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A REAL OPTIONS DESIGN FOR QUALITY CONTROL CHARTS

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ABSTRACT

We develop a financial model for a manufacturing process where quality can be affected by an assignable cause. We evaluate the options associated with applying a statistical process control chart using pentanomial lattice and Monte Carlo simulation methods. By connecting the aspects of market dynamics with the manufacturing operational aspects, we now have a way to help decision makers address the bottom-line profitability associated with the quality control decision.

1 INTRODUCTION

In the forward to *Real Options* (Trigeorgis, 1999), Scott Mason writes: "Flexibility has value. While this statement is obvious at the conceptual level, it is surprisingly subtle at the applied level." The question then becomes: Precisely how valuable is flexibility? The financial arena was the original ground for the application of the options-based framework to the valuation of flexibility. More recently, managerial operating flexibility has been likened to financial options. The goal of our research is to view the flexibility surrounding manufacturing operations using financial options.

In this paper, we specifically consider the manufacturing decision to introduce statistical process control (SPC) charts to monitor quality. Precise methods to design control charts that minimize the cost or maximize the profit of a process have been proposed by a number of authors. These methods yield control chart designs known as economic designs (Collani et al. 1994; Lorenzen and Vance, 1986). They result in models that help determine the control chart parameters that will best suit a process. However, they do not account for the dynamic market conditions that have effect on the decisions manufacturers will make and hence the profitability of the manufacturing operation.

We use the options approach to find the value of applying an SPC chart during a specified length of time, considering future uncertain market variables. The problem is analyzed using a pentanomial lattice and Monte Carlo simulation. Results of the two approaches are compared with numerical examples. Using the proposed design, a company will be able to answer questions about the long-term value of implementing control charts. This will go beyond the traditional statements such as "SPC improves quality" or "the process now produces less scrap" to an ability to determine the bottom-line dollar value to the organization that can be brought about by using (or not using) control charts.

This paper is organized as follows. Some of the approaches for multivariate option valuation are discussed in Section 2. The financial model that will be used to find the option value of control charts is defined in Section 3. Section 4 defines the two numerical valuation procedures, which are pentanomial lattice approach and Monte Carlo simulation. Examples and numerical results are given in Section 5. We make some concluding remarks in Section 6.

2 OPTION MODELS

Fundamentally, an option is the right, but not the obligation, to take an action in the future (Amram and Kulatilaka, 1999). Sometimes, options are associated with investment opportunities that are not financial instruments. These operational options are often termed *real options* to emphasize that they involve real activities or real commodities, as opposed to purely financial commodities, as in the case, for instance, of stock options (Luenberger, 1998).

Here, we formulate the control chart problem as a series of European options. A European option gives the right to exercise the option on the expiration date. In our context, this means that the control chart can be used or not used (which is the option) in any time period.

In most manufacturing systems, there are multiple sources of uncertainty. Valuing real options for such an environment will require the analysis of projects whose values depend on multiple state variables.

In the case of just one state variable, the binomial lattice approach of Cox, Ross, and Rubinstein (CRR) (1979) is a powerful and flexible method for valuing American options. Boyle (1988) developed an extension of the CRR procedure for option valuation in the case of two state variables. Boyle, Evnine and Gibbs (BEG) (1989) developed an *n*-dimensional extension of the CRR procedure. Kamrad and Ritchken (KR) (1991) developed a similar technique for valuing projects for one or more state variables. The KR model results in a pentanomial lattice and is more general than BEG model because it allows for horizontal jumps. The ability to model horizontal jumps is important for applications where the state variables may not change during a time interval.

Hull (1997) gives the method for Monte Carlo simulation that can be used for valuing options with more than one state variable. In our model, the manufacturer can change the decision about applying control charts at each time interval. Therefore, a decision given in one time interval does not affect the succeeding intervals and thus does not start a new path. This feature makes it relatively straight-forward to use Monte Carlo simulation in this problem.

3 APPLYING FINANCIAL MODELS TO MANUFACTURING QUALITY CONTROL

In this section, we provide a framework for the financial model that will be used in this paper. Total sales revenue of a product has two main sources of uncertainty: price and demand.

Let $S_1(t)$ be the price of the product at time t, and let $S_2(t)$ be the demand for the product at time t for a specified time interval. Then, total sales revenue $R(t)$ of that product per time interval at time t is

$$R(t) = S_1(t)\, S_2(t)$$

Total profit $P(t)$ per time interval at time t can be defined as

$$P(t) = R(t) - (\text{Fixed cost} + \text{Variable cost})$$

If we denote the fixed cost per time interval as F, and the variable cost per unit product as C, then $P(t)$ can be defined as

$$P(t) = S_1(t)\, S_2(t) - F - S_2(t)\, C$$

Let $\{S_1(t), S_2(t)\}$ define the state variable value at time t. The manufacturer has an option to apply (or not apply) control charts when desired.

The fundamentals of SPC are addressed in several introductory texts (e.g., see Montgomery (1997) or Grant and Levenworth (1996)). We consider the $\overline{X}$ chart here because it is the standard "workhorse" in many SPC applications.

Applying $\overline{X}$ control charts will result in a fixed cost of b per time interval. Let a be the variable sampling cost per unit (Collani *et al.* 1994). Let g be the cost of not applying $\overline{X}$ control charts per unit product, represented as a fraction of price (due to production scrap, product returns, loss of market share, etc.).

With these definitions, the profit per time interval can be determined as

$$P(t)=(1-g)S_1(t)\, S_2(t) - F - S_2(t)\, C \quad \text{without } \overline{X} \text{ chart} \quad (1a)$$

or

$$P(t)=S_1(t)\, S_2(t)-(F+b)-S_2(t)\,(C+a) \quad \text{with } \overline{X} \text{ chart.} \quad (1b)$$

4 NUMERICAL PROCEDURES

We will use two numerical procedures, pentanomial lattice and the Monte-Carlo simulation, to value the control chart option over a decision horizon. Both the procedures use Equation (1). The pentanomial lattice approach has the advantage of giving a more accurate numerical solution compared to Monte Carlo simulation. Monte Carlo simulation has the advantage of providing an estimate of the variability of the option value, and maximum, average, and minimum profits.

4.1 Using a Pentanomial Lattice

First, the two-state KR model will be used to find the option value of $\overline{X}$ control charts. Two state variables under consideration were given as $S_1(t)$ (price), and $S_2(t)$ (demand) in Section 3.

The two-state KR model is as follows. Assume the joint density of the two state variables $S_1(t)$ and $S_2(t)$ is bivariate lognormal. For state variable i (i=1,2), let the instantaneous mean be $\mu_i = r - \sigma_i^2/2$, where r is the risk free interest rate, and let the instantaneous variance be σ_i^2. Let ρ be the correlation coefficient between the two state variables. For each state variable over $[t, t+\Delta t]$, we have

$$\ln S_i(t+\Delta t) = \ln S_i(t) + \zeta_i(t)$$

where $\zeta_i(t)$ is a normal random variable with mean $\mu_i\Delta t$ and variance $\sigma_i^2\Delta t$. The instantaneous correlation between $\zeta_1(t)$ and $\zeta_2(t)$ is ρ.

The joint normal random variable $\{\zeta_1(t), \zeta_2(t)\}$ is approximated by a pair of multinomial discrete random variables with the following distribution

$\zeta^a_1(t)$	$\zeta^a_2(t)$	Probability
υ_1	υ_2	p_1
υ_1	$-\upsilon_2$	p_2
$-\upsilon_1$	$-\upsilon_2$	P_3
$-\upsilon_1$	υ_2	P_4
0	0	P_5

where $\upsilon_i = \lambda\sigma_i\sqrt{\Delta t}$ (i = 1,2), and $\lambda \geq 1$.

The convergence of the approximating distribution to the true distribution as $\Delta t \to 0$ is ensured by setting the first two moments of the approximating distribution to the true moments of the continuous distribution. Specifically, this means

$$\begin{aligned}
E\{\zeta^a_1(t)\} &= \upsilon_1 (p_1 + p_2 - p_3 - p_4) = \mu_1 \Delta t \\
E\{\zeta^a_2(t)\} &= \upsilon_2 (p_1 - p_2 - p_3 + p_4) = \mu_1 \Delta t \\
\mathrm{Var}\{\zeta^a_1(t)\} &= \upsilon_1^2 (p_1 + p_2 + p_3 + p_4) = \sigma_1^2 \Delta t + O(\Delta t) \\
\mathrm{Var}\{\zeta^a_2(t)\} &= \upsilon_2^2 (p_1 + p_2 + p_3 + p_4) = \sigma_2^2 \Delta t + O(\Delta t)
\end{aligned}$$

In addition, the covariance terms must also be equal. This is achieved by equating the expected value of two variables:

$$E\{\zeta^a_1(t)\, \zeta^a_2(t)\} = \upsilon_1 \upsilon_2 (p_1 - p_2 + p_3 - p_4) = \sigma_1 \sigma_2 \rho \Delta t + O(\Delta t).$$

Substituting $\upsilon_i = \lambda \sigma_i \sqrt{\Delta t}$ (i=1,2), and for sufficiently small Δt, the above equations yield expressions for p_1, p_2, p_3, and p_4:

$$p_1 = \frac{1}{4}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right\} \tag{2}$$

$$p_2 = \frac{1}{4}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right\} \tag{3}$$

$$p_3 = \frac{1}{4}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right) + \frac{\rho}{\lambda^2}\right\} \tag{4}$$

$$p_4 = \frac{1}{4}\left\{\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(-\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right) - \frac{\rho}{\lambda^2}\right\} \tag{5}$$

Since $p_1 + p_2 + p_3 + p_4 + p_5 = 1$, we have

$$p_5 = 1 - \frac{1}{\lambda^2}. \tag{6}$$

There are five paths leaving each node in the pentanomial lattice. An up move for each state variable is denoted by u, and a down move is denoted by d. It is convenient to impose the condition that $ud = 1$, so that an up followed by a down is equal to 1. The total number of nodes after n iterations is

$$\sum_{i=0}^{n}\left[(i+1)^2 + i^2\right] = \left(4n^3 + 12n^2 + 14n + 6\right)/6 \quad \text{if } ud = 1$$

$$\sum_{i=0}^{n} 5^i = \left(5^{n+1} - 1\right)/4 \quad \text{if } ud \neq 1$$

In just 10 iterations, a pentanomial lattice will have 12,207,031 nodes if $ud \neq 1$, whereas there will be only 891 nodes if we impose $ud = 1$. Figure 1 shows the first two iterations of a pentanomial lattice with the condition ud=1. The first element in parenthesis shows the change in the first state variable, and the second element shows the change in the second state variable. As can be seen in Figure 1, there are a total of 19 nodes in the pentanomial lattice for two iterations when $ud = 1$.

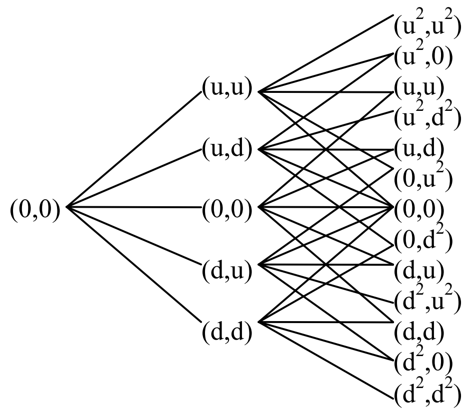

Figure 1: Pentanomial Lattice

We will construct a pentanomial lattice to show the profit associated with the option to apply a control chart. In that lattice, each node contains a profit value per time step. We first construct a pentanomial lattice that contains the profits for the case that the control chart is not applied. We then construct another pentanomial lattice for the case that the control chart is used. We then construct a new lattice that contains the profit differences as shown in the following equation:

$$Max\left[(\text{profit by using chart} - \text{profit by not using chart}), 0\right]$$

Backward discounting is applied on the lattice starting from the last time interval. First, an expected value is found by multiplying the jump probabilities with the corresponding profit values in the five nodes. Next, this expected value is discounted with the risk free interest rate in one interval. This value is the expected amount of profit in the last step originated from one of the nodes in the preceding time interval. The expected discounted value is added to the profit value in the origin node of the five following nodes, so that the expected total profit is found for those two time intervals. When this calculation is done for all nodes in one time interval, and then for all remaining nodes going back one time interval in each iteration, the expected discounted value for time zero is determined.

The program code was written in JavaScript so that it could be run using Netscape Navigator 2.0 or Microsoft Internet Explorer 3.0 or later versions. First, the user enters all parameter values into the input boxes. These values are checked to ensure that they are valid. Next, two pentanomial lattices are formed with the given number of intervals. One of these lattices contains the profit values for the case when the control chart is used at all time intervals, and the other lattice contains the profit values for the case when the control chart is not used. The program stores the values for each lattice in separate arrays.

Table 1 gives a list of the input values required for pentanomial lattice program. Section 5 gives the input and output windows of the program and results on the sensitivity of the option value against g, b, and a, the three cost parameters related with control charts.

Table 1: Inputs for Pentanomial Lattice Program

Input	Explanation
T	total time until expiration (in unit time)
N	total number of time intervals until expiration
r	percent interest rate per unit time
$S_1(0)$	initial price of the product
$S_2(0)$	initial demand per time interval
σ_1	(percent volatility of price per unit time)/100
σ_2	(percent volatility of demand per unit time)/100
ρ	correlation between price and demand
λ	stretch parameter ($\lambda \geq 1$)
F	fixed production cost per time interval
C	variable production cost per product
g	cost of not applying $\overline{X}$ charts (as % of price)
b	fixed cost of applying $\overline{X}$ charts per time interval
a	variable sampling cost per product

4.2 Using Monte Carlo Simulation

Simulation models may be used to give numerous possible paths of evolution for underlying state variables from the present to the final date in the option. In the commonly used Monte Carlo simulation method, the optimal strategy on each path is determined and the payoff calculated (Amram and Kulatilaka, 1999).

Suppose that the process followed by the underlying variable in a risk-neutral world is

$$dS = \hat{\mu} S dt + \sigma S dz \tag{7}$$

where dz is a Weiner process, $\hat{\mu}$ is the expected return in a risk-neutral world ($\hat{\mu}=r$), and σ is the volatility. To simulate the path followed by S, we divide the life of the derivative into N short intervals of length Δt and approximate Equation (7) as

$$S(t+\Delta t) - S(t) = \hat{\mu} S(t)\Delta t + \sigma S(t)\varepsilon\sqrt{\Delta t} \tag{8}$$

where $S(t)$ denotes the value of S at time t, ε is a random sample from a normal distribution with zero mean and unit standard deviation. This enables the value of S at time Δt to be calculated from the initial value of S, the value at time $2\Delta t$ to be calculated from the value at time Δt, and so on. One simulation trial involves constructing a complete path for S using N random samples from a normal distribution (Hull, 1997).

From Ito's lemma (see Hull, 1997) for a discussion of Ito, 1951), the process followed by ln S is

$$d\ln S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

so that

$$S(t+\Delta t) = S(t)\exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}\right] \tag{9}$$

This equation is used to construct a path for S in a similar way to Equation (8). Whereas Equation (8) is true only in the limit as Δt tends to zero, Equation (9) is exactly true for all Δt (Hull, 1997).

As in our problem that was modeled by pentanomial lattice, we continue to let $S_1(t)$ be the price of the product at time t, and $S_2(t)$ be the demand per time interval of that product at time t. Profit per time interval can be modeled with Equation (1). Required $S_1(t)$ and $S_2(t)$ values will be obtained from Monte Carlo simulation by using equation (9).

Two correlated samples ε_1 and ε_2 are found using a two-step procedure. First, independent samples x_1 and x_2 from a univariate standardized normal distribution are obtained generating U_1 and U_2 as IID U(0,1), then setting $x_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ and $x_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ (Law and Kelton, 2000). Second, the required samples ε_1 and ε_2 are calculated as:

$$\varepsilon_1 = x_1$$
$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

where ρ is the coefficient of correlation (Hull, 1997).

The program code for the Monte Carlo simulation was also written in Javascript, and it can be executed with Microsoft Internet Explorer 3.0 or Netscape Navigator 2.0, or later versions.

When the program is executed, the user is asked to enter all required parameter values into the input boxes. After the parameter values are entered, the program generates two arrays. The first array contains the $S_1(t)$ (price) values of the product for all time steps until the expiration date of the option, and the second array contains the corresponding $S_2(t)$ (demand) values of the product. Then, profit values when $\overline{X}$ chart is used, and profit values when $\overline{X}$ chart is not used are obtained for each time interval. These profit amounts are compared for each time step, and the best strategy in each time step is found. Then, present value of profit for the best strategy, and the option value are calculated.

One of the required inputs is the number of simulation runs. A single option value is simulated in each simulation run. The expected option value, which is the average of all option values obtained from simulation runs, is given in the output window. The output window also gives the minimum, the maximum, and the average profit obtained during the simulation runs and the standard deviation of the option value. Table 2 describes all of the required inputs for the Monte Carlo simulation program.

Table 2: Inputs for Monte Carlo Simulation

Input	Explanation
n	number of simulation runs
T	total time until expiration (in unit time)
N	total number of time intervals until expiration
r	percent interest rate per unit time
$S_1(0)$	initial price of the product
$S_2(0)$	initial demand per time interval
σ_1	percent volatility of price per unit time
σ_2	percent volatility of demand per unit time
ρ	correlation between price and demand
F	fixed production cost per time interval
C	variable production cost per product
g	cost of not applying $\bar{X}$ charts (as % of price)
b	fixed cost of applying $\bar{X}$ charts per time interval
a	variable sampling cost per product

5 EXAMPLES AND NUMERICAL RESULTS

In this section, we use the programs discussed above to compare the pentanomial lattice and Monte-Carlo approaches for valuing the control chart option for a specific numerical case. We also investigate the sensitivity of the option value to the three key control chart parameters.

5.1 A Comparative Case for Pentanomial Lattice and Monte-Carlo Valuation

Table 1 shows the pentanomial lattice input parameters. In this case, we use the specific input values and realize the output values given in Figure 2a. This allows us to evaluate a one-year (12 month) horizon. The *option value* is \$14,181 as given in Figure 2b.

Table 2 shows the Monte-Carlo simulation input parameters. In this case, we use the specific input values and realize the output values given in Figure 3a and again evaluate a one-year horizon. The *option value*, based on 10,000 simulation runs, is \$14,153 (Figure 3b). This simulation average represents a 0.20% difference compared to the more accurate pentanomial lattice solution. The minimum and maximum option values obtained from 10,000 simulation runs, and the standard deviation of the option value are also reported (Figure 3b).

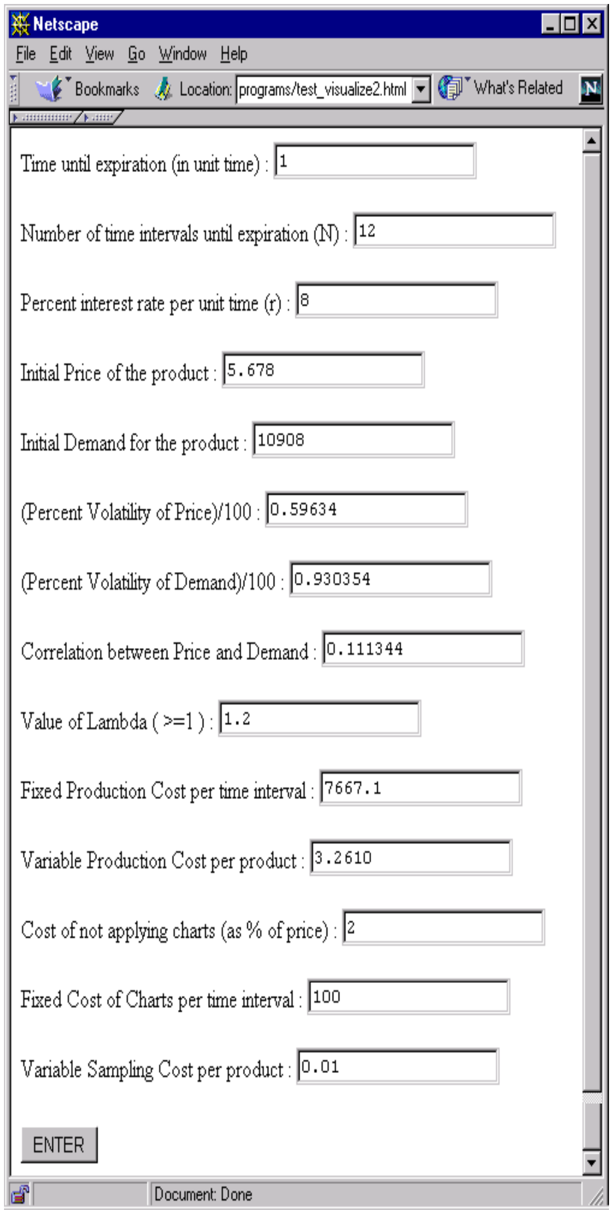

Figure 2a: Input for the Pentanomial Lattice Program.

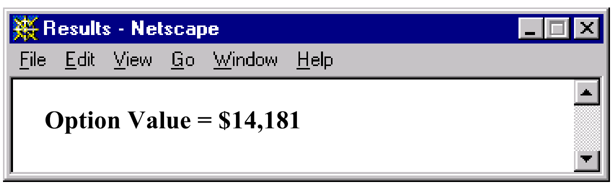

Figure 2b: Output of Pentanomial Lattice Program.

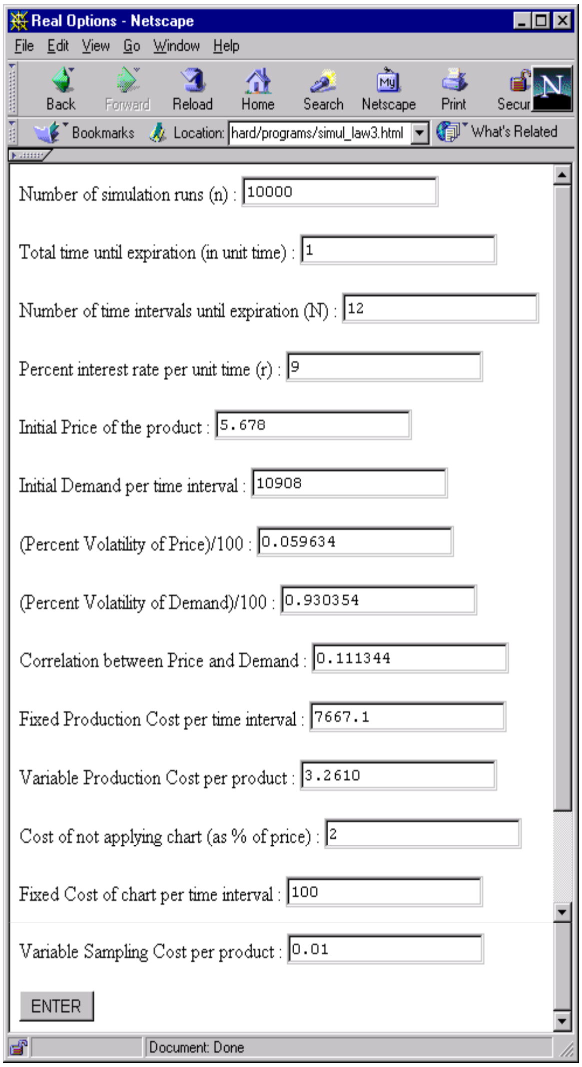

Figure 3a: Input for Monte-Carlo Simulation Program

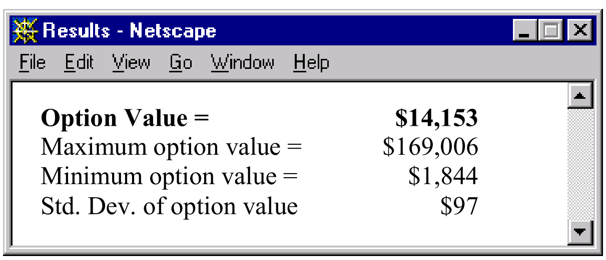

Figure 3b: Output of Monte-Carlo Simulation Program.

5.2 Sensitivity of Option Value to Key Control Chart Parameters

The last three parameters shown in Tables 1 and 2 – *g* (cost of not applying chart), *b* (fixed cost of applying chart) and *a* (variable sampling cost per product) – are characteristic to the control chart process. Figures 4-6 illustrate how the option value changes for our numerical example when each of these key factors are independently varied over a range (with all other factors held constant. Our analysis of these results suggested that there was no substantial interaction among the factors.

Specifically, Figure 4 suggests that we derive more benefit from the option when the cost of not applying control charts is relatively high. Figure 5 suggests that greater value is achieved when the variable sampling costs are small. Figure 6 suggests that the option is preferable when the fixed costs are small.

In general, this type of sensitivity analysis may guide the decision maker on which ranges suggest that the control charts option should always be used, never used, or used with caution.

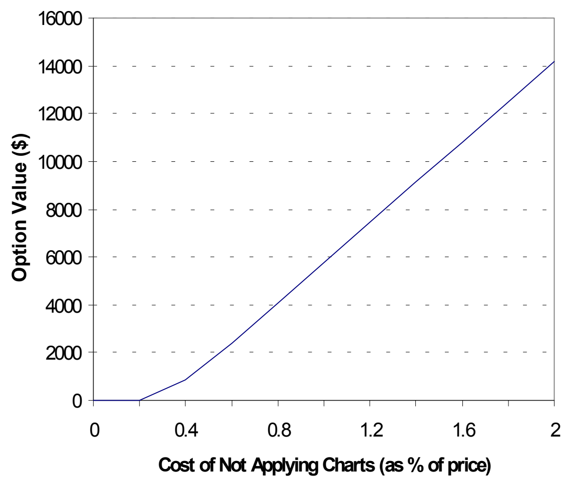

Figure 4: Cost of Not Applying Chart vs. Option Value

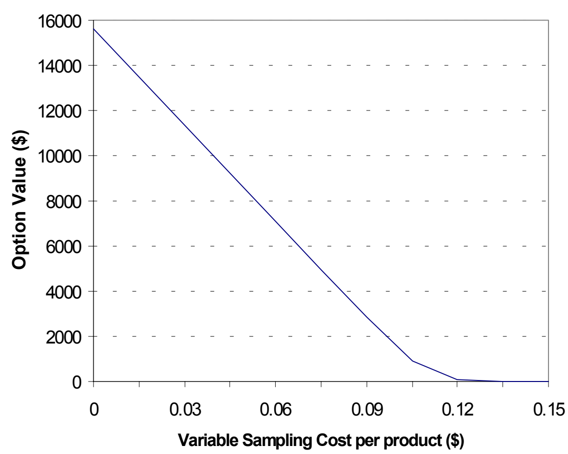

Figure 5: Variable Sampling Cost vs. Option Value

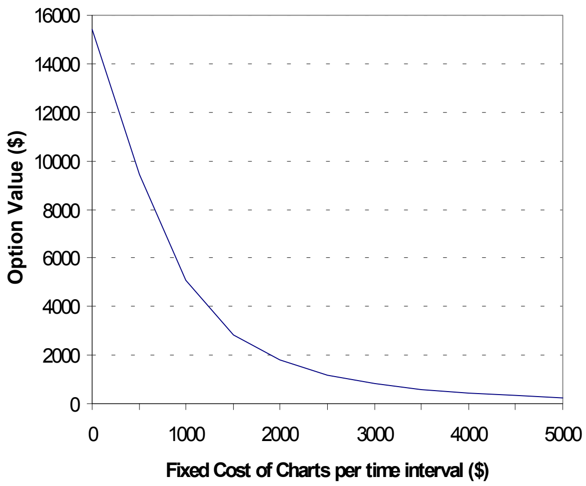

Figure 6: Fixed Cost of Charts vs. Option Value

6 SUMMARY AND CONCLUDING REMARKS

In this paper, we have shown how the value of using control charts can be determined using a real options framework. The need for this approach was motivated by the inability of existing economic control chart methods to address dynamics in the market condition with respect to price and demand.

By connecting the dynamic aspects with the manufacturing operational aspects, we now have a way to address a key issue: the bottom-line profitability associated with the control decision. Monte-Carlo simulation was key in this study because it provided a means for assessing the variability associated with an option value.

Related work is discussed in Nembhard, Shi, and Park (2000). Future work will address the possibility of generalizing these results, extending them to other control decisions, and using them to evaluate issues of flexibility in manufacturing operations.

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