

## PORTFOLIO OPTIMIZATION FOR CAPITAL INVESTMENT PROJECTS

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### ABSTRACT

The new portfolio optimization engine, OptFolio™, simultaneously addresses financial return goals, catastrophic loss avoidance, and performance probability. The innovations embedded in OptFolio enable users to confidently design effective plans for achieving financial goals, employing accurate analysis based on real data. Traditional analysis and prediction methods are based on mean variance analysis -- an approach known to be faulty. OptFolio takes a much more sophisticated and strategic direction. State-of-the-art technology integrates optimization and simulation techniques and a new surface methodology based on linear programming into a global system that guides a series of evaluations to reveal truly optimal investment scenarios. OptFolio is currently being used to optimize project portfolio performance in oil and gas applications and in capital allocation and budgeting for investments in technology.

### 1 INTRODUCTION

Portfolio optimization for capital investment is often too complex to allow for tractable mathematical formulations. Nonetheless, many analysts force these problems into standard forms that can utilize traditional optimization technologies such as quadratic programming. Unfortunately, such formulations omit key aspects of real world settings resulting in flawed solutions based on invalid assumptions. In this paper we focus on a flexible modeling approach that overcomes these limitations.

### 2 BACKGROUND

The customers for OptFolio include C level executives responsible for deciding capital investments and accountable for their performance, finance department analysts charged with developing the capital budget analysis and a project portfolio management plan, and technology managers responsible for planning and implementing projects. Their

needs, which provide compelling reasons to buy the technology, are:

- Technology managers and corporate financial executives are dissatisfied with current way they address risk tolerance.
- They are under continual pressure to improve capital investment performance.
- They need technology that improves the understanding of the analysis and clearly identifies the reasons to make specific investment decisions.
- They are concerned that their competition may be adopting a new and more advanced technology.

Capital investment within commercial firms is primarily accomplished with traditional analyses that include net present value analysis and mean-variance analysis. Although there are many methods being used to enable capital decisions, there are certain conventions that have become standardized through implementation practices. Consequently, many organizations use similar methods to evaluate and select capital spending options and monitor their performance.

Many organizations evaluate their capital projects by estimating their "net present value." Net present value (NPV) is calculated by projecting the future cash flows the investment is likely to generate, "discounting" the future cash flows by the cost of capital, and then subtracting the initial investment.

According to conventional wisdom, it makes economic sense to undertake projects if their NPV's are positive. But this does not guarantee they will be funded. Organizations typically take other factors into consideration, which incorporate their ability to fund the initial investment given their debt position, their current operating expenses and cash flow positions, and their strategic considerations including financial performance expectations.

Determining how to allocate investment capital in order to maximize returns is a ubiquitous challenge where

approaches to solutions cover a very wide spectrum. In organizations both public and private, the decisions of committing limited resources to a variety of uses can either strengthen or deteriorate the very financial foundation of the organization itself. On one end of the spectrum, and at the core of sophisticated financial manuals, capital budgeting procedures many times employ traditional operations research theories and techniques to guide and support decisions. On the other end, and anecdotally, many executives admit that selections come down to mustering intuition, combined with seat-of-the-pants “guestimates”, and peppered with squeaky wheel assignments.

Typically, however, what is common in this arena is building models, which employ pro forma plans centering around measures of the benefits of the investments- the returns, time horizons over which the investments are being made, and estimates of the risks or uncertainty involved. The list of measures expands to include such considerations as cash flow, cost of capital, market share, etc.

Evaluations of alternatives are made as well in a variety of ways. From one-at-a-time comparisons of returns and risks to more sophisticated portfolio optimization and real option theories, organizations run the gamut in the ways they decide to allocate capital. In the companies using these sophisticated methods, which go beyond single project net present value analysis, many portfolio management methods include mean variance analysis.

In a seminal paper in 1952 in the *Journal of Finance*, Harry Markowitz laid down the basis for the modern portfolio theory (Markowitz, H., 1952). For his path-breaking work that has revolutionized investment practice, he was awarded the Nobel Prize in 1990. Markowitz focused the investment profession’s attention to *mean-variance efficient portfolios*. A portfolio is defined as mean-variance efficient if it has the highest expected return for a given variance, or, equivalently, a portfolio is defined as mean-variance efficient if it has smallest variance for a given expected return.

In Figure 1, the curve is known as the *efficient frontier* and contains the mean-variance efficient portfolios. The area below and to the right of this mean-variance efficient frontier contains various risky assets or projects. The mean-variance efficient portfolios are combinations of these risky projects.

*Why are mean-variance portfolios important?* Decision makers are risk-averse. They prefer portfolios with high expected returns and low risk. Another important question: *How is the risk of a portfolio measured?* If portfolio returns are normally distributed, then its risk can be measured by its variance. However, a substantial body of empirical evidence suggests that *actual portfolio returns are not normally distributed* (McVean, J.R., 2000).

If actual portfolio returns are not normally distributed, then variance is not the appropriate risk measure for a portfolio. If not variance, what is an appropriate risk measure

for a portfolio? Before answering this question, consider an alternate paradigm that has been suggested to revive the importance of mean-variance efficient portfolios.

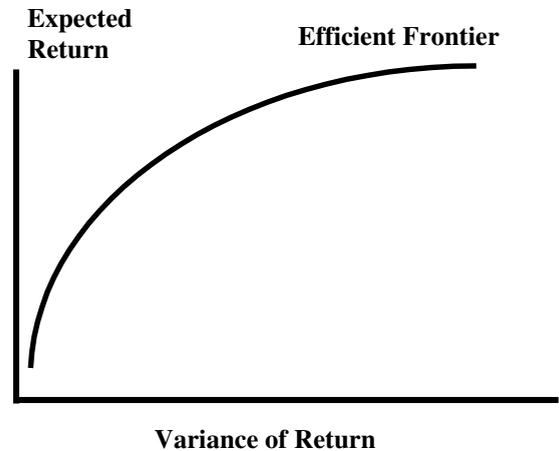


Figure 1: Efficient Frontier

Instead of taking into account the portfolio returns distribution, some finance theorists have suggested that if investments have quadratic utility functions, then portfolio risk can still be appropriately measured by its variance (even if portfolio returns are not normally distributed).

Investor utility functions, in general, describe the rate at which an investor is willing to exchange a unit of risk for a unit of return. In other words, how much additional return would be required to bear an additional unit of risk? The quadratic utility function has a particular shape; it is part of a circle. However, the quadratic utility function has not received much theoretical or empirical support in the literature as a realistic depiction of investor utility functions.

In practice, mean-variance efficient portfolios have been found to be quite unstable: small changes in the estimated parameter inputs lead to large changes in the implied portfolio holdings. The practical implementation of the mean-variance efficient paradigm requires determination of the efficient frontier. This requires three inputs: expected returns of the projects, expected correlation among these projects, and expected variance of these projects (individually). Typically, these input parameters are estimated using either historical data or forecasts. Researchers have found that estimation errors in these input parameters overwhelm the theoretical benefits of the mean-variance paradigm.

Now, as cracks in the foundation are becoming too conspicuous to ignore and capital budgeting participants have been dedicated to traditional ideas for so long that they are not able to pull away, even at the expense of policies that severely hamper their financial growth.

Efforts by more progressive analysts to sound the alert about the crumbling structure underlying mainstream capi-

tal budgeting and investment strategies have not been lacking. Still, the best response has been to cobble together various ad-hoc measures in an attempt to shore up the framework, or erect a makeshift alternative. Recognition that this response is far from ideal has persuaded many to cling to the old ways, in spite of their apparent defects. The inability to devise a more effective alternative has been due in large part to limitations in the technology of decision-making and analysis, not only in the area of investments but in other areas of business alike, which has offered no reliable method to conquer the complexity of problems attended by uncertainty. As a result, the goal of evaluating investments effectively, and to account appropriately for tradeoffs between risk and potential return, has remained incompletely realized and ripe for innovation.

Over the last several years, alternative technologies (methods) have emerged for optimizing decisions under uncertainty. The outcome of this development has begun to penetrate planning and decision-making in many business disciplines, making it possible to study viable solutions to models that are much more flexible and realistic than those treated in the past. In application to the areas of capital budgeting and investment, these alternative technologies are being implemented to create portfolio and asset-allocation strategies to improve performance. Included in these alternative technologies are agent-based modeling for portfolio optimization, genetic algorithms for portfolio optimization, and real options analysis for capital spending. All of these technologies seek to improve on the traditional methods by introducing more flexible, robust, and realistic assumptions and providing more powerful and sophisticated analysis and forecasting tools. Companies marketing these alternative technologies include The Bios Group, Insightful, Merak, United Management Technologies, Glomark, and Portfolio Decisions, Inc.

To date the largest penetration for these technologies have been in academic circles while achieving only a modicum of success in the marketplace. This indicates that commercial applications of alternative technologies are still in the early adoption stages.

### 3 OPTIMIZATION METHODS

The complexities and uncertainties in complex systems are the primary reason that simulation is often chosen as a basis for handling the decision problems associated with those systems. Consequently, decision makers must deal with the dilemma that many important types of real world optimization problems can only be treated by the use of simulation models, but once these problems are submitted to simulation there are no optimization methods that can adequately cope with them.

Advances in the field of metaheuristics—the domain of optimization that augments traditional mathematics with artificial intelligence and methods based on analogs

to physical, biological, or evolutionary processes—have led to the creation of optimization engines that successfully guide a series of complex evaluations with the goal of finding optimal values for the decision variables. One of those engines is the search algorithm embedded in the OptQuest optimization system. OptQuest is designed to search for optimal solutions to the following class of optimization problems:

$$\begin{array}{ll}
 \text{Max or Min} & F(x) \\
 \text{Subject to} & \\
 & Ax \leq b \quad (\text{Constraints}) \\
 & g_l \leq G(x) \leq g_u \quad (\text{Requirements}) \\
 & l \leq x \leq u \quad (\text{Bounds})
 \end{array}$$

where  $x$  can be continuous or discrete.

The objective  $F(x)$  may be any mapping from a set of values  $x$  to a real value. The set of constraints must be linear and the coefficient matrix “A” and the right-hand-side values “b” must be known. The requirements are simple upper and/or lower bounds imposed on a function that can be linear or non-linear. The values of the bounds “ $g_l$ ” and “ $g_u$ ” must be known constants. All the variables must be bounded and some may be restricted to be discrete with an arbitrary step size.

A typical example might be to maximize the net present value for a portfolio by judiciously choosing projects subject to budget restriction and a limit on risk. In this case,  $x$  represents the specific project participation levels  $F(x)$  is the expected net present value. The budget restriction is modeled as  $Ax \leq b$  and the limit on risk is achieved by a requirement modeled as  $G(x) \leq g_u$  where  $G(x)$  is percentile value. Each evaluation, of  $F(x)$  and  $G(x)$  requires a Monte Carlo simulation of the portfolio. By combining simulation and optimization, a powerful design tool results.

The optimization procedure uses the outputs from the system evaluator, which measures the merit of the inputs that were fed into the model. On the basis of both current and past evaluations, the optimization procedure decides upon a new set of input values (see Figure 2).

The optimization procedure is designed to carry out a special “non-monotonic search,” where the successively generated inputs produce varying evaluations, not all of them improving, but which over time provide a highly efficient trajectory to the best solutions. The process continues until an appropriate termination criterion is satisfied (usually based on the user’s preference for the amount of time to be devoted to the search).

### 4 PROJECT PORTFOLIO OPTIMIZATION

In many industries, strategic planning requires executives to select a portfolio of projects for funding that will likely advance the corporate goals. In general, there are many more

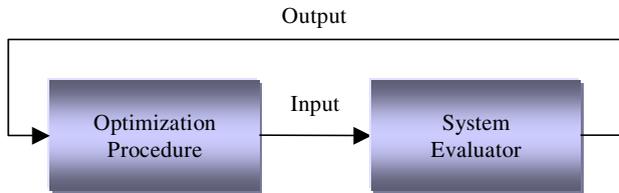


Figure 2. Coordination Between Optimization and System Evaluation

projects than funding can support so the selection process must intelligently choose a subset of projects that meet the companies profit goals while obeying budgetary restrictions. Additionally, executives wish to manage the overall risk of a portfolio of projects and ensure that cash flow and other such “accounting” type constraints are satisfied.

The Petroleum and Energy (P&E) industry uses project portfolio optimization to manage its investments in the exploration and production of oil and gas. Each project’s proforma is modeled as a simulation capturing the uncertainties of production and sales.

The application illustrated here involves five potential projects with ten year models that incorporate multiple types of uncertainty in drilling, production, and market conditions. We examined multiple cases to demonstrate the flexibility of the software to enable a variety of decision alternatives. We present a standard case and one that utilizes the power of simulation optimization.

**Case 1**

In case 1, the decision was to determine participation levels [0,1] in each of the five projects with the objective of maximizing expected net present value of the portfolio while keeping the standard deviation of the net present value of the investment below a specified threshold. This is the traditional Markowitz approach. In this case, all projects must begin in the first year.

- Maximize  $E(NPV)$
- While keeping  $\sigma < 10,000$  M\$
- All projects must start in year 1

In this case, the best investment decision resulted in an expected net present value of approximately \$37,400 M with a standard deviation of \$9,500 M. Figure 3 shows the corresponding non-normal NPV distribution.

**Case 2**

The goal was to determine participation levels in each project where starting times for each project could vary and we would maximize the probability of exceeding the expected net present value of \$47,500 M (which was

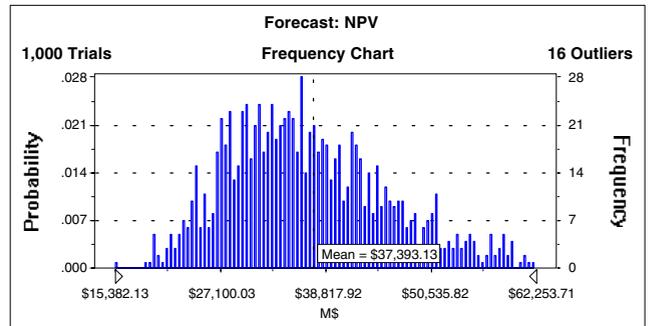


Figure 3. Case 1 NPV Distribution

achieved in a previous analysis). Risk was controlled by limiting the 10<sup>th</sup> Percentile of NPV.

- Maximize Probability( $E(NPV) > 47,455$  M\$)
- While keeping 10<sup>th</sup> Percentile of NPV  $> 36,096$  M\$
- All projects may start in year 1, year 2, or year 3

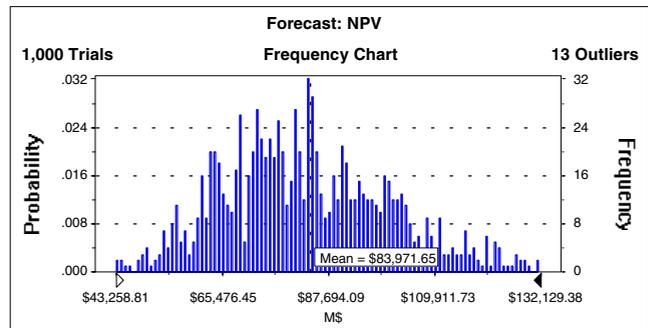


Figure 4. Case 2 NPV Distribution

In this case, where starting times could vary, and we wanted to maximize the chance of exceeding the net present value of \$47,500 M, the best investment decision resulted in an expected net present value of approximately \$84,000 M with a standard deviation of \$18,500 M. The NPV had a 99% probability of exceeding \$47,500 M. This case demonstrates that adopting measures of risk other than standard deviation can result in superior portfolios. Simulation optimization is the only technology that can offer these types of analyses.

The integration of simulation with optimization has been shown to be a powerful approach for portfolio optimization. However, the computational cost of multiple simulations can be quite high. To minimize the number of simulations required to determine the optimal portfolio, our technology utilizes mathematical programming techniques to aid in the optimization process. The layered envelope response method is critical to the success of this project.

## LAYERED ENVELOPE RESPONSE (LEVER)

To produce a method that goes dramatically beyond past efforts to handle problems in the realm we address, a primary step is to create an effective “externalized representation” of the problem objective function.

Many efforts have been undertaken to try to capture an objective function for various types of applications by devising a model to fit the outputs of an evaluation process used in these applications. Prominent examples include linear and quadratic curve fitting, response surface methodology and kriging. However, each of these approaches suffers significant limitations in dealing with the complex objective function surfaces implied by outputs that arise in the context we face. The goal of optimizing over mixed discrete and nonlinear spaces, where uncertainty enters into the picture, can generate structures that popular methods are ill-suited to handle.

The major challenge faced in addressing problems of capital investment and budgeting, with applications to areas such as manufacturing, pharmaceuticals, chemical products, and distribution, is to produce a means to derive a “map” of the objective function, given that no explicit form is provided for the function in advance. Under the condition that the only information about the objective function comes from a computer model (such as a simulation or other iterative computational process), whose operations cannot be translated into a closed form expression, we seek a method to infer an approximation to the outputs that can operate as a proxy for the results generated by the more complex computer-based process. If such a proxy can be reasonably faithful to the results produced by the more complete and complex model, then an enormous amount of computational effort can be saved, and a greatly enhanced search process can be produced. The solution mechanisms devised to work with the outputs from the general computer-based model can be streamlined to work much faster on the proxy model. In addition, the use of the proxy makes it possible to derive trial solutions that are pre-conditioned to be closer to optimality (at an earlier stage of search) for the original model. The result is an appreciable gain in both the efficiency and effectiveness of the search.

The overriding factor is creating such a proxy model is to be able to handle spaces where the objective function has multiple local optima, and does not fit any preconceived format or structure. The need to go beyond limiting assumptions about the nature of the objective function curvature -- and still more generally, to be able to include discontinuities in the space -- makes it essential to find an alternative to the popular approaches that researchers have focused on in the past.

Our approach, called “Layered Envelope Response” (or the LEVER method), is based on the observation that any nonlinear function can be approximated to any degree

of accuracy by a mixed zero-one linear integer programming formulation. While this fact has been known for many years, there has been no method to exploit it. Previous efforts to use mixed zero-one formulations to approximate nonlinear functions have been restricted to very simple cases, where the details of the approximation can be specified by a classical textbook formula, and chiefly where the form of the nonlinear objective is known in advance. In addition, these applications normally involve functions of only one or two variables. Upon reaching the two-variable case, the difficulty of applying the classical representation is already formidable, and the situation becomes progressively worse as additional variables are introduced. Thus, recourse to a mixed zero-one representation is avoided in all but the simplest cases.

The LEVER approach represents a departure that does not depend on the classical rules for generating a mixed zero-one formulation. The use of layered envelopes, generated by successively created linear programs of a special form, replaces the classical approach by a dynamic and adaptive process which is better suited to handling real world complexity. A key advantage of the LEVER method is the ability to avoid identifying in advance the domains of the variables over which the approximation operates -- and more specifically, to avoid identifying the subdomains that characterize regions where the function takes different forms. It is this type of identification that dooms the classical approach. By contrast, the ability to operate without explicitly referring to such subdomains gives rise to a previously unequalled modeling capability.

Our work builds upon and extends work by Roy (2001), Roy et. al. (1995), and McMillan et. al. (1992) on embedding linear programming and symbolic processing in neural networks, work by Barr (2000a, 2000b) on nonlinear discrimination, work by Glover and Laguna (1997a, 1997b) on search methods for mixed integer optimization, and work by Glover (1990) on optimizing models for discriminant analysis.

### LEVER Method

We denote the problem of interest as

$$\begin{array}{ll} \text{Minimize} & x_0 = F(x) \\ \text{subject to} & x \in X \end{array}$$

where  $x \in X$  imposes bounds and possibly other linear inequality restrictions on the vector  $x = (x_1 \ x_2 \ \dots \ x_n)$ , which we implicitly treat as a column vector. (The condition  $x \in X$  may represent a relaxation of a more complex constraining condition  $x \in X_0$ , which may additionally impose discrete requirements on some of the problem variables, and may include other constraints of interest.)

The function  $F(x)$  is not explicitly represented as a mathematical function, but is determined by a “black box”

(such as a simulation or other iterative process), and the goal is to identify a model that gives an approximation to the foregoing problem. The model we generate is a zero-one mixed integer programming problem:

$$\begin{aligned}
 \text{(P)} \quad & \text{Minimize} && x_0 \\
 & \text{subject to} && \\
 & x_0 + A_i x \geq a_{i0} && i \in I_0 \\
 & x_0 + A_i x + U_i z_h \geq a_{i0} && i \in I_{1h} \quad h \in H \\
 & x_0 + A_i x + U_i(1 - z_h) \geq a_{i0} && i \in I_{2h} \quad h \in H \\
 & x \in X && \\
 & z_h \in \{0,1\} && h \in H
 \end{aligned}$$

The vectors  $A_i$  are row vectors of constants given by  $A_i = (a_{i1} \ a_{i2} \ \dots \ a_{in})$ . The scalar constants  $U_i$  are "upper bounds" determined as subsequently specified so that the inequalities over  $I_{1h}$  are redundant unless  $z_h = 0$  and the inequalities over  $i \in I_{2h}$  are redundant unless  $z_h = 1$ . (In the non-redundant cases the multiplier of  $U_i$  is 0, and hence these inequalities take the same form as the inequalities over  $I_0$ .)

The process for generating (P) is based on having determined the objective function values  $x_0 = F(x)$ , or more specifically  $x_0(t) = F(x(t))$ , for a set of trial solutions  $x = x(t)$ ,  $t \in T$ , whereupon (P) represents a piecewise linear approximation to  $F(x)$  that likewise yields the values  $x_0 = x_0(t)$  when  $x = x(t)$ ,  $t \in T$ . The zero-one variables  $z_h$  are required for the case where the surface generated by the  $x_0(t)$  values is non-convex. We identify the components of  $x(t)$  by  $x_j(t)$ ,  $j \in N = \{1, \dots, n\}$ .

Problem (P) is created in iterative stages by solving a collection of linear programming problems over nonnegative variables  $u_t$ ,  $t \in T_0$  (a changing subset of  $T$ ) and unrestricted variables  $a_0$  and  $a_j$ ,  $j \in N$ , of the following form:

$$\begin{aligned}
 \text{(LP)} \quad & \text{Minimize} && \sum (c_t u_t; t \in T_0) \\
 & \text{subject to} && \\
 & x_0(t) + \sum (a_j x_j(t); j \in N) - u_t = a_0 && t \in T_0 \\
 & u_t \geq 0 && t \in T_0
 \end{aligned}$$

Each solution yields one of the inequalities  $x_0 + A_i x \geq a_{i0}$  of (P) by taking  $a_{i0} = a_0$  and  $A_i = (a_1 \ a_2 \ \dots \ a_n)$ . The LEVER method for determining the full set of inequalities of (P) also requires solving additional linear programs of a form slightly different from (LP), identified later. The complete method consists of two component routines, LEVER-1 and LEVER-2. An important feature of these routines is that they can be organized to solve the successive problems (LP) and their variants by a post-optimizing process that allows the collection to be solved much more efficiently than by solving each instance in isolation.

As successive problems (LP) are solved to yield associated inequalities  $x_0 + A_i x \geq a_{i0}$  of (P), we keep track of the number  $n(t)$  of the hyperplanes  $x_0 + A_i x = a_{i0}$  on which each given trial solution  $x(t)$  lies; i.e.,  $n(t) = |L(t)|$ , where

$L(t) = \{i: x_0(t) + A_i x(t) = a_{i0}\}$ . Thus, to begin, before any inequalities are generated,  $n(t) = 0$  for all  $t$ . The first LEVER routine undertakes to generate inequalities that correspond to supporting hyperplanes for the convex region spanned by the points  $(x_0(t), x(t))$ ,  $t \in T_0$ . The process also identifies points that lie in the interior of this region, and hence that require a non-convex representation by means of the inequalities over  $I_{0h}$  and  $I_{1h}$  of (P). These latter points are removed from  $T_0$ , thereby accelerating computation and creating conditions that allow additional supporting hyperplanes to be determined more efficiently. Finally  $T_0$  consists only of supported points that lie on the hyperplanes generated (hence  $n(t) > 0$  for all  $t \in T_0$ ) and the process terminates after assuring that a sufficient set of hyperplanes is generated so that the minimum  $x_0$  over  $x \in X \cap R$ , where  $R$  is the supported region, corresponds to a point  $x(t)$ ,  $t \in T_0$ .

In the complete LEVER method, LEVER-1 also is employed as a subroutine in LEVER-2, to generate the "conditional inequalities"  $x_0 + A_i x + U_i z_h \geq a_{i0}$  and  $x_0 + A_i x + U_i(1 - z_h) \geq a_{i0}$  of (P). An index set  $I$  is maintained that includes the indexes  $i$  for both ordinary and conditional inequalities. (Because some hyperplanes (and associated inequalities) generated in LEVER-1 will be replaced by others generated in LEVER-2, the ultimate composition of  $I$  will have gaps in its sequence of  $i$  values corresponding to those that have been removed.)

### LEVER-1.

*Step 0 (Initialization).* Let  $T_0 = T$ ,  $I = \Phi$ ,  $H = \Phi$ , and set  $h^* = 0$  and  $i^* = 0$ . (This step applies only to the first execution of LEVER-1, and is disregarded when LEVER-1 is called as a subroutine by LEVER-2.)

*Step 1.* Create the coefficients  $c_t$ ,  $t \in T_0$ , of the objective function for (LP) by reference to an ordering  $[t]$  of the indexes  $t \in T_0$  so that

$$c_{[1]} \gg c_{[2]} \gg \dots \gg c_{[r]} = 1, \quad t' = [T_0],$$

where  $p < q$  is implied by  $n[p] < n[q]$  and by  $x_0[p] < x_0[q]$  when  $n[p] = n[q]$ . (Hence to begin, when all  $n(t) = 0$ , the  $c_t$  values are largest for the smallest  $x_0(t)$  values, and subsequently, the  $c_t$  values are largest for the smallest  $n(t)$  values, breaking ties in favor of the smallest  $x_0(t)$  values.)

*Step 2.* Solve the problem (LP).

*Step 3.* Let  $r = \max\{p: u_{[p]} = 0 \text{ for all } h \leq p\}$ , where  $r = 0$  if  $u_{[1]} > 0$ . Then if  $r > 0$ , remove the indexes  $[1]$  to  $[r]$  from  $T_0$ .

*Step 4.* If the solution vector  $A = (a_1 \ a_2 \ \dots \ a_n)$  does not duplicate any previously generated vector  $A_i$ ,  $i \in I$ , then set  $i^* = i^* + 1$ , add  $i^*$  to  $I$ , record  $A_{i^*} = A$  and  $a_{i^*0} = a_0$ , and return to Step 1.

*Step 5.* If A duplicates a previously generated vector  $A_i$ ,  $i \in I$ , solve the linear program

$$(LP1) \quad \text{Minimize} \quad x_0 \\ \text{subject to} \\ x_0 + A_i x \geq a_{i0} \quad i \in I \\ x \in X$$

*Step 6.* Denote the optimal solution for (LP1) by  $(x_0^*, x^*)$ . If  $x_0^* = \text{Min}(x_0(t) : x(t) \in X, t \in T_0)$ , then terminate LEVER-1 and proceed to LEVER-2. Otherwise, solve the problem

$$(LP2) \quad \text{Minimize} \quad \sum (u_t : t \in T_0) + M u_0 \\ \text{subject to} \\ x_0(t) + \sum (a_j x_j(t) : j \in N) - u_t = a_0 \quad t \in T_0 \\ x_0^* + \sum (a_j x_j^* : j \in N) + u_0 = a_0 \\ u_0, u_t \geq 0 \quad t \in T_0$$

where  $M \gg 1$ . Record the solution by setting  $i^* = i^* + 1$ , adding  $i^*$  to I, setting  $A_{i^*} = (a_1 \ a_2 \ \dots \ a_n)$  and  $a_{i^*0} = a_0$ . Then return to Step 1.

## LEVER-2.

*Step 1.* If  $n(t) > 0$  for all  $t \in T$  (all points  $(x_0(t), x(t))$  are supported), the LEVER method is completed and the process of generating (P) terminates. Otherwise, choose a currently unsupported point  $(x_0(t^*), x(t^*))$  by the rule

$$t^* = \text{argmax}(x_0(t) : n(t) = 0, t \in T).$$

*Step 2.* Divide T into two subsets  $T_1$  and  $T_2$  by determining a vector A and hyperplane  $Ax(t^*) = 1$  and defining

$$T_1 = \{ t \in T : Ax(t) < 1 \text{ or } t=t^* \} \text{ and} \\ T_2 = \{ t \in T : Ax(t) \geq 1 \}.$$

(Note that  $t^*$  is an element of both sets, but otherwise they are disjoint.)

*Step 3.* Remove from I the indexes of all hyperplanes  $x_0 + A_i x = a_{i0}$  that contain two points  $(x_0(t_1), x(t_1))$  and  $(x_0(t_2), x(t_2))$ , where  $t_1 \in T_1$  and  $t_2 \in T_2$ . Partition the resulting I into the two sets

$$I_1 = \{ i \in I : \text{hyperplane } x_0 + A_i x = a_{i0} \text{ contains only} \\ \text{points } (x_0(t), x(t)) \text{ for } t \in T_1 \} \\ I_2 = \{ i \in I : \text{hyperplane } x_0 + A_i x = a_{i0} \text{ contains only} \\ \text{points } (x_0(t), x(t)) \text{ for } t \in T_2 \}.$$

Update the values  $n(t)$  to reflect the reduction that occurs for points  $(x_0(t), x(t))$  that lie on the hyperplanes discarded as a result of shrinking I.

*Step 4.* Apply LEVER-1 as a subroutine for the two cases where  $T_0$  begins as  $T_1$  and as  $T_2$  (bypassing the ini-

tialization of Step 0) where each reference to I in LEVER-1 is replaced by reference to  $I_1$  and  $I_2$ , respectively. Denote the final  $I_1$  and final  $I_2$  produced by LEVER-1 for these two cases as  $I_1^*$  and  $I_2^*$ , and set  $h^* = h^* + 1$ , adding  $h^*$  to H. Then, for  $h = h^*$  the sets  $I_{1h}$  and  $I_{2h}$  for (P) are given by  $I_{1h} = I_1^* - I$  and  $I_{2h} = I_2^* - I$  (hence  $I_{1h}$  and  $I_{2h}$  contain only the indexes for the new hyperplanes generated to create  $I_1^*$  and  $I_2^*$ .) The value  $U_i$  is given by

$$U_i = \max(a_{i0} - A_i x(t) : t \in T_2) \text{ for each } i \in I_{1h} \\ U_i = \max(a_{i0} - A_i x(t) : t \in T_1) \text{ for each } i \in I_{2h}.$$

*Step 5.* Let I be updated to be the union of I,  $I_{1h}$  and  $I_{2h}$ . (If I is augmented as specified in Steps 4 and 6 of LEVER-1, as well as augmenting  $I_1$  and  $I_2$  in the role of I, then the final I will have the correct composition. However, the identification of  $I_{1h}$  and  $I_{2h}$  in Step 4 immediately above must be changed so that these sets continue to identify precisely the new indexes added to create  $I_1^*$  and  $I_2^*$ .) Then return to Step 1 of LEVER-2.

The combination of the LEVER method with the general simulation approach produces an effective and efficient application for solving the portfolio optimization problem. The LEVER method is periodically exercised to produce an approximation of the underlying stochastic function being optimized. A mixed integer program is then solved on the approximation producing an excellent solution for evaluation by the simulator. The application of the LEVER method reduces the number of required simulations by as much as an order of magnitude and makes simulation optimization for portfolio selection much more approachable from a computational viewpoint.

## 5 OPTTEK SYSTEMS, INC.

OptTek Systems, Inc. is an optimization software and services company located in Boulder, Colorado. We are the leading optimization software provider to the simulation software market and are confident that our products and services will add significant value to our customers.

OptTek software is recognized throughout the simulation and optimization market for its quality, speed, and customer service. Independent evaluations of our software demonstrate that our technology yields faster and higher quality solutions when compared to other optimization methods currently on the market. The software integrates state-of-the-art metaheuristic procedures, including Tabu Search, Neural Networks, and Scatter Search, into a single composite method. Some of the differences between OptTek's methods and other methods include:

- The ability to avoid being trapped in locally optimal solutions to problems that contain nonlinearities (which commonly are present in real world problems).

- The ability to handle nonlinear and discontinuous relationships that are not specifiable by the kinds of equations and formulas that are used in standard mathematical programming formulations.
- The ability to solve problems that involve uncertainties, such as those arising from uncertain supplies, demands, prices, costs, flow rates, queuing rates and so forth.
- The ability to solve decision support problems for extremely complex systems.

While other methods currently being applied in complex and highly uncertain environments have value, they either identify feasible solutions or locally optimal solutions. Both are typically improvements over the status quo but neither identifies the global optimum or “best” solution.

OptTek’s methods, which are well known in both the simulation and optimization communities, are based on the contributions of Professor Fred Glover, one of the founders of OptTek and a winner of the von Neumann Theory Prize in operations research, who developed the adaptive memory method called Tabu search, and the evolutionary method called Scatter Search, singularly powerful search techniques in global optimization.

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