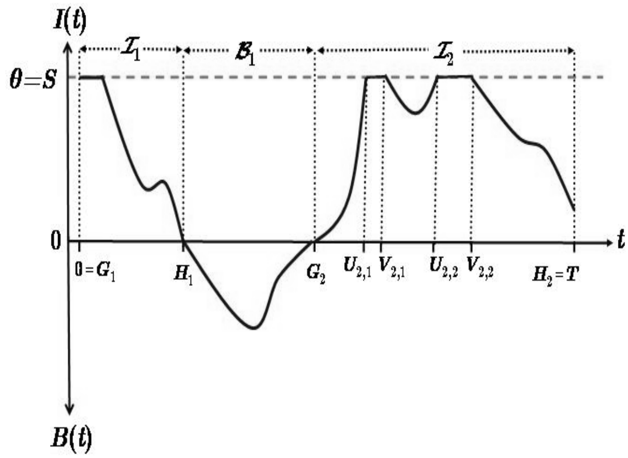


Figure 2: A Generic Sample Path of an MTS System with Backorders

**Proposition 1** *For every $\theta \in \Theta$,*

$$\begin{aligned} L'_I(\theta) &= \frac{1}{T}\sum_{j=1}^{J(\theta)}[H_j(\theta) - G_j(\theta)] \\ &= \frac{1}{T}\int_0^T 1_{\{I(\theta,t)>0\}}\,dt. \end{aligned} \tag{12}$$

**Proof.** From Eq. (4),

$$L_I(\theta) \;=\; \frac{1}{T}\sum_{j=1}^{J(\theta)}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta, t)\,dt.$$

Since $J(\theta)$ is locally independent of $\theta$, differentiating the equation above with respect to $\theta$ yields

$$L'_I(\theta) \;=\; \frac{1}{T}\sum_{j=1}^{J(\theta)}\frac{d}{d\theta}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta, t)\,dt. \tag{13}$$

Next, for each $j = 1, \ldots, J(\theta)$,

$$\begin{aligned} \frac{d}{d\theta}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta, t)\,dt \;&=\; -\,I(\theta, G_j(\theta))\frac{dG_j(\theta)}{d\theta} \\ + I(\theta, H_j(\theta))\frac{dH_j(\theta)}{d\theta} &+ \int_{G_j(\theta)}^{H_j(\theta)}\frac{dI(\theta, t)}{d\theta}\,dt. \end{aligned} \tag{14}$$

For $j = 1$, $dG_1(\theta)/d\theta = 0$, since $G_1 = 0$ is independent of $\theta$, while for $j = 2, \ldots, J(\theta)$, $I(\theta, G_j(\theta)) = 0$ by definition. Similarly, $I(\theta, H_j(\theta)) = 0$ for $j < J(\theta)$ and for $j = J(\theta)$ with $H_{J(\theta)}(\theta) < T$ by definition. For $j = J(\theta)$ with $H_{J(\theta)}(\theta) = T$ one has $dH_{J(\theta)}(\theta)/d\theta = 0$, since in this case, $H_{J(\theta)}(\theta)$ is locally independent of $\theta$. Hence, the first two terms on the right-hand side of Eq. (14) vanish for all $j = 1, \dots, J(\theta)$. Furthermore, $I(\theta, t) = S(\theta) - X(t)$ during the surplus periods by Eq. (9), and $X(t)$ is independent of $\theta = S(\theta)$ by Eq. (11) and the initial condition $X(0) = 0$, whence $\frac{dI(\theta,t)}{d\theta} = 1$ over $(G_j(\theta), H_j(\theta))$. It follows that

$$\int_{G_j(\theta)}^{H_j(\theta)} \frac{dI(\theta,t)}{d\theta}\, dt \;=\; [H_j(\theta) - G_j(\theta)]. \tag{15}$$

Eq. (12) now follows from Eq. (15) in view of Eqs. (13) - (15).

We point out that Proposition 1 agrees with the result obtained in Paschalidis et al. (2004) by other methods.

Next, observe that $\mathcal{B}_k(\theta) = [H_k(\theta), G_{k+1}(\theta))$, $k = 1, \dots, K(\theta)$, denotes the $k$-th shortage period in $[0, T]$, where $H_k(\theta)$ is its start point and $G_{k+1}(\theta)$ is its end point (note the dependence on the base-stock level $\theta$). If $I(\theta, T) > 0$, then $K(\theta) = J(\theta) - 1$, while if $B(\theta, T) > 0$, then $K(\theta) = J(\theta)$. We use the convention that $G_{K(\theta)+1}(\theta) = T$ when $B(\theta, T) > 0$ (i.e., when a shortage period is still in progress at time $T$). By Assumption 1 and 2, the realizations of $H_k(\theta)$ and $G_{k+1}(\theta)$ are locally differentiable functions of $\theta$, and the number of shortage periods, $K(\theta)$, is locally independent of $\theta$, w.p.1.

**Proposition 2** *For every $\theta \in \Theta$,*

$$\begin{aligned} L'_B(\theta) \;&=\; -\frac{1}{T}\sum_{k=1}^{K(\theta)} [G_{k+1}(\theta) - H_k(\theta)] \\ &=\; -\frac{1}{T}\int_0^T 1_{\{B(\theta,t)>0\}}\, dt. \end{aligned} \tag{16}$$

**Proof.** From Eq. (8),

$$B(\theta, t) = I(\theta, t) - S(\theta) + X(t), \quad t \in [0, T].$$

Substituting the formula above into Eq. (5), one has

$$\begin{aligned} L_B(\theta) \;&=\; \frac{1}{T}\int_0^T [I(\theta,t) - S(\theta) + X(t)]\, dt \\ &=\; L_I(\theta) - S(\theta) + \frac{1}{T}\int_0^T X(t)\, dt. \end{aligned} \tag{17}$$

Since $S(\theta) = \theta$ and $\{X(t)\}$ does not depend on $\theta$, differentiating Eq. (17) with respect to $\theta$ yields,

$$L'_B(\theta) \;=\; L'_I(\theta) - 1.$$

Eq. (16) now follows by substituting Eq. (12) into the above, since by Eq. (3), $[0, T]$ can be partitioned into surplus and shortage periods.

To show that the IPA gradients are unbiased, consider any $\theta, \theta + \Delta\theta \in \Theta$. Since $\{X(t)\}$ is independent of $\theta$ and $S(\theta) = \theta$, we have from Eq.(9),

$$\begin{aligned} &|I(\theta + \Delta\theta, t) - I(\theta, t)| \\ &= |[S(\theta + \Delta\theta) - X(t)]^+ - [S(\theta) - X(t)]^+| \\ &\le |\Delta\theta|, \end{aligned} \tag{18}$$

and from Eq. (10),

$$\begin{aligned} &|B(\theta + \Delta\theta, t) - B(\theta, t)| \\ &= |[X(t) - S(\theta + \Delta\theta)]^+ - [X(t) - S(\theta)]^+| \\ &\le |\Delta\theta|. \end{aligned} \tag{19}$$

**Proposition 3** *Under Assumption 1 and 2, the IPA derivatives $L'_I(\theta)$ and $L'_B(\theta)$ are unbiased.*

**Proof.** To show that $L'_I(\theta)$ and $L'_B(\theta)$ are unbiased, we use Lemma 1. First, Condition (a) of Lemma 1 is satisfied by part (b) of Assumption 2 for both $L'_I(\theta)$ and $L'_B(\theta)$. Next, by Eq. (18),

$$\begin{aligned} &|L_I(\theta + \Delta\theta) - L_I(\theta)| \\ &= \frac{1}{T}\left|\int_0^T [I(\theta + \Delta\theta, t) - I(\theta, t)]\, dt\right| \\ &\le \frac{1}{T}\int_0^T |I(\theta + \Delta\theta, t) - I(\theta, t)|\, dt \;\le\; |\Delta\theta|, \end{aligned} \tag{20}$$

and by Eq. (19),

$$\begin{aligned} &|L_B(\theta + \Delta\theta) - L_B(\theta)| \\ &= \frac{1}{T}\left|\int_0^T [B(\theta + \Delta\theta, t) - B(\theta, t)]\, dt\right| \\ &\le \frac{1}{T}\int_0^T |B(\theta + \Delta\theta, t) - B(\theta, t)|\, dt \;\le\; |\Delta\theta|. \end{aligned} \tag{21}$$

Eqs. (20) and (21) establish that Condition (b) of Lemma 1 holds for both $L'_I(\theta)$ and $L'_B(\theta)$. The proof of the proposition is complete.

### 3.2 IPA Gradients with Respect to the Production Rate

In this section we treat the IPA derivatives, $L'_I(\theta)$ and $L'_B(\theta)$, where $\theta$ is a parameter of the production rate, $\mu(\theta, t)$, such that for all $\theta \in \Theta$ and all $t \in [0, T]$,

$$\frac{d\mu(\theta,t)}{d\theta} \;=\; \mu'(\theta, t) = 1. \tag{22}$$

This functional form corresponds to a linear relationship between $\theta$ and the production rate. More picturesquely, $\theta$ can be viewed as a “knob” whose “turning" scales the production rate.

The following assumptions are made throughout this section.

**Assumption 3**

**(a)** *The process $\{\alpha(t)\}$ and the base-stock level, S, are independent of the parameter $\theta$.*
**(b)** *The random derivatives $L'_I(\theta)$ and $L'_\varsigma(\theta)$ exist, w.p.1.*

The surplus periods, $\mathcal{I}_j(\theta) = [G_j(\theta), H_j(\theta))$, generally depend on $\theta$ for all $j = 1, \ldots, J(\theta)$, except for the initial $G_1 = 0$. Let $[U_{j,m}(\theta), V_{j,m}(\theta))$, $m = 1, \ldots, M_j(\theta)$, be the subintervals in $\mathcal{I}_j(\theta)$ during which $I(t) = S$, with the notational conventions $V_{1,0}(\theta) = G_1 = U_{1,1}$, $V_{j,0}(\theta) = G_j(\theta)$ for $j > 1$, and $U_{j,M_j(\theta)+1}(\theta) = H_j(\theta)$, for all $j = 1, \ldots, J(\theta)$. We let $H_{J(\theta)}(\theta) = T$ if $I(\theta, T) > 0$. Assumption 1 and 3 imply that all $U_{j,m}(\theta)$ and $V_{j,m}(\theta)$ are locally differentiable functions with respect to $\theta$. For all $t \in [V_{j,m}(\theta), U_{j,m+1}(\theta))$ in surplus period, $\mathcal{I}_j(\theta)$, let $F_{j,m}(\theta)$ be the (common) most recent time point at which the on-hand inventory was $S$. Note that the $F_{j,m}(\theta)$ are well-defined by the assumption $I(0) = S$ (though they may fall in the current or preceding surplus periods). Assumption 1 and 3 imply that each $F_{j,m}(\theta)$ is a locally continuously-differentiable function of $\theta$.

From the definitions above it follows that for all $1 \le j \le J(\theta)$ and $1 \le m \le M_j(\theta)$,

$$\int_{U_{j,m}(\theta)}^{V_{j,m}(\theta)} \frac{dI(\theta,t)}{d\theta}\, dt \;=\; 0, \tag{23}$$

while Eq. (9) implies for all $1 \le j \le J(\theta)$ and $1 \le m \le M_j(\theta)$,

$$I(\theta,t) \;=\; S - X(\theta,t) > 0, \quad t \in (V_{j,m}(\theta), U_{j,m+1}(\theta)). \tag{24}$$

Furthermore, Eq. (11) implies for all $1 \le j \le J(\theta)$ and $1 \le m \le M_j(\theta)$,

$$\frac{dX(\theta,t)}{dt^+} \;=\; \alpha(t) - \mu(\theta,t), \quad t \in (F_{j,m}(\theta), U_{j,m+1}(\theta)), \tag{25}$$

whence,

$$\begin{aligned} X(\theta,t) \;&=\; X(\theta, F_{j,m}(\theta)) + \int_{F_{j,m}(\theta)}^{t} [\alpha(\tau) - \mu(\theta,\tau)]\, d\tau \\ &= \int_{F_{j,m}(\theta)}^{t} [\alpha(\tau) - \mu(\theta,\tau)]\, d\tau, \quad t \in (F_{j,m}(\theta), U_{j,m+1}(\theta)), \end{aligned} \tag{26}$$

since $X(\theta, F_{j,m}(\theta)) = 0$.

**Proposition 4** *For every $\theta \in \Theta$,*

$$\begin{aligned} L'_I(\theta) &= \frac{1}{2T}\sum_{m=1}^{M_1(\theta)} [U_{1,m+1}(\theta) - V_{1,m}(\theta)]^2 \\ &\quad + \frac{1}{2T}\sum_{j=2}^{J(\theta)} \{ [U_{j,1}(\theta) - F_{j,1}(\theta)]^2 - [G_j(\theta) - F_{j,1}(\theta)]^2 \\ &\quad + \sum_{m=1}^{M_j(\theta)} [U_{j,m+1}(\theta) - V_{j,m}(\theta)]^2 \}. \end{aligned} \tag{27}$$

**Proof.** From Eqs. (4), we can write

$$L'_I(\theta) = \frac{1}{T}\sum_{j=1}^{J(\theta)} \frac{d}{d\theta}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta,t)\, dt, \tag{28}$$

since $J(\theta)$ is locally independent of $\theta$. Differentiating each term in Eq. (28) with respect to $\theta$ yields,

$$\begin{aligned} \frac{d}{d\theta}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta,t)\, dt \;&=\; -I(\theta, G_j(\theta))\, \frac{dG_j(\theta)}{d\theta} \\ &\quad + I(\theta, H_j(\theta))\, \frac{H_j(\theta)}{d\theta} + \int_{G_j(\theta)}^{H_j(\theta)} \frac{dI(\theta,t)}{d\theta}\, dt \\ &= \int_{G_j(\theta)}^{H_j(\theta)} \frac{dI(\theta,t)}{d\theta}\, dt, \quad j = 1, \ldots, J(\theta), \end{aligned} \tag{29}$$

and the proof already appears in Proposition 1. Furthermore, from Eq.(23), we can rewrite Eq. (29) for each $j = 1, \ldots, J(\theta)$ as

$$\frac{d}{d\theta}\int_{G_j(\theta)}^{H_j(\theta)} I(\theta,t)\, dt \;=\; \sum_{m=0}^{M_j(\theta)} \int_{V_{j,m}(\theta)}^{U_{j,m+1}(\theta)} \frac{dI(\theta,t)}{d\theta}\, dt, \tag{30}$$

since the $M_j(\theta)$ is locally independent of $\theta$.

Next, differentiate Eq. (24), and substitute Eq. (25) into the result. In view of Eq. (26) we can now deduce for every $t \in (V_{j,m}(\theta), U_{j,m+1}(\theta))$ the representation,

$$\begin{aligned} \frac{dI(\theta,t)}{d\theta} \;&=\; -\frac{dX(\theta,t)}{d\theta} \\ &= -\frac{d}{d\theta}\int_{F_{j,m}(\theta)}^{t} [\alpha(\tau) - \mu(\theta,\tau)]\, d\tau \\ &= [\alpha(F^+_{j,m}(\theta)) - \mu(\theta, F^+_{j,m}(\theta))]\, \frac{dF_{j,m}(\theta)}{d\theta} \\ &\quad + \int_{F_{j,m}(\theta)}^{t} d\tau \;=\; t - F_{j,m}(\theta). \end{aligned} \tag{31}$$

To see that, note that only two cases are possible for $\alpha(t) - \mu(\theta, t)$: (i) either it jumps at $t = F_{j,m}(\theta)$ (implying $dF_{j,m}(\theta)/d\theta = 0$, since the jumps of both $\{\alpha(t)\}$ and $\{\mu(t)\}$ are locally independent of $\theta$), or (ii) it crosses zero continuously at $t = F_{j,m}(\theta)$ (implying $\alpha(F^+_{j,m}(\theta)) - \mu(\theta, F^+_{j,m}(\theta)) = \alpha(F_{j,m}(\theta)) - \mu(\theta, F_{j,m}(\theta)) = 0$). Either way, the first term on the third right-hand equality of (31) vanishes, and the rest follows from Eq. (22).

Next, substituting Eq. (31) into Eq. (30), we can now write for $j = 1, \ldots, J(\theta)$,

$$\int_{G_j(\theta)}^{H_j(\theta)} \frac{dI(\theta,t)}{d\theta}\,dt \;=\; \sum_{m=0}^{M_j(\theta)} \int_{V_{j,m}(\theta)}^{U_{j,m+1}(\theta)} [t - F_{j,m}(\theta)]\,dt$$
$$= \sum_{m=0}^{M_j(\theta)} \{\frac{1}{2}[U^2_{j,m+1}(\theta) - V^2_{j,m}(\theta)] - F_{j,m}(\theta)[U_{j,m+1}(\theta) - V_{j,m}(\theta)]\}. \tag{32}$$

Eq. (27) now follows by substituting Eq. (32) into Eq. (29), and then substituting Eq. (29) into Eq. (28), and noting the identities $V_{1,0} = U_{1,1} = G_1 = 0$ (independent of $\theta$), and $F_{j,m}(\theta) = V_{j,m}(\theta)$ for all $j = 1, \ldots, J(\theta)$ and $m = 2, \ldots, M_j(\theta)$.

We next derive $L'_B(\theta)$. For $\theta \in \Theta$, the shortage periods, $\mathcal{B}_k(\theta) = [H_k(\theta), G_{k+1}(\theta))$, generally depend on $\theta$ for all $k = 1, \ldots, K(\theta)$. By Assumption 1 and 3, its end points are locally differentiable functions with respect to $\theta$. For each shortage period, $\mathcal{B}_k(\theta)$, let $\tilde{F}_k(\theta)$ be the most recent time point at which the on-hand inventory was $S$. The existence of the $\tilde{F}_k(\theta)$ follows from the initial condition $I(0) = S$. In view of Assumption 1 and 3, $\tilde{F}_k(\theta)$ is a locally continuously-differentiable function of $\theta$. Observe that $\tilde{F}_k(\theta)$ may or may not belong to the immediately preceding surplus period.

Eq.(10) implies for all $1 \le k \le K(\theta)$,

$$B(\theta, t) = X(\theta, t) - S > 0, \quad t \in (H_k(\theta), G_{k+1}(\theta)). \tag{33}$$

Furthermore, Eq. (11) implies for all $1 \le k \le K(\theta)$,

$$\frac{dX(\theta,t)}{dt^+} = \alpha(t) - \mu(\theta, t), \quad t \in (\tilde{F}_k(\theta), G_{k+1}(\theta)), \tag{34}$$

whence, since $X(\theta, \tilde{F}_k(\theta)) = 0$,

$$X(\theta, t) \;=\; X(\theta, \tilde{F}_k(\theta)) + \int_{\tilde{F}_k(\theta)}^{t} [\alpha(\tau) - \mu(\theta, \tau)]\,d\tau$$
$$= \int_{\tilde{F}_k(\theta)}^{t} [\alpha(\tau) - \mu(\theta, \tau)]\;d\tau, \quad t \in (\tilde{F}_k(\theta), G_{k+1}(\theta)). \tag{35}$$

**Proposition 5** *For every $\theta \in \Theta$,*

$$L'_B(\theta) \;=\; -\frac{1}{2T} \sum_{k=1}^{K(\theta)} \{[G_{k+1}(\theta) - \tilde{F}_k(\theta)]^2 - [H_k(\theta) - \tilde{F}_k(\theta)]^2\}. \tag{36}$$

**Proof.** From Eq. (5) we obtain by differentiation with respect to $\theta$,

$$L'_B(\theta) \;=\; \frac{1}{T} \sum_{k=1}^{K(\theta)} \frac{d}{d\theta} \int_{H_k(\theta)}^{G_{k+1}(\theta)} B(\theta, t)\,dt, \tag{37}$$

since $K(\theta)$ is locally independent of $\theta$ as a consequence of Assumption 1 and 3. Differentiating each term in Eq. (37) with respect to $\theta$ yields,

$$\frac{d}{d\theta} \int_{H_k(\theta)}^{G_{k+1}(\theta)} B(\theta, t)\,dt \;=\; -B(\theta, H_k(\theta))\,\frac{dH_k(\theta)}{d\theta} + B(\theta, G_{k+1}(\theta))\,\frac{dG_{k+1}(\theta)}{d\theta} + \int_{H_k(\theta)}^{G_{k+1}(\theta)} \frac{dB(\theta,t)}{d\theta}\,dt$$
$$= \int_{H_k(\theta)}^{G_{k+1}(\theta)} \frac{dB(\theta,t)}{d\theta}\,dt, \quad k = 1, \ldots, K(\theta). \tag{38}$$

To see that, note that if $B(\theta, T) = 0$, then by assumption, $B(\theta, H_k(\theta)) = B(\theta, G_{k+1}(\theta)) = 0$ for all $k = 1, \ldots, K(\theta)$, while if $B(\theta, T) > 0$, then $\dfrac{dG_{K(\theta)+1}(\theta)}{d\theta} = 0$ because $G_{K(\theta)+1}(\theta) = T$ is locally independent of $\theta$.

Next, differentiate Eq. (33), and substitute Eq. (34) into the result. In view of Eq. (35), we can now deduce for every $t \in (H_k(\theta), G_{k+1}(\theta))$ the representation,

$$\frac{dB(\theta,t)}{d\theta} \;=\; \frac{dX(\theta,t)}{d\theta}$$
$$= \frac{d}{d\theta} \int_{\tilde{F}_k(\theta)}^{t} [\alpha(\tau) - \mu(\theta, \tau)]\,d\tau$$
$$= -[\alpha(\tilde{F}^+_k(\theta)) - \mu(\theta, \tilde{F}^+_k(\theta))]\,\frac{d\tilde{F}_k(\theta)}{d\theta} - \int_{\tilde{F}_k(\theta)}^{t} d\tau \;=\; \tilde{F}_k(\theta) - t. \tag{39}$$

To see that, note that only two cases are possible for $-[\alpha(t) - \mu(\theta, t)]$: (i) either it jumps at $t = \tilde{F}_k(\theta)$ (implying $d\tilde{F}_k(\theta)/d\theta = 0$, since the jumps of both $\{\alpha(t)\}$ and $\{\mu(t)\}$ are locally independent of $\theta$), or (ii) it crosses zero continuously at $t = \tilde{F}_k(\theta)$ (implying $\alpha(\tilde{F}^+_k(\theta)) - \mu(\theta, \tilde{F}^+_k(\theta)) = \alpha(\tilde{F}_k(\theta)) - \mu(\theta, \tilde{F}_k(\theta)) = 0$). Either way, the first term on the third right-hand equality of (39) vanishes, and the rest follows from Eq. (22).

Next, substituting Eq. (39) into Eq. (38), we can now write for $k = 1, \ldots, K(\theta)$,

$$\frac{d}{d\theta}\int_{H_k(\theta)}^{G_{k+1}(\theta)} B(\theta,t)\,dt \;=\; \int_{H_k(\theta)}^{G_{k+1}(\theta)} [\tilde{F}_k(\theta) - t]\,dt$$
$$= \sum_{k=1}^{K(\theta)}\{\tilde{F}_k(\theta)[G_{k+1}(\theta) - H_k(\theta)]$$
$$- \frac{1}{2}[G_{k+1}^2(\theta) - H_k^2(\theta)]\}. \qquad (40)$$

Eq. (36) now follows by substituting Eq. (40) into Eq. (37).

**Proposition 6** *Under Assumption 1 and 3, the IPA derivatives $L_I'(\theta)$ and $L_B'(\theta)$ are unbiased.*

**Proof.** Condition (a) of Lemma 1 is satisfied by part (b) of Assumption 3, so it remains to prove Condition (b) of Lemma 1.

Observe that Eq. (11) is a special case (for an unlimited buffer capacity) of Eq. (2.1) in Wardi and Melamed (2001). Since our initial condition is the same as that of Wardi and Melamed (2001), Proposition 3.2 (ibid.) implies that for any $\theta, \theta + \Delta\theta \in \Theta$,

$$|X(\theta+\Delta\theta,t) - X(\theta,t)| \;\le\; \int_0^t |\Delta\theta|\,d\tau \;=\; |\Delta\theta|\,t. \quad (41)$$

From (41), we readily obtain by appeal to Eq. (9),

$$|I(\theta+\Delta\theta,t) - I(\theta,t)|$$
$$= |[S - X(\theta+\Delta\theta,t)]^+ - [S - X(\theta,t)]^+|$$
$$\le \max\{|X(\theta+\Delta\theta,\tau) - X(\theta,\tau)| : 0 \le \tau \le t\}$$
$$\le |\Delta\theta|\,t, \qquad (42)$$

and similarly,

$$|B(\theta+\Delta\theta,t) - B(\theta,t)| \;\le\; |\Delta\theta|\,t. \qquad (43)$$

Finally, by Eq. (42),

$$|L_I(\theta+\Delta\theta) - L_I(\theta)|$$
$$= \frac{1}{T}\left|\int_0^T [I(\theta+\Delta\theta,t) - I(\theta,t)]\,dt\right|$$
$$\le \frac{1}{T}\int_0^T |I(\theta+\Delta\theta,t) - I(\theta,t)|\,dt$$
$$\le \frac{1}{T}\int_0^T |\Delta\theta| t\,dt \;=\; \frac{T}{2}\,|\Delta\theta|, \qquad (44)$$

and by Eq. (43),

$$|L_B(\theta+\Delta\theta) - L_B(\theta)| \;\le\; \frac{T}{2}\,|\Delta\theta|. \qquad (45)$$

## 4 CONCLUSIONS

We have derived IPA gradient formulas for the MTS system with backorders in SFM setting for the time averaged inventory level and backorders level with respect to the base-stock level and a parameter of the production rate process. The IPA gradients derived are unbiased, nonparametric and their formulas are easy to compute and intuitive.

The methodology employed in this paper holds out the promise of generalizations and extensions. First, in order to implement IPA-based control schemes, the formulas need to be generalized to arbitrary initial conditions. This is so, because the system may potentially be in any state when a control action is applied, at which point the IPA gradient computation needs to be restarted from that (general) state. Second, control schemes can be devised and readily implemented, based on both system state and IPA gradient information. Finally, our methodology can be extended to MTS systems with lost sales and beyond. Future extensions and generalizations will be reported elsewhere.

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